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MEASURE OF NON STRICT SINGULARITY OF SCHECHTER ESSENTIAL SPECTRUM OF TWO BOUNDED OPERATORS AND APPLICATION

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Abstract. In this paper, we discuss the essential spectrum of sum of two bounded operators using measure of non strict singularity. Based on this new investigation, a problem of one-speed neutron transport operator is presented.
Keywords: Fredholm operators, lower (respectively upper) semi-Fredholm operators, essential spectrum, measure of non-strict-singularity, one-speed neutron transport operator.
MSC(2010): Primary: 47A10; Secondary: 47A53, 34K08.

1. Introduction

One of the interesting problems in the study of essential spectrum of linear operators on Banach spaces is the invariance of the essential spectrum under (additive) perturbation. This problem has attracted the attention of several authors and has produced man important results in the spirit of [8, 14, 15].

In 1972, M. Schechter [23] defined a new concept of measuring of a bounded operator acting on Banach spaces, called, the measure of non strict singularity and denote by g (see Definition 2.6) which has been successfully applied in many areas such as: topology, functional analysis, matrix theory and operator theory. Later, N. Moalla [17] proved that $I - T$ is a Fredholm operator with index null, for every bounded operator T satisfying $g(T^m) < 1$, for some $m > 0$ (see [17, Proposition 2.3]).

Our interest concentrates on characterizing the Schechter essential spectrum, $\sigma_{ess}(.)$, to the sum of two bounded operators acting on Banach spaces involving the concept of left and right Fredholm inverse (see Definition 2.4). This

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motivation is based on a new concept of measure of non strict singularity recently studied in [17]. More precisely, we derive some new conditions on the operators A and B to obtain the following characterization on the Schechter essential spectrum

$$\sigma_{ess}(A+B) \subseteq \sigma_{ess}(A),$$

which represents an amelioration of some earlier works. Hence, our results improve and give more supplements to those in [1,9,14,15,17,25]. Furthermore, a typical example of a problem of one-speed neutron transport operator is given for showing efficiency and accuracy of this work on L_1-space based on the regularity of the collision operator introduced by B. Lods in [16] as follows:
$A_H\psi(x,v) := -v_3\dfrac{\partial\psi}{\partial x}(x,v) - \sigma(x,v)\psi(x,v) + \displaystyle\int_K \kappa(x,v,v')\psi(x,v')dv'$, on D,
where $D = (0,1)\times K$ with K is the unit sphere of $\mathbb{R}^3$, $x \in (0,1)$, $v = (v_1, v_2, v_3) \in K$, $\kappa(.,.,.)$ is a nonlinear function and $\sigma(.,.)$ is a positive bounded function. This equation describes the transport solution $\psi(.,.)$ in the vertical direction and characterizes the possible leakage of energy at boundary of the channel.

Now, let us outline the content of this paper. In Section 2, we gather some results and notations from Fredholm perturbation theory connected with the notion of measure of non strict singularity. In Section 3, we use the notion of measure of non strict singularity to establish the invariance of the essential spectrum of two bounded linear operators. Finally, to illustrate the applicability of this new investigation, we introduce a problem of one-speed neutron transport operator (see Theorem 4.5).

2. Notations and definitions

We start this section by giving some basic definitions and notations that we will need in the sequel. Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ (respectively $\mathcal{C}(X,Y)$) the set of all bounded (respectively closed, densely defined) linear operators from X into Y. The subset of all compact operators of $\mathcal{L}(X,Y)$ is designated by $\mathcal{K}(X,Y)$. For $A \in \mathcal{C}(X,Y)$, we write $\mathcal{D}(A) \subset X$ for the domain, $N(A) \subset X$ for the null space and $R(A) \subset Y$ for the range of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y. The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) = \{A \in \mathcal{C}(X,Y);\ \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } Y\},$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X,Y) = \{A \in \mathcal{C}(X,Y);\ \beta(A) < \infty \text{ and } R(A) \text{ is closed in } Y\}.$$

The set of bounded upper (respectively lower) semi-Fredholm operators from X into Y is defined by

$\Phi^b_+(X,Y) = \Phi_+(X,Y) \cap \mathcal{L}(X,Y)$ (respectively $\Phi^b_-(X,Y) = \Phi_-(X,Y) \cap \mathcal{L}(X,Y)$).

$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ (respectively $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$) denotes the set of Fredholm (respectively semi-Fredholm) operators from X into Y. The set of bounded Fredholm operators from X into Y is defined by $\Phi^b(X,Y) = \Phi(X,Y) \cap \mathcal{L}(X,Y)$. If $X = Y$, then the sets $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{C}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ are replaced, respectively, by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\mathcal{C}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\Phi^b(X)$, $\Phi^b_+(X)$ and $\Phi^b_-(X)$. A complex number λ is in Φ_A, Φ_{+A} or Φ_{-A}, that is, $\lambda - A$ is in $\Phi(X)$, $\Phi_+(X)$ or $\Phi_-(X)$ respectively. The index of an operator $A \in \Phi_\pm(X)$ is defined by $i(A) := \alpha(A) - \beta(A)$.
The set of Fredholm operators defines the corresponding Schechter essential spectrum (see [22, 24])

$$\sigma_{ess}(A) := \mathbb{C} \setminus \{\lambda \in \Phi_A;\ i(\lambda - A) = 0\}.$$

