ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 6, pp. 1671–1677

Title:

Gorenstein hereditary rings with respect to a semidualizing module

Author(s):

G. Zhao and B. Zhang

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 6, pp. 1671–1677
Online ISSN: 1735-8515

GORENSTEIN HEREDITARY RINGS WITH RESPECT TO A SEMIDUALIZING MODULE

G. ZHAO AND B. ZHANG*

(Communicated by Mohammad-Taghi Dibaei)

ABSTRACT. Let C be a semidualizing module. We first investigate the properties of finitely generated G_C-projective modules. Then, relative to C, we introduce and study the rings over which every submodule of a projective (flat) module is G_C-projective (flat), which we call C-Gorenstein (semi)hereditary rings. It is proved that every C-Gorenstein hereditary ring is both coherent and C-Gorenstein semihereditary.
Keywords: Semidualizing module, G_C-projective module, G_C-flat module, G_C-(semi)hereditary ring, coherent ring.
MSC(2010): Primary: 18G25; Secondary: 13D02, 13D05.

1. Introduction

Throughout this work R is a commutative ring with unity. For an R-module T, let $\mathrm{add}_R T$ be the subclass of R-modules consisting of all modules isomorphic to direct summands of finite direct sums of copies of T. We define $\mathrm{gen}^*(T) = \{M$ is an R-module | there exists an exact sequence $\cdots \to T_n \to \cdots \to T_1 \to T_0 \to M \to 0$ with each $T_i \in \mathrm{add}_R T$ and $\mathrm{Hom}_R(T, -)$ leaves it exact $\}$ (see [14]). $\mathrm{cogen}^*(T)$ is defined dually. Recall that an R-module $C \in \mathrm{gen}^*(R)$ is said to be *semidualizing* if $\mathrm{Ext}^i_R(C, C) = 0$ for any $i \geq 1$, and the map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism.

In the following, we always assume that C is a semidualizing R-module. Recall from [15] that an R-module M is called *G_C-projective* if there exists an exact sequence of R-modules $\cdots \to P_1 \to P_0 \to C \otimes_R P_{-1} \to C \otimes_R P_{-2} \to \cdots$ with all P_i projective, such that $M \cong \mathrm{Im}(P_0 \to C \otimes_R P_{-1})$ and $\mathrm{Hom}_R(-, C \otimes_R P)$ leaves the sequence exact for any projective R-module P. The G_C-injective modules are defined in a dual manner. An R-module M is called *G_C-flat* [9] if there is an exact sequence of R-modules $\cdots \to F_1 \to F_0 \to C \otimes_R F_{-1} \to C \otimes_R F_{-2} \to \cdots$ with all F_i flat, such that $M \cong \mathrm{Im}(F_0 \to C \otimes_R F_{-1})$ and

Article electronically published on 30 November, 2017.
Received: 3 December 2015, Accepted: 8 September 2016.
*Corresponding author.

$\mathrm{Hom}_R(C, I) \otimes_R -$ leaves the sequence exact for any injective R-module I. The G_C-projective, flat, and injective dimensions of an R-module M are defined in terms of G_C-projective, flat resolutions, and injective coresolutions, and denoted by G_C-$\mathrm{pd}_R(M)$, G_C-$\mathrm{fd}_R(M)$ and G_C-$\mathrm{id}_R(M)$, respectively. The G_C-projective dimension was first introduced by Golod in [7] for finitely generated modules over a commutative Noetherian ring, and was extended by Holm and Jørgensen in [9] to arbitrary modules. Later, White further extended in [15] these concepts to the non-Noetherian setting, and showed that they share many common properties with the Gorenstein homological dimensions [8]. Since then these notions have been extensively studied (see also [2, 12] for a new trends in relative homological algebra).

It is well-known that, the classical global dimensions of rings play an important role in the theory of rings. Motivated by Bennis and Mahdou's [3] ideas to study the global dimensions of a ring R in terms of Gorenstein homological dimensions, recently, Zhao and Sun studied in [16] the global dimensions of R defined by some relative homological dimensions with respect to C, and proved that $\sup\{G_C\text{-pd}_R(M) | M \text{ is an } R\text{-module}\} = \sup\{G_C\text{-id}_R(M) | M \text{ is an } R\text{-module}\}$. The common value, denoted by G_C-gl.dim(R), is named as the C-Gorenstein global dimension of R. Similarly, the C-Gorenstein weak global dimension of R is also defined as $\mathrm{G}_C\text{-}w\mathrm{gl.dim}(R) = \sup\{G_C\text{-fd}_R(M) | M \text{ is an } R\text{-module}\}$.

