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DUALITY FOR VECTOR EQUILIBRIUM PROBLEMS WITH CONSTRAINTS

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Abstract. In the paper, we study duality for vector equilibrium problems using a concept of generalized convexity in dealing with the quasi-relative interior. Then, their applications to optimality conditions for quasi-relative efficient solutions are obtained. Our results are extensions of several existing ones in the literature when the ordering cones in both the objective space and the constraint space have possibly empty interior.
Keywords: Equilibrium problem, duality, optimality conditions, quasi-relative efficient solution, generalized subconvexlikeness.
MSC(2010): Primary: 49N15; Secondary: 32F17, 54C60, 90C46, 91B50.

1. **Introduction**

Vector equilibrium problems and their applications have been intensively developed since they include other problems, e.g., vector variational inequalities, vector optimization problems, vector Nash equilibrium problems, etc. Significant results in the literature are mainly related to the existence of solutions ([6, 27]), stability ([1, 2, 15, 17]), solving algorithms ([18, 31]), optimality conditions ([19, 24, 29, 30, 32, 33]), duality ([25]), etc. Almost all of the above-mentioned results, especially optimality conditions, were established with ordering cones having nonempty interior. Nevertheless, in several infinite-dimensional spaces, such as l^p or $L^p(\Omega)$, the interior of the positive cone is empty. To overcome this case, Borwein and Lewis introduced the quasi-relative interior in [11], which can be considered as a further extension among other generalized interior, e.g., the core, the intrinsic core, and the strong quasi relative interior, see [20, 34]. By virtue of this concept, the quasi-relative efficient solution was introduced. This solution has been recently involved in some topics of optimization problems, especially optimality conditions (see, e.g., [8, 22]).

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To the best of our knowledge, there have not been results on duality for equilibrium problems in terms of quasi-relative efficient solutions. Inspired by [13, 21], in this paper we study duality for vector equilibrium problems using the quasi-relative interior. The layout of the paper is as follows. Section 2 is devoted to preliminaries. In Section 3, we establish duality theorems for quasi-relative efficient solutions of a constrained equilibrium problem and its dual problem. Their applications to optimality conditions for set-valued optimization problems are obtained to show that our results are generalizations of several existing ones in the literature. Some concluding remarks are contained in Section 4.

2. **Preliminaries**

Let X, Y be two normed spaces, $C \subseteq Y$ be a convex cone. X^* is the dual space of X and $\langle ., . \rangle$ is the canonical paring. $\mathbb{N}$ and $\mathbb{R}^k_+$ stand for the set of natural numbers and the nonnegative orthant of the k-dimensional space, respectively (respectively, for short). For a subset $A \subseteq X$, $\text{int}A$, $\text{cl}A$, $\text{cone}A$, $\text{aff}A$, and $\text{lin}A$ denote the interior, closure, conic hull, affine hull, and linear hull of A, respectively. The notion $\text{lin}_0 A$ is used for the linear space parallel with the affine hull of A, that is, for some (every) $a \in A$,

$$\text{lin}_0 A := \text{aff}A - a = \text{lin}(A - a) = \text{lin}(A - A).$$

For the above C, the dual cone of C is defined by

$$C^* := \{c^* \in Y^* | \langle c^*, c \rangle \geq 0, \forall\, c \in C\}.$$

The domain, image, and graph of a set-valued mapping $F : X \to 2^Y$ are denoted by, respectively,

$$\text{dom}F := \{x \in X | F(x) \neq \emptyset\}, \quad \text{Im}F := \{y \in Y | y \in F(x)\},$$
$$\text{gr}F := \{(x, y) \in X \times Y | y \in F(x)\}.$$

We denote $F_+(.) := F(.) + C$.

Recall that the quasi-relative interior of a convex subset $S \subseteq X$, see [11], is defined by

$$\text{qri}S := \{x \in S | \text{cl}\,\text{cone}(S - x) \text{ is a linear subspace of } X\}.$$

Some properties of the quasi-relative interior are collected in the following.