Let $A \in \mathcal{C}(X)$. It follows from the closedness of A that $\mathcal{D}(A)$ endowed with the graph norm $\|.\|_A$ ($\|x\|_A = \|x\| + \|Ax\|$) is a Banach space denoted by X_A. Clearly, for $x \in \mathcal{D}(A)$ we have $\|Ax\| \le \|x\|_A$; so $A \in \mathcal{L}(X_A, X)$. Let B be a linear operator. If $\mathcal{D}(A) \subset \mathcal{D}(B)$, then B will be called A-defined. If B is A-defined operator, hence we will denote its restriction to $\mathcal{D}(A)$ by $\hat{B}$. Moreover, if $\hat{B}$ belongs to $\mathcal{L}(X_A, X)$, we say that B is an A-bounded operator. Furthermore, we have the obvious relations:

$$\left\{\begin{array}{l} \alpha(\hat{A}) = \alpha(A),\ \beta(\hat{A}) = \beta(A),\ R(\hat{A}) = R(A), \\ \alpha(\hat{A} + \hat{B}) = \alpha(A + B), \\ \beta(\hat{A} + \hat{B}) = \beta(A + B) \text{ and } R(\hat{A} + \hat{B}) = R(A + B). \end{array}\right. \tag{2.1}$$

Hence, $A \in \Phi(X)$ (respectively $\Phi_+(X)$) if and only if $\hat{A} \in \Phi(X_A, X)$ (respectively $\Phi_+(X_A, X)$).

We will recall some basic definitions for bounded operators in Banach spaces that are useful in the reminder of this paper.

Definition 2.1. Let X and Y be two Banach spaces. An operator $A \in \mathcal{L}(X,Y)$ is said to be weakly compact if $A(B)$ is relatively weakly compact in Y for every bounded $B \subset X$.

The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$ the family of weakly compact operators on $X, \mathcal{W}(X) := \mathcal{W}(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (see [7]).

Definition 2.2. Let X and Y be two Banach spaces. An operator $S \in \mathcal{L}(X,Y)$ is said to be strictly singular if the restriction of S to any infinite-dimensional subspace of X is not an homeomorphism. Let $S(X,Y)$ denote the set of strictly singular operators from X to Y.

The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [12] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators we refer to [12]. Note that $S(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$. In general, if $X=Y$, one has $S(X):=S(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space, then $S(X)=\mathcal{K}(X)$. The class of weakly compact operators in L_1-spaces (respectively $\mathcal{C}(\Omega)$- spaces with Ω is a compact Hausdroff space) is nothing else than the family of strictly singular operators on L_1-spaces (respectively $\mathcal{C}(\Omega)$-spaces) (see [20, Theorem 1]).

Remark 2.3. Notice that according to Theorem 1 established by A. Pelczynski in [20] the class of weakly compact operators on L_1-spaces is nothing else but the family of strictly singular operators on L_1-spaces.

If $1<p<\infty$, X_p (where X_p denotes the space $L_p(\Omega,d\mu)$ for $1\leq p\leq\infty$ and (Ω,Σ,μ) stands for a positive measure space) is reflexive and then $\mathcal{L}(X_p)=\mathcal{W}(X_p)$. On the other hand, it follows from [6, Theorem 5.2] that $\mathcal{K}(X_p)\subsetneqq\mathcal{S}(X_p)\subsetneqq\mathcal{W}(X_p)$ with $p\neq 2$. For $p=2$ we have $\mathcal{K}(X_p)=\mathcal{S}(X_p)=\mathcal{W}(X_p)=\mathcal{F}(X_p)$.

In this work, we are interested to discuss the invariance of essential spectrum of two bounded operators involving the theory of Fredholm inverse. For this purpose, we need to review the following definition due to V. Müller; see [19].

Definition 2.4. Let X and Y be two Banach spaces.

(i) An operator $A\in\mathcal{L}(X,Y)$ is said to have a left Fredholm inverse if there are maps $A_l\in\mathcal{L}(Y,X)$ and $K\in\mathcal{K}(X)$ such that I_X+K extends A_lA. The operator A_l is called left Fredholm inverse of A.

(ii) An operator $A\in\mathcal{L}(X,Y)$ is said to have a right Fredholm inverse if there is a map $A_r\in\mathcal{L}(Y,X)$ such that $I_Y-AA_r\in\mathcal{K}(Y)$. The operator A_r is called right Fredholm inverse of A.

(iii) An operator $A\in\mathcal{C}(X,Y)$ is said to have a left Fredholm inverse (respectively right Fredholm inverse) if $\hat{A}$ has a left Fredholm inverse (respectively right Fredholm inverse) of A.