On the other hand, in classical homological algebra, the rings of (weak) global dimensions at most 1, called (semi)hereditary rings [13], are important classes of rings, and the following are well-known: (1) every hereditary ring is coherent and semihereditary; (2) a ring R is semihereditary if and only if every finitely generated submodule of a projective R-module is projective. Rings of small Gorenstein homological dimensions were introduced in [4, Section 5] which ends with the following question "whether G-hereditary rings are coherent?". This question is recently resolved positively in [6] (see also [1, 10, 11] where some results on these kind of rings were established). Then, naturally, relative Gorenstein rings will be of interest. According to the terminology of the classical theory of homological algebra and the one of Gorenstein homological algebra started in [4, Section 5], we introduce the following notions: A ring R is called *C-Gorenstein hereditary* (G_C-hereditary for short) if every submodule of a projective R-module is G_C-projective (i.e., G_C-gl.dim$(R) \leq 1$), and R is said to be *C-Gorenstein semihereditary* (G_C-semihereditary for short) if R is coherent and every submodule of a flat R-module is G_C-flat. In this paper, we are mainly concerned with the following natural questions:

Question A. Is it true that every C-Gorenstein hereditary ring is coherent and C-Gorenstein semihereditary?

Question B. Is it true that R is C-Gorenstein semihereditary if and only if every finitely generated submodule of a G_C-projective R-module is G_C-projective?

It is shown that Question A has an affirmative answer (see Corollary 2.4 and Theorem 2.8). Also, a partial answer to Question B is provided at the end of this paper.

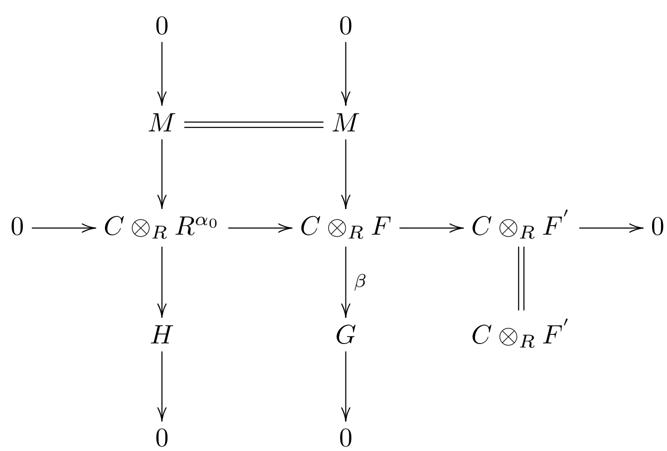
2. C-Gorenstein hereditary and semihereditary rings

We use $(-)^C$ to denote the functor $\mathrm{Hom}_R(-, C)$. The following result is the relative version of [6, Lemma 2.3 and Corollary 2.4], but the proof is slightly different.

Lemma 2.1. *Assume that M is a finitely generated G_C-projective R-module. Then*

(1) $M \in cogen^*(C)$.
(2) $M^C \in gen^*(R)$.

Proof. (1) Because M is G_C-projective, there is an exact sequence $0 \to M \to C \otimes_R F \to G \to 0$, in which F is free and G is G_C-projective by [15, Observation 2.3 and Proposition 2.9]. Since M is also finitely generated, there exists a finitely generated free submodule R^{α_0} with α_0 an integer, and a free submodule F' of F, such that $M \subseteq C \otimes_R R^{\alpha_0} \cong C^{\alpha_0}$ and $F = R^{\alpha_0} \oplus F'$. Setting $H = \mathrm{Coker}(M \to C \otimes_R R^{\alpha_0})$ yields a commutative diagram with exact row



with middle row is split. By the snake lemma, we get an exact sequence $0 \to H \to G \to C \otimes_R F^{'} \to 0$, which implies that H is finitely generated G_C-projective [15, Theorem 2.8]. Repeating this process to H and so on, one has an exact sequence

$$0 \to M \to C^{\alpha_0} \to C^{\alpha_1} \to \cdots \qquad (*)$$

with each image is finitely generated G_C-projective, which implies that the sequence $(*)$ is exact after applying $\mathrm{Hom}_R(-,C)$. Therefore, $M \in \mathrm{cogen}^*(C)$.