Proposition 2.1. *Let* $S \subseteq X$ *be convex and* $\text{qri}S \neq \emptyset$. *Then,*

(i) *if* $\text{int}S \neq \emptyset$, *then* $\text{int}S = \text{qri}S$;
(ii) $\text{qri}(\text{qri}S) = \text{qri}S$;
(iii) $\text{cl}\,\text{qri}S = \text{cl}S$;
(iv) $\lambda \text{qri}S + (1 - \lambda)S \subseteq \text{qri}S$ *for all* $\lambda \in (0, 1]$, *whence* $\text{qri}S$ *is convex;*
(v) *if, additionally,* S *is a pointed cone, then* $0 \notin \text{qri}S$ *and* $\text{qri}S \cup \{0\}$ *is a cone;*
(vi) *if* $U \subseteq X$ *is convex and* $\text{qri}U \neq \emptyset$, *then* $\text{qri}(S \times U) = \text{qri}S \times \text{qri}U$;

(vii) $\text{qri}S = \{x \in S | \text{cl cone}(S - x) = \text{cl lin}_0 S\}$.

Properties (i)-(vi) can be referred to [10,11], while the proof of (vii) is implied by [34, Proposition 1.2.7].

Proposition 2.2 ([12,16])**.** *Let $S \subseteq X$ be convex, $\text{qri}S \neq \emptyset$, and $x_0 \in S$. If $x_0 \notin \text{qri}S$, then there exists $x^* \in X^* \setminus \{0\}$ such that $\langle x^*, x_0\rangle \leq \langle x^*, x\rangle$ for all $x \in S$.*

Definition 2.3. Let $S \subseteq X$, $\text{qri}C \neq \emptyset$, and $F : S \to 2^Y$. The mapping F is said to be generalized C-subconvexlike on S if $\text{cone}_+(F(S)) + \text{qri}C$ is convex, where $\text{cone}_+(F(S)) := \{ry | r > 0, y \in F(S)\}$.

The above definition proposes a generalized concept of convexity for a set-valued mapping which are weaker than earlier ones, see [26, Definition 3.1 and Lemma 3.2].

The following proposition gives us a sufficient condition for the generalized C-subconvexlikeness of F.

Proposition 2.4. *Let $S \subseteq X$, $\text{qri}C \neq \emptyset$, and $F : S \to 2^Y$. Suppose that for all $\lambda \in [0, 1]$, $x_1, x_2 \in S$, there are $x \in S, r > 0$ such that*

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subseteq rF(x) + C. \tag{2.1}$$

Then, the mapping F is generalized C-subconvexlike on S.

Proof. We need to prove that $\text{cone}_+(F(S))+\text{qri}C$ is convex. Let any $\lambda \in (0, 1)$, $k_1, k_2 \in \text{cone}_+(F(S)) + \text{qri}C$. Then, there exist $x_i \in S$, $y_i \in F(x_i)$, $r_i > 0$, $c_i \in \text{qri}C$ such that $k_i = r_i y_i + c_i$, $i = 1, 2$. So, we get

$$\begin{aligned}\lambda k_1 + (1 - \lambda)k_2 &= \lambda(r_1y_1 + c_1) + (1 - \lambda)(r_2y_2 + c_2)\\ &= (\lambda r_1y_1 + (1 - \lambda)r_2y_2) + \lambda c_1 + (1 - \lambda)c_2\\ &= (\lambda r_1 + (1 - \lambda)r_2)\Big(\frac{\lambda r_1}{\lambda r_1 + (1 - \lambda)r_2}y_1 + \frac{(1 - \lambda)r_2}{\lambda r_1 + (1 - \lambda)r_2}y_2\Big)\\ &\quad +\lambda c_1 + (1 - \lambda)c_2\\ &= r(\beta y_1 + (1 - \beta)y_2) + c',\end{aligned}$$

where $r := \lambda r_1 + (1-\lambda)r_2$, $\beta := \lambda r_1/(\lambda r_1 + (1-\lambda)r_2)$, and $c' := \lambda c_1 + (1-\lambda)c_2$. It follows from the convexity of C and Proposition 2.1(iv) that $c' \in \text{qri}C$.

By the assumption, for all $\alpha \in (0, 1), x_1, x_2 \in S$, there are $x \in S$, $t > 0$ with

$$\alpha F(x_1) + (1 - \alpha)F(x_2) \subseteq tF(x) + C. \tag{2.2}$$

Putting $\alpha := \beta$ in (2.2), we have

$$\begin{aligned}\beta y_1 + (1 - \beta)y_2 &\in \beta F(x_1) + (1 - \beta)F(x_2)\\ &\subseteq tF(x) + C,\end{aligned}$$

i.e., there exist $y \in F(x)$ and $c_0 \in C$ such that $\beta y_1 + (1-\beta)y_2 = ty + c_0$. Hence,

$$\begin{aligned} \lambda k_1 + (1-\lambda)k_2 &= r(ty + c_0) + c' \\ &= rty + rc_0 + c' \\ &\in \text{cone}_+(F(S)) + C + \text{qri}C \\ &\subseteq \text{cone}_+(F(S)) + \text{qri}C \text{ (Proposition 2.1(iv),(v))}. \end{aligned}$$

Therefore, $\text{cone}_+(F(S)) + \text{qri}C$ is convex. □

The condition (2.1) is not necessary, by the following example.