We know by the classical theory of Fredholm operators, see for example [11], that A belongs to $\Phi_+(X),\Phi_-(X)$ or $\Phi(X)$ if it possesses a left, right or two-sided Fredholm inverse, respectively.
Several measures of non compactness were defined in the literature (see for example [2]). Among them, let us mention the first one introduced in 1930 by K. Kuratawski [13]. Another one called the Hausdroff measure of non compactness was defined and studied by V. Rakočević in [21]:

Definition 2.5. For a nonempty bounded subset Ω of X, consider the Hausdorff measure of non compactness of Ω as follows: $q(\Omega)=\inf\{r>0,\ \Omega$ can be covred by a finite set of open balls of radius $r\}$. The Hausdroff measure of non compactness of $A \in \mathcal{L}(X,Y)$ is defined by:

$$q(A) = q[A(B_X)],$$

where B_X denotes the closed unit ball in X; that is, the set of all $x \in X$ satisfying $\|x\| \leq 1$.

Later then, the notion of the measure of non compactness was generalized to a new concept of measuring, so called measure of non strict singularity, by M. Schechter in [23], which is related to define this kind of measure as well:

Definition 2.6. For $A \in \mathcal{L}(X,Y)$, set $g_M(A) = \inf_{N \subset M} q(A|_N)$ and $g(A) = \sup_{M \subset X} g_M(A)$, where M and N represent infinite dimensional subspaces of X, and $A_{|N}$ denotes the restriction of A to the subspace N. The semi-norm g is called a measure of non strict singularity.

3. Essential spectrum of the sum of two operators

The goal of this section is to provide a characterization of Schechter essential spectrum of the sum of two bounded operators by means of measure of non strict singularity.

Theorem 3.1. *Let X be a Banach space, and let A and B be two operators in $\mathcal{L}(X)$. We have:*

(i) *If for each $\lambda \in \Phi_A$, there exists a left Fredholm inverse $A_{\lambda l}$ of $\lambda - A$ such that $g((BA_{\lambda l})^n) < 1$, for some $n > 0$, then $\sigma_{ess}(A+B) \subseteq \sigma_{ess}(A)$.*

(ii) *If for each $\lambda \in \Phi_A$, there exists a right Fredholm inverse $A_{\lambda r}$ of $\lambda - A$ such that $g((A_{\lambda r}B)^n) < 1$, for some $n > 0$, then $\sigma_{ess}(A+B) \subseteq \sigma_{ess}(A)$.*

Proof. (i) Suppose that $\lambda \notin \sigma_{ess}(A)$ that is $\lambda - A \in \Phi(X)$ with $i(\lambda - A) = 0$. Since $A_{\lambda l}$ is a Fredholm inverse of $\lambda - A$, then there exists $F \in \mathcal{K}(X)$ such that

$$A_{\lambda l}(\lambda - A) = I - F \text{ on } X, \tag{3.1}$$

so, $A_{\lambda l}(\lambda - A) + F = I_X$. It follows from equation (3.1) that the operator $\lambda - A - B$ can be written in the following form $\lambda - A - B = \lambda - A - B(A_{\lambda l}(\lambda - A) + F) = (I_X - BA_{\lambda l})(\lambda - A) - BF$.

Therefore, by [17, Proposition 2.3], one has $g((BA_{\lambda l})^n) < 1$. Then, we infer that $I_X - BA_{\lambda l} \in \Phi(X)$ with $i(I_X - BA_{\lambda l}) = 0$. The use of [19, Theorem 12, p. 153], allows us to conclude that

$$(I_X - BA_{\lambda l})(\lambda - A) \in \Phi(X)$$

and

$$\begin{aligned} i[(I_X - BA_{\lambda l})(\lambda - A)] &= i(I_X - BA_{\lambda l}) + i(\lambda - A) \\ &= i(\lambda - A). \end{aligned}$$

Taking into account the fact that $BF \in \mathcal{K}(X)$, we have $\lambda - A - B \in \Phi(X)$ and $i(\lambda - A - B) = i(\lambda - A)$. This proves that $\lambda \notin \sigma_{ess}(A + B)$. Hence, we find $\sigma_{ess}(A + B) \subseteq \sigma_{ess}(A)$. (ii) Let $\lambda \in \mathbb{C}$ and $A_{\lambda r}$ be a right Fredholm inverse of $\lambda - A$. Following Definition 2.4 (ii), we conclude that there exists a compact operator $K \in \mathcal{K}(X)$, such that $(\lambda - A)A_{\lambda r} = I - K$ on X. Thus, the operator $\lambda - A - B$ may be written as follows: $\lambda - A - B = \lambda - A - ((\lambda - A)A_{\lambda r} + F)B = (\lambda - A)(I_X - A_{\lambda r}B) - FB$. Arguing as above, we can easily derive the rest of the proof of this assertion in the same way as (i). □

Remark 3.2.
- (i) Theorem 3.1 remains true if we replace the assumptions $g((BA_{\lambda l})^n) < 1$ and $g((A_{\lambda r}B)^n) < 1$ by $q((BA_{\lambda l})^n) < 1$ and $q((A_{\lambda r}B)^n) < 1$, (respectively $\|(BA_{\lambda l})^n\| < 1$ and $\|(A_{\lambda r}B)^n\| < 1$), for some $n > 0$.
- (ii) The result of the Theorem 3.1 remains valid for closed, densely defined linear operator A and A-bounded operator B on X. It is sufficient to replace A by $\hat{A} \in \mathcal{L}(X_A, X)$ and B by $\hat{B} \in \mathcal{L}(X_A, X)$ as in proof of Theorem 3.1.
- (iii) In view of [17, Proposition 2.2], Theorem 3.1 may be viewed as an extension of many known results in the literature, in particular it extends the results obtained in [1, 14, 15, 24] to a wide classes of perturbing operators under measuring of non strict singularity.