(2) Applying $\mathrm{Hom}_R(-,C)$ to the sequence $(*)$ in (1) provides an exact sequence

$$\cdots \to (C^{\alpha_1})^C \to (C^{\alpha_0})^C \to M^C \to 0$$

Because $(C^{\alpha_i})^C \cong \mathrm{Hom}_R(C,C)^{\alpha_i} \cong R^{\alpha_i}$, the desired result follows. □

The following theorem plays a crucial role in proving the main result in this paper.

Theorem 2.2. *A ring R is coherent if every finitely generated submodule of a G_C-projective R-module is G_C-projective.*

Proof. Let M be a finitely generated submodule of a projective R-module. By the hypothesis, M is G_C-projective since every projective R-module is G_C-projective [15, Proposition 2.6]. It follows from Lemma 2.9(2) that $M^C \in \mathrm{gen}^*(R)$, and hence it is finitely generated. On the other hand, since M is finitely generated, there is an exact sequence

$$0 \to K \to F_0 \to M \to 0,$$

where $F_0 = R^\alpha$ is finitely generated free. Applying $\mathrm{Hom}_R(-,C)$ to this short exact sequence gives rise to a monomorphism: $0 \to M^C \to (R^\alpha)^C$. Since $(R^\alpha)^C = \mathrm{Hom}_R(R^\alpha, C) \cong C^\alpha$ is G_C-projective by [15, Proposition 2.6] again, the assumption yields that M^C is G_C-projective. Replacing M with M^C in Lemma 2.9(2), we get that $M^{CC} \in \mathrm{gen}^*(R)$, and hence finitely presented.

On the other hand, from Lemma 2.9(1), we know that $M \in \mathrm{cogen}^*(C)$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & M & \to & C^{\alpha_0} & \to & C^{\alpha_1} & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M^{CC} & \to & (C^{\alpha_0})^{CC} & \to & (C^{\alpha_1})^{CC} & \to & \cdots . \end{array}$$

As $(C^{\alpha_i})^{CC} \cong (R^{\alpha_i})^C \cong C^{\alpha_i}$ for each $i \geq 0$, $M \cong M^{CC}$ is finitely presented. Thus, R is coherent. □

To prove the coherence of G_C-hereditary rings, we need the following result, which gives some other descriptions of G_C-hereditary rings.

Proposition 2.3. *Let R be a ring. The following are equivalent.*

(1) *R is G_C-hereditary.*
(2) *Every submodule of a G_C-projective R-module is G_C-projective.*
(3) *Every quotient module of a G_C-injective R-module is G_C-injective.*

Proof. (1) ⇒ (2) Follows from [15, Proposition 2.12].

(2) ⇒ (1) Evident.

(2) ⇔ (3) The assertion holds by [16, Theorem 4.4]. □

Corollary 2.4. *Every G_C-hereditary ring is coherent.*

Proof. It follows from Theorem 2.10 and Proposition 2.7. □

In the special case that $C = R$, we obtain the main result of [6, Theorem 2.5].

Corollary 2.5. *All Gorenstein hereditary rings are coherent.*

Before starting to study the G_C-semihereditary rings, we first give some equivalent characterizations of modules with finite G_C-flat dimension. We write $(-)^+ = \mathrm{Hom}_R(-, E)$, where E is an injective cogenerator for the categories of R-modules.

Lemma 2.6. *Suppose that R is a coherent ring, and M an R-module with $G_C\text{-fd}_R(M) < \infty$. For a nonnegative integer n, the following are equivalent.*

(1) $G_C\text{-fd}_R(M) \leq n$.
(2) $\mathrm{Tor}^R_{i>n}(M, \mathrm{Hom}_R(C, I)) = 0$ *for any injective module I.*
(3) *In every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, with G_i are G_C-flat, K_n is also G_C-flat.*

Proof. (1) ⇔ (2) Since R is coherent, it follows from [17, Theorem 3.8] that $G_C\text{-fd}_R(M) \leq n$ if and only if $G_C\text{-id}_R(M^+) \leq n$. This, by the dual version of [15, Proposition 2.12], is equivalent to that $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, I), M^+) = 0$ for any injective module I and $i > n$. Because $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, I), M^+) \cong (\mathrm{Tor}^R_i(M, \mathrm{Hom}_R(C, I)))^+$ by [5, Chapter VI, Proposition 5.1], we get the desired result.