Example 2.5 ([26, Example 3.2])**.** Let $X = Y = \mathbb{R}^2$, $S = \{(x_1, x_2) \in X : x_1 + x_2 = 1\}$, $C = \mathbb{R}_+ \times \{0\} \subseteq Y$, and $F : X \to 2^Y$ be defined by $F(x_1, x_2) := \{(x_1, x_2), (1/2, 1/2)\}$ for all $(x_1, x_2) \in S$.

It is easy to see that $\text{int}C = \emptyset$, $\text{qri}C = \{(y, 0) \in Y | y > 0\}$, and $\text{cone}_+(F(S)) + \text{qri}C = \{(y_1, y_2) \in Y | y_1 + y_2 > 0\}$. Thus, $\text{cone}_+(F(S)) + \text{qri}C$ is convex, i.e., the mapping F is generalized C-subconvexlike on S. However, (2.1) does not hold. Indeed, by choosing $\lambda := 1/2 > 0$, $x := (-1, 2)$, and $u := (2, -1)$ $(x, u \in S)$, one gets

$$\lambda F(x) + (1-\lambda)F(u) = \{(1/2, 1/2),\ (-1/4, 5/4),\ (5/4, -1/4)\}.$$

It is enough to show that for all $z = (z_1, z_2) \in S$, $r > 0$,

$$\{(1/2, 1/2),\ (-1/4, 5/4),\ (5/4, -1/4)\} \nsubseteq rF(z) + C.$$

In fact, for any $z = (z_1, z_2) \in S$, we have the following three cases

- if $z_1, z_2 \geq 0$, then $\{(5/4, -1/4), (-1/4, 5/4)\} \nsubseteq rF(z) + C$ for all $r > 0$,
- if $z_1 > 0$, $z_2 < 0$, then $(-1/4, 5/4) \notin rF(z) + C$ for all $r > 0$,
- if $z_1 < 0$, $z_2 > 0$, then $(5/4, -1/4) \notin rF(z) + C$ for all $r > 0$.

Proposition 2.6. *Let $S \subseteq X$, $\text{qri}C \neq \emptyset$, and $F : S \to 2^Y$. Suppose that F is generalized C-subconvexlike on S and $\text{qri}(\text{cone}_+ F(S) + \text{qri}C) \neq \emptyset$. If $F(S) \cap -\text{qri}C = \emptyset$, then there exists $c^* \in C^* \setminus \{0\}$ such that $\langle c^*, y\rangle \geq 0$ for all $y \in F(S)$.*

Proof. We first prove that $0 \notin \text{cone}_+(F(S)) + \text{qri}C$. Suppose to the contrary, i.e., there exist $x \in S$, $y \in F(x)$, and $r > 0$ such that $0 \in ry + \text{qri}C$. Thus, we get $-y \in (1/r)\text{qri}C \subseteq \text{qri}C$, which contradicts the assumption that $F(S) \cap -\text{qri}C = \emptyset$. Hence, $0 \notin \text{cone}_+(F(S)) + \text{qri}C$, which implies $0 \notin \text{qri}(\text{cone}_+(F(S)) + \text{qri}C)$.

By Proposition 2.2, there exists $c^* \in X^* \setminus \{0\}$ such that $\langle c^*, y\rangle \geq 0$ for all $y \in \text{cone}_+(F(S)) + \text{qri}C$. So, for any $x \in S$, $y \in F(x)$, $c \in \text{qri}C$, $r > 0$,

$$\langle c^*, ry + c\rangle \geq 0. \tag{2.3}$$

Letting $r = 1/n$ in (2.3) and taking $n \to +\infty$, we get $\langle c^*, c\rangle \geq 0$ for all $c \in \text{qri}C$.