An immediate consequence of Theorem 3.1 in terms of power strictly singular operators is expressed as well.

Corollary 3.3. *Let X be a Banach space, and let A and B be two operators in $\mathcal{L}(X)$.*
- (i) *If for each $\lambda \in \Phi_A$, there exists a Fredholm inverse $A_{\lambda l}$ of $\lambda - A$ such that, for some $n > 0$,*
$$(BA_{\lambda l})^n \in \mathcal{S}(X),$$
then $\sigma_{ess}(A + B) \subseteq \sigma_{ess}(A)$.
- (ii) *If for each $\lambda \in \Phi_A$, there exists a Fredholm inverse $A_{\lambda r}$ of $\lambda - A$ such that, for some $n > 0$,*
$$(A_{\lambda r}B)^n \in \mathcal{S}(X),$$
then $\sigma_{ess}(A + B) \subseteq \sigma_{ess}(A)$.

Remark 3.4.
- (i) Corollary 3.3 remains true if we replace the assumptions $(BA_{\lambda l})^n \in \mathcal{S}(X)$ and $(A_{\lambda r}B)^n \in \mathcal{S}(X)$ by $(BA_{\lambda l})^n \in \mathcal{K}(X)$ and $(A_{\lambda r}B)^n \in \mathcal{K}(X)$ $\big($respectively $(BA_{\lambda l})^n \in \mathcal{W}(X)$ and $(A_{\lambda r}B)^n \in \mathcal{W}(X)\big)$, for some $n > 0$.
- (ii) The results of Theorem 3.1 and Corollary 3.3 remain true for other types of essential spectrum. But, it provides sufficient conditions to ensure it for Browder essential spectrum.

We now discuss a typical example motivating the abstract theoretical results.

4. **Application to transport operator**

Our aim in the present work is to establish the invariance of the essential spectrum via the concept of measure of non strict singularity of n-power strictly bounded operator on L_1-space.

In order to state our general framework, we first make the functional setting of the problem precise. Let the following space:

$X := L_1(D, dxdv)$, where $D = (0,1) \times K$ with K the unit sphere of $\mathbb{R}^3$, $x \in (0,1)$ and $v = (v_1, v_2, v_3) \in K$.
Define the following sets representing the incoming D^i and the outgoing D^0 boundary of the phase space D

$$\begin{array}{lll} D^i = & D_1^i \cup D_2^i = & \{0\} \times K^1 \cup \{1\} \times K^0, \\ D^0 = & D_1^0 \cup D_2^0 = & \{0\} \times K^0 \cup \{1\} \times K^1, \end{array}$$

for

$$K^0 = K \cap \{v_3 < 0\} \ \text{ and } \ K^1 = K \cap \{v_3 > 0\}.$$

Furthermore, we introduce the following boundary spaces

$$\begin{aligned} X^i &:= L_1(D^i, |v_3|dv) := L_1(D_1^i, |v_3|dv) \oplus L_1(D_2^i, |v_3|dv) \\ &:= X_1^i \oplus X_2^i \end{aligned}$$

and

$$\begin{aligned} X^0 &:= L_1(D^0, |v_3|dv) := L_1(D_1^0, |v_3|dv) \oplus L_1(D_2^0, |v_3|dv) \\ &:= X_1^0 \oplus X_2^0. \end{aligned}$$

We define the partial Sobolev space $\mathcal{W}$ by:

$$\mathcal{W} = \left\{ \psi \in X,\ v_3 \frac{\partial \psi}{\partial x} \in X \right\}.$$

It is well-known that any function $\psi \in \mathcal{W}$ has traces (see, for instance, [4]) on the spatial boundary denoted respectively by ψ^o and ψ^i which represent respectively the outgoing and the incoming fluxes related by the boundary operator H; namely:

$$\left\{ \begin{array}{l} H : X_1^0 \times X_2^0 \longrightarrow X_1^i \times X_2^i \\ \\ H \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{array} \right.$$

with $k, l \in \{1, 2\}$, $H_{kl} : X_l^0 \longrightarrow X_k^i$, $H_{kl} \in \mathcal{L}(X_l^0, X_k^i)$, defined such that the boundary conditions can be written as $\psi^i = H\psi^o$.