(3) ⇒ (1) is trivial. Conversely, let $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ be an exact sequence with all G_i are G_C-flat. Then $0 \to M^+ \to G_0^+ \to \cdots \to G_{n-1}^+ \to K_n^+ \to 0$ is exact with all G_i^+ are G_C-injective. The assumption implies that $G_C\text{-id}_R(M^+) \leq n$, and so K_n^+ is G_C-injective by the dual version of [15, Proposition 2.12] again. Thus K_n is G_C-flat. □

By Lemma 2.6 and a standard argument, it is not difficult to get the following result.

Proposition 2.7. *Let R be a ring. The following are equivalent.*

(1) *R is G_C-semihereditary.*
(2) *R is coherent and $G_C\text{-wgl.dim}(R) \leq 1$.*
(3) *R is coherent and every submodule of a G_C-flat R-module is G_C-flat.*

The next result, together with Corollary 2.4, gives an affirmative answer to Question A.

Theorem 2.8. *If R is a G_C-hereditary ring, then it is G_C-semihereditary.*

Proof. The coherence of R follows from Corollary 2.4. It follows from [16, Corollary 4.6] that $\mathrm{G}_C\text{-wgl.dim}(R) \leq \mathrm{G}_C\text{-gl.dim}(R) \leq 1$. Thus R is G_C-semihereditary by Proposition 2.7. □

Recall that an R-module M is called FP-injective if $\mathrm{Ext}_R^1(N, M) = 0$ for all finitely presented R-modules N. The FP-injective dimension of M, denoted by $FP\text{-id}(M)$, is defined to be the least nonnegative integer n such that $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for all finitely presented R-modules N. If no such n exists, set $FP\text{-id}(M) = \infty$.

Lemma 2.9. *Let R be a coherent ring. The following are equivalent.*

(1) $FP\text{-id}(C \otimes_R P) \leq n$ *for any projective R-module P.*
(2) $\mathrm{fd}(\mathrm{Hom}_R(C, I)) \leq n$ *for any injective R-module I.*

Proof. (1) $\Rightarrow$ (2) Let P be a projective R-module. The hypothesis implies that $\mathrm{Ext}_R^{i>n}(N,\, C \otimes_R P) = 0$ for all finitely presented R-modules N. Since R is coherent, $N \in \mathrm{gen}^*(R)$. Thus $\mathrm{Tor}^R_{i>n}((C \otimes_R P)^+, N) \cong (\mathrm{Ext}_R^{i>n}(N, C \otimes_R P))^+ = 0$ by [5, Chapter VI, Proposition 5.3]. This implies that $\mathrm{fd}(C \otimes_R P)^+ \leq n$, and so $\mathrm{fd}(\mathrm{Hom}_R(C, P^+)) \leq n$ by the adjoint isomorphism.

For any injective R-module I, since P^+ is also an injective cogenerator, I is a direct summands of $\prod P^+$. Thus $\mathrm{Hom}_R(C, I)$ is a direct summands of $\mathrm{Hom}_R(C, \prod P^+) \cong \prod \mathrm{Hom}_R(C, P^+)$, and hence $\mathrm{fd}(\mathrm{Hom}_R(C, I)) \leq \mathrm{fd}(\prod \mathrm{Hom}_R(C, P^+)) \leq n$ from the coherence of R.

(2) $\Rightarrow$ (1) Suppose F is a flat R-module, then F^+ is injective, and so $\mathrm{fd}(\mathrm{Hom}_R(C, F^+)) \leq n$. The adjoint isomorphism $\mathrm{Hom}_R(C, F^+) \cong (C \otimes_R F)^+$ yields that $\mathrm{fd}(C \otimes_R F)^+ \leq n$. Thus, for any finitely presented R-module N, $(\mathrm{Ext}_R^{i>n}(N, C \otimes_R F))^+ \cong \mathrm{Tor}^R_{i>n}((C \otimes_R F)^+, N) = 0$, and hence $\mathrm{Ext}_R^{i>n}(N, C \otimes_R F) = 0$. Therefore, $FP\text{-id}(C \otimes_R F) \leq n$, which completes the proof. □

The following result gives a partial answer to Question B.