We next show that $c^* \in C^*$. If not, there is $c' \in C$ with $\langle c^*, c'\rangle < 0$. For given $c_0 \in \mathrm{qri}C$, it follows from Proposition 2.1(iv) that for all $t > 0$,

$$\frac{1}{2}c_0 + \frac{1}{2}(tc') \in \mathrm{qri}C.$$

Thus, $\langle c^*, c_0 + tc'\rangle \geq 0$ for all $t > 0$, which is impossible since $\langle c^*, c'\rangle < 0$. Hence, $c^* \in C^*$.

Moreover, taking $r := 1$ and $c := c_0$ (with some $c_0 \in \mathrm{qri}C$) in (2.3), then, for each $y \in F(S)$,

$$\langle c^*, y\rangle + \frac{1}{n}\langle c^*, c_0\rangle \geq 0.$$

When $n \to +\infty$, we get that $\langle c^*, y\rangle \geq 0$, and the proof is completed. □

To illustrate Proposition 2.6, we consider the following example.

Example 2.7. Let $X = l^2$, $Y = \mathbb{R} \times l^2$, $C = \mathbb{R}_+ \times l^2_+$, and $S = l^2_+$, where

$$l^2 := \left\{ x = \{x_n\}_{n\in\mathbb{N}} \,|x_n \in \mathbb{R}, \forall n \in \mathbb{N}, \text{ and } \sum_{n=1}^{+\infty} x_n^2 < +\infty \right\},$$

$$l^2_+ := \left\{ x = \{x_n\}_{n\in\mathbb{N}} \in Z | x_n \geq 0, \forall n \in \mathbb{N} \right\}.$$

Let $F : X \to 2^Y$ be given by, for all $x = \{x_n\}_{n\in\mathbb{N}} \in X$,

$$F(x) := \begin{cases} \{(y_1, y_1) \in Y | y_1 \geq \langle \lambda, x\rangle,\ y_2 = -x\}, & \text{if } x \in S, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n = \dfrac{1}{n}$ for all $n \in \mathbb{N}$.

It is easy to check that $\mathrm{qri}C = \mathrm{int}\mathbb{R}_+ \times \{\{x_n\}_{n\in\mathbb{N}} \subseteq l^2 | x_n > 0, \forall n \in \mathbb{N}\}$, and all assumptions of Proposition 2.6 are satisfied. We can choose $c^* = (1, \lambda) \in C^* \setminus \{(0, 0)\}$ such that $\langle c^*, y\rangle \geq 0$ for all $y \in F(S)$.

The generalized C-subconvexlikeness of F in Proposition 2.6 cannot be dispensed as illustrated by

Example 2.8. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and consider a map $F : X \to 2^Y$ defined by $F(x) = \{(y_1, y_2) \in Y | y_1 = x,\ y_2 \geq -x^3\}$ for all $x \in X$. It is obvious to see that $\mathrm{qri}C = \{(y_1, y_2) \in Y | y_1 > 0,\ y_2 > 0\}$ and $F(X) \cap -\mathrm{qri}C = \emptyset$. However, F is not generalized C-subconvexlike, since $\mathrm{cone}_+(F(X)) + \mathrm{qri}C$ is not convex. Thus, the conclusion of Proposition 2.6 is not fulfilled. Indeed, let any $c^* = (c_1^*, c_2^*) \in C^*(= \mathbb{R}^2_+)$, $y = (y_1, y_2) \in F(X)$ such that $\langle c^*, y\rangle \geq 0$, then

$$c_1^* y_1 - c_2^* y_1^3 \geq 0.$$

By taking $y_1 \in X$, the above inequality implies $c_1^* = y_1^2 c_2^*$. Hence, $c_1^* = c_2^* = 0$.

3. **Duality**

In this section, let X, Y, Z be normed spaces, $S \subseteq X$, and $C \subseteq Y$, $D \subseteq Z$ be convex cones with $\mathrm{qriC} \times \mathrm{qri}D \neq \emptyset$. We consider the following vector equilibrium problem (VEP)

$$\text{find } x_0 \in S \text{ such that, for all } x \in \Omega := \{x \in S | G(x) \cap -D \neq \emptyset\},$$

$$F(x_0, x) \cap -\mathrm{qri}C = \emptyset,$$

where $F : S \times S \to 2^Y$, $G : S \to 2^Z$.