Now, we define the streaming operator T_H (with a domain including the boundary conditions) by:

$$\begin{cases} T_H : \mathcal{D}(T_H) \subseteq X \longrightarrow X \\ \qquad \psi \longrightarrow T_H\psi(x,v) = -v_3 \dfrac{\partial \psi}{\partial x}(x,v) - \sigma(x,v)\psi(x,v) \\ \qquad \mathcal{D}(T_H) = \\ \{\psi \in \mathcal{W};\ \psi_{|D^i} := \psi^i \in X^i;\ \psi_{|D^0} := \psi^o \in X^0 \text{ and } \psi^i = H(\psi^o)\}. \end{cases}$$

Let λ^* be the real defined by

$$\lambda^* := \text{ess-}\inf\{\sigma(x,v),\ \ (x,v) \in D\}$$

and

$$\lambda_0 := \begin{cases} -\lambda^* & \text{if } \|H\| \leq 1 \\ -\lambda^* + \log(\|H\|) & \text{if } \|H\| > 1. \end{cases}$$

Note that, if H is strictly singular operator, then

$$\sigma(T_H) = \{\lambda \in \mathbb{C};\ Re\lambda \leq -\lambda^*\}.$$

In fact, we can easily show that $\sigma(T_H)$ reduces to $\sigma C(T_H)$, the continuous spectrum of T_H, that is

$$\sigma(T_H) = \sigma C(T_H) = \{\lambda \in \mathbb{C};\ Re\lambda \leq -\lambda^*\}. \tag{4.1}$$

On the other hand, if $\lambda \in \sigma C(T_H)$ then $\mathcal{R}(\lambda - T_H)$ is not closed (otherwise $\lambda \in \rho(T_H)$). So, $\lambda \in \sigma_{ess}(T_H)$. This implies that $\sigma C(T_H) \subseteq \sigma_{ess}(T_H)$.

Thus according to equation (4.1), we obtain

$$\sigma_{ess}(T_H) = \{\lambda \in \mathbb{C};\ Re\lambda \leq -\lambda^*\}.$$

The transport operator can be formulated as follows: $\qquad A_H = T_H + K,$

where K is the following collision operator:

$$\begin{cases} K : X \quad \longrightarrow X \\ \qquad \psi \quad \longrightarrow \displaystyle\int_K \kappa(x,v,v')\psi(x,v')dv' \end{cases}$$

and the kernel $\kappa : (0,1) \times K \times K \longrightarrow \mathbb{R}$ is assumed to be measurable.

Observe that the collision operators K acts only on the variables v', so x may be viewed merely as a parameter in $(0,1)$. Then, we will consider K as the function

$$K(.) : x \in (0,1) \to K(x) \in \mathcal{L}(L_1(K,dv)).$$

Remark 4.1. Note that, to verify that the assumptions $g((K(\lambda-T_H)^{-1})^n)<1$ and $g((\lambda-T_H)^{-1}K)^n)<1$, for some $n>0$, we shall prove that the operators $K(\lambda-T_H)^{-1}$ and $(\lambda-T_H)^{-1}K$ are weakly compact on X.

From the previous remark, the problem is equivalent to finding a solution ψ (must be sought in $\mathcal{D}(T_H)$) of the equation:

$$(\lambda - T_H)\psi \quad = \quad \varphi, \tag{4.2}$$

for $\lambda\in\mathbb{C}$ and $\varphi\in X$. Thus, for $Re\lambda>-\lambda^*$, the solution of equation (4.2) is formally given by:

$$\psi(x,v)=\psi(0,v)\,e^{-\int_0^x \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}+ \frac{1}{|v_3|}\int_0^x e^{-\int_{x'}^x \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}\,\varphi(x',v)dx',\;\; v\in K^1 \tag{4.3}$$

$$\psi(x,v)=\psi(1,v)e^{-\int_x^1 \frac{\sigma(s,v)+\lambda}{|v_3|}ds}+ \frac{1}{|v_3|}\int_x^1 e^{-\int_x^{x'} \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}\,\varphi(x',v)dx',\;\; v\in K^0 \tag{4.4}$$

whereas $\psi(1,v)$ and $\psi(0,v)$ are given by:

$$\psi(1,v)=\psi(0,v)e^{-\int_0^1 \frac{\sigma(s,v)+\lambda}{|v_3|}ds}+ \frac{1}{|v_3|}\int_0^1 e^{-\int_{x'}^1 \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}\,\varphi(x',v)dx',\;\; v\in K^1 \tag{4.5}$$

$$\psi(0,v)=\psi(1,v)e^{-\int_0^1 \frac{\sigma(s,v)+\lambda}{|v_3|}ds}+ \frac{1}{|v_3|}\int_0^1 e^{-\int_0^{x'} \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}\,\varphi(x',v)dx', \qquad v\in K^0. \tag{4.6}$$