Theorem 2.10. *Let R be a ring with G_C-wgl.dim$(R) < \infty$. If every finitely generated submodule of a G_C-projective R-module is G_C-projective, then R is G_C-semihereditary.*

Proof. Let M be a finitely presented R-module. There is an exact sequence

$$0 \to K \to P \to M \to 0$$

with P finitely generated projective. Note that R is coherent by Theorem 2.2, it follows that K is finitely generated. By the hypothesis, K is G_C-projective. Thus, one has that $G_C\text{-pd}_R(M) \leq 1$ for every finitely presented R-module M, and so $\mathrm{Ext}_R^{i>1}(M, C \otimes_R Q) = 0$ for any projective R-module Q by [15, Proposition 2.12]. This means that $FP\text{-id}(C \otimes_R Q) \leq 1$. By Lemma 2.9, $\mathrm{fd}(\mathrm{Hom}_R(C, I)) \leq 1$ for any injective R-module I, which implies that $\mathrm{Tor}^R_{i>1}(N, \mathrm{Hom}_R(C, I)) = 0$ for any R-module N. Since $G_C\text{-fd}(N) < \infty$, Lemma 2.6 yields that $G_C\text{-fd}(N) \leq 1$. Therefore, G_C-wgl.dim$(R) \leq 1$, and hence R is G_C-semihereditary by Proposition 2.7. □

Acknowledgements

The research was partially supported by the National Natural Science Foundation of China (Grant Nos. 11401147, 11201220, 11531002) and Natural Science Foundation of Shandong Province of China (Grant No. ZR2017BA028). The authors would like to thank the referee for helpful comments and detailed suggestions.

References

[1] D. Bennis, A note on Gorenstein global dimension of pullback rings, *Int. Electron. J. Algebra* **8** (2010) 30–44.
[2] D. Bennis, J.R. García Rozas and L. Oyonarte, Relative Gorenstein dimensions, *Mediterr. J. Math.* **13** (2016) 65–91.
[3] D. Bennis and N. Mahdou, Global Gorenstein dimensions, *Proc. Amer. Math. Soc.* **138** (2010), no. 2, 461–465.
[4] D. Bennis and N. Mahdou, Gorenstein homological dimensions of commutative rings, Arxiv:0611358v1 [math.AC].
[5] H. Cartan and S. Eilenberg, Homological Algebra, Reprint of the 1956 original, Princeton Landmarks in Math. Princeton Univ. Press, Princeton, 1999.
[6] Z.H. Gao and F.G. Wang, All Gorenstein hereditary rings are coherent, *J. Algebra Appl.* **13** (2014), no. 4, Article 1350140, 5 pages.
[7] E.S. Golod, G-dimension and generalized perfect ideals (Russian), in: Algebraic Geometry and its Applications, Collection of Articles, *Tr. Mat. Inst. Steklova* **165** (1984) 62–66.
[8] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra* **189** (2004) 167–193.
[9] H. Holm and P. Jørgensen, Semidualizing modules and related Gorenstein homological dimensions, *J. Pure Appl. Algebra* **205** (2006) 423–445.
[10] K. Hu and F.G. Wang, Some results on Gorenstein Dedekind domains and their factor rings, *Comm. Algebra* **41** (2013) 284–293.
[11] K. Hu, F. Wang, L. Xu and S. Zhao, On overrings of Gorenstein Dedekind domains, *J. Korean Math. Soc.* **50** (2013), no. 5, 991–1008.
[12] Z. Liu, Z. Huang and A. Xu, Gorenstein projective dimension relative to a semidualizing bimodule, *Comm. Algebra* **41** (2013) 1–18.
[13] J.J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.
[14] T. Wakamatsu, Tilting modules and Auslander's Gorenstein property, *J. Algebra* **275** (2004) 3–39.
[15] D. White, Gorensten projective dimension with respect to a semidualizing module, *J. Commut. Algebra* **2** (2010) 111–137.
[16] G.Q. Zhao and J.X. Sun, Global dimensions of rings with respect to a semidualizing module (preprint), Arxiv:1307.0628 [math.RA].
[17] G.Q. Zhao and X.G. Yan, Resolutions and stability of C-Gorenstein flat modules, *Rocky Mountain J. Math.* **46** (2016), no. 5, 1739–1753.

(Guoqiang Zhao) Department of Mathematics, Hangzhou Dianzi University, Hangzhou, 310018, P.R. China.
E-mail address: gqzhao@hdu.edu.cn

(Bo Zhang) School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, 454000, P.R. China.
E-mail address: bzhang1980@126.com