Assume that $F(x_1, x_2) \neq \emptyset$, $G(x) \neq \emptyset$ for all $x_1, x_2, x \in S$. By setting $F_{x_0}(x) := F(x_0, x)$ for $x_0, x \in S$ and $F_{x_0}(S) := \bigcup_{x \in S} F_{x_0}(x)$, a feasible point $x_0 \in \Omega$ is said to be a quasi-relative efficient solution of (VEP) if $F_{x_0}(\Omega) \cap -\mathrm{qri}C = \emptyset$. When $\mathrm{int}C \neq \emptyset$, this point is called a weakly efficient solution.

Inspired by [21], we consider the Mond-Weir dual of Lagrange type of (VEP), denoted by ($\mathrm{D_{MW}}$VEP), as follows.

$$\text{maximize } y,$$

$$\inf_{(v,w) \in \left(F_{x_0}, G\right)_+ (S)} \{\langle c^*, v \rangle + \langle d^*, w \rangle\} \geq \langle c^*, y \rangle\,, \tag{3.1}$$

$$(c^*, d^*) \in (C^* \times D^*) \setminus \{(0, 0)\}, \tag{3.2}$$

$$x \in \Omega,\ y \in F_{x_0}(x). \tag{3.3}$$

Remark 3.1. (i) From ($\mathrm{D_{MW}}$VEP), we can get the Mond-Weir dual type expressed in terms of generalized derivatives, for example, the contingent variation in [24], the weak contingent epiderivative in [14], the contingent derivative in [28], the Studniarski derivaitve in [3], etc. To illustrate this statement, we now establish the Mond-Weir dual type in terms of the Studniarski derivative. Recall that the Studniarski derivative of a set-valued mapping $F : X \to 2^Y$ at $(x_0, y_0) \in \mathrm{gr}F$ is defined by (see [3])

$$D^m_S F(x_0, y_0)(u) := \{v \in Y | \exists t_n \to 0^+, (u_n, v_n) \to (u, v),$$
$$y_0 + t_n^m v_n \in F(x_0 + t_n u_n)\}.$$

Assume that $(\mathrm{dom}F_{x_0}) \cap (\mathrm{dom}G) \subseteq S$. Letting $x \in \Omega$ and $(y, z) \in (F_{x_0}, G)(x)$, one needs to imply that

$$\begin{cases} \langle c^*, y' \rangle + \langle d^*, z' \rangle \geq 0, \forall (y', z') \in D^m_S(F_{x_0}, G)_+(x, y, z)(u), \forall u \in X, \\ \langle d^*, z \rangle \geq 0, \\ (c^*, d^*) \in (C^* \times D^*) \setminus \{(0, 0)\}. \end{cases} \tag{3.4}$$

Indeed, putting $v := y$ and $w := z$ in (3.1), one has $\langle d^*, z \rangle \geq 0$. In fact, we get $\langle d^*, z \rangle = 0$ since $z \in G(x) \cap -D$ $(x \in \Omega)$. On the other hand, with any $(y', z') \in D^m_S(F_{x_0}, G)_+(x, y, z)(u)$, there exist $t_n \to 0^+$, $(u_n, v_n, w_n) \to (u, y', z')$ such that

$$(y, z) + t_n^m(v_n, w_n) \in (F_{x_0}, G)(x + t_n u_n) + C \times D,$$

so

$$(y, z) + t_n^m(v_n, w_n) \in (F_{x_0}, G)(S) + C \times D.$$

It follows from (3.1) that

$$\langle c^*, y + t_n^m v_n \rangle + \langle d^*, z + t_n^m w_n \rangle \geq \langle c^*, y \rangle ,$$

i.e., $\langle c^*, y' \rangle + \langle d^*, z' \rangle \geq 0$. Hence, the constraints (3.4) are obtained.

However, (3.1) cannot be implied from the conditions (3.4) by illustrating in Example 3.2 below.

(ii) We can also propose the Wolfe dual of Lagrange type of (VEP) by

$$(\mathrm{D_W VEP}) \quad \begin{cases} \text{maximize } \langle c^*, y \rangle + \langle d^*, z \rangle , \\ \inf\limits_{(v,w) \in \left(F_{x_0}, G\right)_+(S)} \{\langle c^*, v \rangle + \langle d^*, w \rangle\} \geq \langle c^*, y \rangle , \\ (c^*, d^*) \in (C^* \times D^*) \setminus \{(0, 0)\}, \\ x \in \Omega, \ y \in F_{x_0}(x), \ z \in G(x) \cap -D. \end{cases}$$

The Wolfe dual type presented in terms of generalized derivatives, see [14, 24], can be implied from ($\mathrm{D_W}$VEP) similarly.