For the clarity of our subsequent analysis, we introduce the following bounded operators depending on the parameter λ,

$$\begin{cases} N_\lambda : X^i \longrightarrow X^0, N_\lambda u := (N_\lambda^+ u, N_\lambda^- u), \text{ with} \\ \\ (N_\lambda^+ u)(0,v) := u(1,v)\,e^{-\int_0^1 \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}, \qquad v\in K^0 \\ (N_\lambda^- u)(1,v) := u(0,v)\,e^{-\int_0^1 \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}, \qquad v\in K^1 \end{cases}$$

$$\begin{cases} B_\lambda : X^i \longrightarrow X, B_\lambda u := \chi_{K^0}(v)B_\lambda^+ u + \chi_{K^1}(v)B_\lambda^- u, \text{ with} \\ \\ (B_\lambda^- u)(x,v) := u(0,v)\,e^{-\int_0^x \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}, \qquad v\in K^1 \\ (B_\lambda^+ u)(x,v) := u(1,v)\,e^{-\int_x^1 \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}, \qquad v\in K^0 \end{cases}$$

$$\begin{cases} G_\lambda : X \longrightarrow X^0, G_\lambda\varphi := (G_\lambda^+\varphi, G_\lambda^-\varphi), \text{ with} \\ G_\lambda^-\varphi := \dfrac{1}{|v_3|}\displaystyle\int_0^1 e^{-\int_x^1 \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}\,\varphi(x,v)\,dx, \qquad v \in K^1 \\ G_\lambda^+\varphi := \dfrac{1}{|v_3|}\displaystyle\int_0^1 e^{-\int_0^x \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}\,\varphi(x,v)\,dx, \qquad v \in K^0 \end{cases}$$

and

$$\begin{cases} F_\lambda : X \longrightarrow X, F_\lambda\varphi := \chi_{K^0}(v)F_\lambda^+\varphi + \chi_{K^1}(v)F_\lambda^-\varphi, \text{ with} \\ F_\lambda^-\varphi := \dfrac{1}{|v_3|}\displaystyle\int_0^x e^{-\int_{x'}^x \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}\,\varphi(x',v)\,dx', \quad v \in K^1 \\ F_\lambda^+\varphi := \dfrac{1}{|v_3|}\displaystyle\int_x^1 e^{-\int_x^{x'} \frac{\sigma(s,v)+\lambda}{|v_3|}\,ds}\,\varphi(x',v)\,dx', \quad v \in K^0, \end{cases}$$

where $\chi_{K^0}(.)$ and $\chi_{K^1}(.)$ denote, respectively, the characteristic functions of the sets K^0 and K^1. The operators $N_\lambda, B_\lambda, G_\lambda$ and F_λ are bounded on their respective spaces. Their norms are bounded above, respectively by $e^{-(Re\lambda+\lambda^*)}, (\lambda^* + Re\lambda)^{-1}, (\lambda^* + Re\lambda)^{-1}$ and $(Re\lambda + \lambda^*)^{-1}$.
Let us now explicit the resolvent of T_H. To this aim, for any $Re\lambda > -\lambda^*$, equations (4.5) and (4.6) revel that

$$\psi^0 = N_\lambda H\psi^0 + G_\lambda\varphi$$

and

$$(I - N_\lambda H)\psi^0 = G_\lambda\varphi.$$

On the other hand, if $Re\lambda > \lambda_0$, the solution of the last equation is reduced to the following form:

$$\psi^0 = \sum_{n\geq 0}(N_\lambda H)^n G_\lambda\varphi. \tag{4.7}$$

Moreover, equations (4.3) and (4.4) can be rewritten as:

$$\psi = B_\lambda H\psi^0 + F_\lambda\varphi. \tag{4.8}$$

Substituting equation (4.7) into equation (4.8), we get the resolvent of T_H by the following form:

$$(\lambda - T_H)^{-1} = \sum_{n\geq 0} B_\lambda H(N_\lambda H)^n G_\lambda + F_\lambda. \tag{4.9}$$

Observe that the operator F_λ is nothing else but $(\lambda - T_0)^{-1}$ where T_0 designate the operator T_H with a boundary condition $H = 0$.

Thoughout the sequel, we shall assume that K is a regular operator in the following sense.

Definition 4.2 ([16])**.** Let K be the collision operator defined above. Then, K is said to be regular if $\{\kappa(x,.,v'),\ (x,v') \in (0,1)\times K\}$ is a relatively weakly compact subset of $L_1(K,dv)$.

Remark 4.3. (i) From the definition of the regularity of the collision operator introduced by B. Lods in [16, Definition 2.1], we can provide the invariance of the Schechter essential spectrum in these models of transport equation. This weak compactness assumption is more general than the ones used by M. Mokhtar-Kharroubi in [18].

(ii) Definition 4.2 asserts that for every $x \in (0,1)$,

$$f \in L^1(K) \longrightarrow \int_K \kappa(x,v,v')f(v')dv' \in L^1(K)$$

is a weakly compact operator and this weak compactness holds collectively in $x \in (0,1)$.

Now, we claim the following weak compactness result.