Example 3.2. Let $X = \mathbb{R}^2$, $Y = Z = \mathbb{R}$, $S = X$, $C = D = \mathbb{R}_+$, and $F : X \times X \to 2^Y$, $G : X \to 2^Z$ be defined by

$$F(x_1, x_2) := \begin{cases} \{x_1^2 + x_2^2\}, & \text{if } (x_1, x_2) \in \mathbb{R}^2_+, \\ \{-2\}, & \text{otherwise,} \end{cases}$$

$$G(x) := \begin{cases} \mathbb{R}_+, & \text{if } x \in \mathbb{R}_+, \\ \{-1\}, & \text{otherwise.} \end{cases}$$

Let $x = 0$, $y = 0 \in F_0(x)$, and $z = 0 \in (G(x) \cap (-D))$. By calculating, we get

$$D^2_S(F_0, G)_+(x, y, z)(u) = \begin{cases} \{u^2\} \times \mathbb{R}_+, & \text{if } u \in \mathbb{R}_+, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is obvious that three relations in $(*)$ are fulfilled for all $(c^*, d^*) \in \mathbb{R}^2_+ \setminus \{(0, 0)\}$. However, (3.1) does not hold since $\langle c^*, y \rangle = 0$, while

$$\inf_{(v,w) \in \left(F_{x_0}, G\right)_+(S)} \{\langle c^*, v \rangle + \langle d^*, w \rangle\} < 0.$$

In this paper, we discuss only duality theorems for (VEP)-($\mathrm{D_{MW}}$VEP) problems. Results in the Wolfe dual type can be obtained by similar proofs.

A feasible point (c^*, d^*, y_0) is said to be a quasi-relative efficient solution of $(\mathrm{D_{MW}VEP})$ with respect to x_0 if $(\Delta - y_0) \cap \mathrm{qri}C = \emptyset$, where

$$\Delta := \{y \in Y | \exists (c^*, d^*) \in (C^* \times D^*) \setminus \{(0,0)\} \text{ such that } (c^*, d^*, y) \text{ satisfies the constraints of } (\mathrm{D_{MW}VEP})\}.$$

The weak, strong, and converse duality theorems for (VEP) and $(\mathrm{D_{MW}VEP})$ are established as follows.

Theorem 3.3 (Weak duality)**.** *If $\hat{x}$ is a feasible point of* (VEP) *and $(c^*, d^*, \overline{y})$ is a feasible point* $(\mathrm{D_{MW}VEP})$ *with respect to x_0, then $\langle c^*, \overline{y}\rangle \leq \langle c^*, y\rangle$ for all $y \in F_{x_0}(\hat{x})$.*

Proof. Since $\hat{x} \in \Omega$, there exists $\hat{z} \in G(\hat{x}) \cap -D$. Besides, it follows from the feasibility of $(c^*, d^*, \overline{y})$ that

$$\begin{aligned}
\langle c^*, \overline{y}\rangle &\leq \inf_{(y,z)\in \left(F_{x_0}, G\right)_+(S)} \{\langle c^*, y\rangle + \langle d^*, z\rangle\} \\
&\leq \langle c^*, y\rangle + \langle d^*, z\rangle \quad \text{for all } (y,z) \in (F_{x_0}, G)(S) \\
&\leq \langle c^*, y\rangle + \langle d^*, \hat{z}\rangle \quad \text{for all } y \in F_{x_0}(\hat{x}) \\
&\leq \langle c^*, y\rangle \quad (\text{since } \hat{z} \in -D \text{ and } d^* \in D^*),
\end{aligned}$$

and the proof is completed. □

Theorem 3.4 (Strong duality)**.** *Let $x_0 \in \Omega$, $y_0 \in F_{x_0}(x_0) : y_0 \in -C$, and $z_0 \in G(x_0) \cap -D$. Suppose that x_0 is a quasi-relative efficient solution of* (VEP)*, $(F_{x_0} - y_0, G)_+$ is generalized $(C \times D)$-subconvexlike on S, and $\mathrm{qri}(\mathrm{cone}_+((F_{x_0} - y_0, G)_+(S)) + \mathrm{qri}(C \times D)) \neq \emptyset$. Then, there exists $(c^*, d^*) \in (C^* \times D^*) \setminus \{(0,0)\}$ such that (c^*, d^*, y_0) is a feasible point of* $(\mathrm{D_{MW}VEP})$ *with respect to x_0. If, additionally, the inequality of* (3.1) *is strict for all feasible points, then (c^*, d^*, y_0) is a quasi-relative efficient solution.*