Lemma 4.4. *Assume that the collision operator K is non-negative. We have:*

(i) *If $\left\{\frac{\kappa(x,.,v')}{|v_3'|},\ (x,v') \in (0,1)\times K\right\}$ is a relatively weakly compact subset of $L_1(K,dv)$, then for $Re\lambda > -\lambda^*$, the operator $K(\lambda - T_H)^{-1}$ is weakly compact on X.*

(ii) *If K is regular, then for $Re\lambda > -\lambda^*$, the operator $(\lambda - T_H)^{-1}K$ is weakly compact on X.*

Proof. (i) In view of equation (4.9), the operator $K(\lambda - T_H)^{-1}$ is given by

$$K(\lambda - T_H)^{-1} = \sum_{n\geq 0} KB_\lambda H(N_\lambda H)^n G_\lambda + KF_\lambda.$$

Then, to prove the weak compactness of $K(\lambda - T_H)^{-1}$, it suffices to prove the weak compactness of the operators KB_λ and KF_λ. Following [10, Lemma 4.2], we need only to prove the claim for the operator KB_λ. Indeed, let $\varphi \in X^i$.

$$\begin{aligned}
(KB_\lambda\varphi)(x,v) &= \int_K K(x,v,v')(B_\lambda\varphi)(x,v')dv' \\
&= \int_{K^0} K(x,v,v')B_\lambda^+\varphi(x,v')dv' \\
&\quad + \int_{K^1} K(x,v,v')B_\lambda^-\varphi(x,v')dv' \\
&= \widetilde{K}\widetilde{B}_\lambda\varphi(x,v),
\end{aligned}$$

where

$$\begin{aligned} \widetilde{K} : X &\longrightarrow X \\ \varphi &\longrightarrow \int_K \frac{\kappa(x,v,v')}{|v_3'|}\ \varphi(x,v')\ dv', \end{aligned}$$

and $\widetilde{B}_\lambda := |v_3'|B_\lambda$.

Thus, we can restrict ourselves to claim that $\widetilde{K}\widetilde{B}_\lambda$ depends continuously on $\widetilde{K}$. Let $\varphi \in X$. We have:

$$\begin{aligned} \|\widetilde{B}_\lambda\varphi\| &\leq \int_0^1 \int_{K^0} |v_3'|dv'\ e^{-\int_x^1 \frac{\sigma(s,v')+\lambda}{|v_3'|}ds}\ |\varphi(1,v')|\ dx \\ &\quad + \int_0^1 \int_{K^1} |v_3'|dv'\ e^{-\int_0^x \frac{\sigma(s,v')+\lambda}{|v_3'|}ds}\ |\varphi(0,v')|\ dx \\ &\leq \int_0^1 \int_{K^0} |v_3'|dv'\ e^{-\frac{Re\lambda+\lambda^*}{|v_3'|}|1-x|}\ |\varphi(1,v')|\ dx \\ &\quad + \int_0^1 \int_{K^1} |v_3'|dv'\ e^{-\frac{Re\lambda+\lambda^*}{|v_3'|}|x|}\ |\varphi(0,v')|\ dx \\ &\leq \int_0^1 \int_{K^0} |\varphi(1,v')||v_3'|dv'dx + \int_0^1 \int_{K^1} |\varphi(0,v')|\ |v_3'|dv'dx \\ &\leq \|\varphi\|. \end{aligned}$$

Then,

$$\|\widetilde{K}\widetilde{B}_\lambda\| \leq \|\widetilde{K}\|.$$

We derive from the approximation property of regular operators established in [16], that the kernel of $\widetilde{K}$ may be decomposed as follows:

$$\widetilde{\kappa}(v,v) = \frac{\kappa(x,v,v')}{|v_3'|} = \kappa_1(v)\kappa_2(v'),$$

where $\kappa_1(\cdot) \in L_1(K)$ and $\kappa_2(\cdot) \in L_\infty(K)$.

Let $\varphi \in X^i$; we have:

$$\begin{aligned} \widetilde{K}\widetilde{B}_\lambda\varphi(x,v) &= \kappa_1(v)\Big[\int_{K^0} \kappa_2(v')|\varphi(1,v')|e^{-\int_x^1 -\frac{\sigma(s,v')+\lambda}{|v_3'|}ds}|v_3'|dv' \\ &\quad + \int_{K^1} \kappa_2(v')|\varphi(0,v')|e^{-\int_0^x -\frac{\sigma(s,v')+\lambda}{|v_3'|}ds}|v_3'|dv'\Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} |\widetilde{K}\widetilde{B}_\lambda\varphi(x,v)| &\leq \|\kappa_2\|_\infty|\kappa_1(v)|\Big[\int_{K^0} |\varphi(1,v')|e^{-\frac{\lambda^*+Re\lambda}{|v_3'|}|1-x|}|v_3'|dv' \\ &\quad + \int_{K^1} |\varphi(0,v')|e^{-\frac{\lambda^*+Re\lambda}{|v_3'|}|x|}|v_3'|dv'\Big]. \end{aligned}$$

Let $\mathcal{O}$ be a bounded set of X^i, and let $\psi \in \mathcal{O}$. It follows, for $Re\lambda > -\lambda^*$, and for all measurable subset E of D, that:

$$\int_E \mid \widetilde{K}\widetilde{B}_\lambda \ \psi(x,v) \mid dx \ dv \leq \|\kappa_2\|_\infty \ \| \ \psi\|_1 \int_E |\kappa_1(v)| \ dx \ dv.$$

Since

$$\lim_{|E|\longrightarrow 0} \int_E |\kappa_1(v)| \ dx \ dv = 0, \quad (\kappa_1 \subseteq L_1(K, dv))$$

the weak compactness of the set $\widetilde{K}\widetilde{B}_\lambda(\mathcal{O})$ can be obtained by the use of [5, Corollary 11, p. 294]. Our claim follows and the proof of this item is thereby completed.