Proof. Since x_0 is a quasi-relative efficient solution of (VEP), we get

$$(F_{x_0}, G)(S) \cap -\mathrm{qri}(C \times D) = \emptyset.$$

Since $y_0 \in -C$, we have

$$(F_{x_0} - y_0, G)_+(S) \cap -\mathrm{qri}(C \times D) = \emptyset.$$

By Proposition 2.6, there exists $(c^*, d^*) \in (Y^* \times Z^*) \setminus \{(0,0)\}$ such that for all $(y,z) \in (F_{x_0} - y_0, G)_+(S)$,

$$\langle c^*, y\rangle + \langle d^*, z\rangle \geq 0,$$

equivalently, for all $(y,z) \in (F_{x_0}, G)(S)$, $(c,d) \in C \times D$,

$$\langle c^*, y - y_0 + c\rangle + \langle d^*, z + d\rangle \geq 0. \tag{3.5}$$

Let $y = y_0$, $c = 0$, and $z = z_0$ in (3.5), we get that $\langle d^*, z_0 + d\rangle \geq 0$. Because D is a cone, it implies that $\langle d^*, d\rangle \geq 0$ for all $d \in D$, i.e., $d^* \in D^*$. Besides, it follows from (3.5) that $\langle d^*, z_0\rangle \geq 0$ (with $y = y_0$, $c = 0$, and $d = 0$). Thus, $\langle d^*, z_0\rangle = 0$ (since $z_0 \in -D$ and $d^* \in D^*$).

Taking $y = y_0$, $z = z_0$, and $d = 0$ in (3.5), we get that $\langle c^*, c\rangle \geq 0$ for all $c \in C$, i.e., $c^* \in C^*$. Moreover, (3.5) implies that (3.1) holds for (c^*, d^*, y_0). Hence, (c^*, d^*, y_0) is a feasible point of ($\mathrm{D_{MW}VEP}$) with respect to x_0.

We next prove that (c^*, d^*, y_0) is a quasi-relative efficient solution. Suppose to the contrary, i.e., there is $(\hat{c}^*, \hat{d}^*, \hat{y})$ satisfying (3.1), (3.2) and $\hat{y} - y_0 \in \mathrm{qri}C$. By the assumption and the feasibility of $(\hat{c}^*, \hat{d}^*, \hat{y})$, we get

$$\langle \hat{c}^*, y_0 - \hat{y}\rangle > 0,$$

which contradicts that $\langle \hat{c}^*, y_0 - \hat{y}\rangle \leq 0$ since $y_0 - \hat{y} \in -\mathrm{qri}C$ and $\hat{c}^* \in C^*$. $\square$

Theorem 3.5 (Converse duality)**.** *Let $x_0 \in \Omega$, $y_0 \in F_{x_0}(x_0) : y_0 \in C$. Suppose that (c^*, d^*, y_0) is a feasible point of* ($\mathrm{D_{MW}VEP}$) *with respect to x_0 such that* (3.1) *is a strict inequality. Then, x_0 is a quasi-relative efficient solution of* (VEP).

Proof. Suppose that x_0 is not a quasi-relative efficient solution of (VEP), i.e., there exist $\overline{x} \in S$, $\overline{y} \in F_{x_0}(\overline{x})$, and $\overline{z} \in G(\overline{x}) \cap -D$ such that $\overline{y} \in -\mathrm{qri}C$, which implies that

$$\overline{y} - y_0 \in -\mathrm{qri}C - C \subseteq -\mathrm{qri}C.$$

Thus, $\langle c^*, \overline{y} - y_0\rangle \leq 0$. By the assumption, we get

$$\langle c^*, \overline{y} - y_0\rangle > -\langle d^*, \overline{z}\rangle \geq 0,$$

which is a contradiction. $\square$

We now apply the above duality results to equilibrium conditions for quasi-relative efficient solutions of (VEP) in dealing with Fritz-John and Kuhn-Tucker types. The next theorem can be expressed in terms of all generalized derivatives defined in the primal space, such as the contingent derivative (see [7]), variants of (generalized) contingent epiderivatives (see [9, 14, 23, 32]), radial sets and radial derivatives (see [4,5]), etc. Here, we prove the statement using the higher-order contingent derivaitve, known as the first and the most popular derivative for set-valued mappings. Recall that the mth-order contingent derivative of a set-valued mapping $F : X \to 2^Y$ at $(x_0, y_0) \in \mathrm{gr}F$ with respect to $(u_i, v_i) \in X \times Y$, $i = 1, \ldots, m-1$, is defined by