(ii) According to [10, Lemma 4.3], to prove the weak compactness of $(\lambda - T_H)^{-1}K$ it sufficient to add the claim for the operator $G_\lambda K$.
Following the approximation property for the class of regular operators introduced in [16], the kernel of the collision operator K becomes:

$$\kappa(v,v') = \kappa_1(v)\kappa_2(v'), \quad \text{where} \quad \kappa_1(.) \in L_1(K), \ \text{and} \ \kappa_2(.) \in L^\infty(K).$$

Now, we will show that $G_\lambda K$ is weakly compact.

For $\varphi \in X$, we have $(G^+_\lambda K\varphi)(x,v) = \dfrac{1}{|v_3|} \displaystyle\int_0^1 e^{-\int_0^x \frac{\sigma(s,v)+\lambda}{|v_3|}ds} \ K\varphi(x,v)dx$

$$= \frac{1}{|v_3|} \int_0^1 \int_K e^{-\int_0^x \frac{\sigma(s,v)+\lambda}{|v_3|}ds} \ \kappa_1(v)\kappa_2(v')\varphi(x,v')dx \ dv' \qquad = J_\lambda U_\lambda \varphi$$

where $v \in K^0$, J_λ and U_λ denote the following bounded operators

$$U_\lambda : X \longrightarrow L_1((0,1), dx)$$

$$\varphi \ \longrightarrow \int_K \kappa_2(v')\varphi(x,v') \ dv',$$

and

$$J_\lambda : L_1((0,1), dx) \longrightarrow X^0_1$$

$$\varphi \ \longrightarrow \frac{1}{|v_3|} \int_0^1 e^{-\int_0^x \frac{\sigma(s,v)+\lambda}{|v_3|} \ ds} \ \kappa_1(v)\varphi(x) \ dx.$$

Now, it suffices to show that J_λ is weakly compact. To do this, let $\mathcal{O}$ be a bounded set of $L_1((0,1), dx)$. Let $\psi \in \mathcal{O}$. It follows for all measurable subset E of K^0 that

$$\int_E |J_\lambda\psi(v)||v_3| \ dv \leqslant \|\psi\| \int_E |\kappa_1(v)| \ dv.$$

Since

$$\lim_{|E|\to 0}\int_E |\kappa_1(v)|\ dv = 0, \quad (\kappa_1(.)\in L_1(K,dv))$$

where $|E|$ is the measure of E, the applicability of [5, Corollary 11] asserts that the set $J_\lambda(\mathcal{O})$ is weakly compact. Hence the weak compactness of G_λ^+K is checked.
The proof of the weak compactness of the operator G_λ^-K is similar to the operator G_λ^+K. □

Now, we are ready to state and prove the precise picture of the essential spectrum of A_H:

Theorem 4.5. *If the operator H is strictly singular, K is a non negative, regular collision operator. If in addition*

$$\left\{\frac{\kappa(x,.,v')}{|v_3'|},\ (x,v)\in(0,1)\times K\right\}$$

is a relatively weakly compact subset of $L_1(K,dv)$, then

$$\sigma_{ess}(A_H)\subseteq\sigma_{ess}(T_H)=\{\lambda\in\mathbb{C};\ Re\lambda\le-\lambda^*\}.$$

Proof. Since T_H generates a strongly continuous semi-group on X, then it follows that $\lim\limits_{Re\lambda\to\infty}\|(\lambda-T_H)^{-1}\|=0$ (see [4,18]). Hence, there exists $\eta>-\lambda^*$ such that for $Re\lambda>\eta$, we have $r_\sigma(K(\lambda-T_H)^{-1})<1$ ($r_\sigma(.)$ is the spectral radius). Therefore, the open half plane $\mathfrak{O}=\{\lambda\in\mathbb{C}\ ;\ Re\lambda>\eta\}$ is contained in $\Phi_{T_H}\cap\Phi_{A_H}$; then $\Phi_{T_H}\cap\Phi_{A_H}\neq\emptyset$.

On the other hand, Proposition 4.4 with [5, Corollary 13] assert that $[K(\lambda-T_H)^{-1}]^2$ is compact on X for all $\lambda\in\rho(T_H)$.

Consequently, [17, Proposition 2.2] revels that

$$g((K(\lambda-T_H)^{-1})^2)<1\quad\text{and}\quad g(((\lambda-T_H)^{-1}K)^2)<1.$$

Now, applying Theorem 3.1, we get

$$\sigma_{ess}(A_H)\subseteq\sigma_{ess}(T_H)=\{\lambda\in\mathbb{C};\ Re\lambda\le-\lambda^*\}.$$ □

5. Conclusion

In this paper, new sufficient conditions are derived to characterize the essential spectrum of the sum of two linear operators. The obtained result exploits the concept of measuring of non strict singularity involving an elegant use of the notion of Fredholm inverse properties of linear operators. The validity of the main result is illustrated by an example of one-speed neutron transport operator on L_1-space.

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