$$D^m F(x_0, y_0, u_1, v_1, \ldots, u_{m-1}, v_{m-1})(u) := \{v \in Y | \exists t_n \to 0^+, \exists (u_n, v_n) \to (u, v),$$

$$y_0 + t_n v_1 + \ldots + t_n^{m-1} u_{m-1} + t_n^m v_n \in F(x_0 + t_n u_1 + \ldots + t_n^{m-1} u_{m-1} + t_n^m u_n)\}.$$

Theorem 3.6. *Let $x_0 \in \Omega$, $y_0 \in F_{x_0}(x_0)$, $z_0 \in G(x_0) \cap -D$, and $(u_i, v_i, w_i) \in X \times Y \times Z$, $i = 1 \ldots m-1$.*

(i) (*Necessary condition*) *Suppose that* $y_0 \in -C$, $(v_i, w_i) \in (-C)\times(-D)$, $i = 1 \dots m-1$, x_0 *is a quasi-relative efficient solution of* (VEP), $(F_{x_0} - y_0, G)_+$ *is generalized* $(C\times D)$*-subconvexlike on* S, *and* $\mathrm{qri}(\mathrm{cone}((F_{x_0} - y_0, G)_+(S)) + \mathrm{qri}(C\times D)) \neq \emptyset$. *Then, there exists* $(c^*, d^*) \in (C^*\times D^*)\setminus\{(0,0)\}$ *such that for all* $(y,z) \in D^m(F_{x_0}, G)_+(x_0, y_0, z_0, u_1, v_1, w_1 \dots u_{m-1}, v_{m-1}, w_{m-1})(\Upsilon)$, *where* $\Upsilon := \mathrm{dom}D^m(F_{x_0}, G)_+(x_0, y_0, z_0, u_1, v_1, w_1 \dots u_{m-1}, v_{m-1}, w_{m-1})$,

$$\langle c^*, y\rangle + \langle d^*, z\rangle \geq 0, \tag{3.6}$$

and

$$\langle d^*, z_0\rangle = 0. \tag{3.7}$$

If, additionally, there exists $\overline{x} \in S$: $\langle d^*, G(\overline{x})\rangle \cap -\mathrm{int}\mathbb{R}_+ \neq \emptyset$, *then* $c^* \neq 0$.

(ii) (*Sufficient condition*) *Assume that* $y_0 \in C$ *and the following condition is satisfied*

$$F(x) - y_0 \subseteq D^m(F_{x_0}, G)_+(x_0, y_0, z_0, u_1, v_1, w_1 \dots u_{m-1}, v_{m-1}, w_{m-1})(x - x_0). \tag{3.8}$$

If there is $(c^*, d^*) \in (C^*\times D^*)\setminus\{(0,0)\}$ *such that* (3.6) *(being a strict inequality) and* (3.7) *hold, then* x_0 *is a quasi-relative efficient solution of* (VEP).

Proof. (i) By the proof of Theorem 3.4, there exists $(c^*, d^*) \in (C^*\times D^*)\setminus\{(0,0)\}$ such that $\langle d^*, z_0\rangle = 0$ and for all $(y,z) \in (F_{x_0}, G)_+(S)$,

$$\langle c^*, y - y_0\rangle + \langle d^*, z - z_0\rangle \geq 0. \tag{3.9}$$

Let $u \in \mathrm{dom}D^m(F_{x_0}, G)_+(x_0, y_0, z_0, u_1, v_1, w_1 \dots u_{m-1}, v_{m-1}, w_{m-1})$, $(y,z) \in D^m(F_{x_0},\ G)_+(x_0, y_0, z_0, u_1, v_1, w_1 \dots u_{m-1}, v_{m-1}, w_{m-1})(u)$, then there exist $t_n \to 0^+$, $\{x_n\}_{n\in\mathbb{N}} \subseteq S$, and $(y_n, z_n) \in (F_{x_0}, G)(x_n)$ for all n such that

$$\frac{x_n - x_0 - t_n u_1 - \cdots - t_n^{m-1}u_{m-1}}{t_n^m} \to u,$$

$$\frac{y_n - y_0 - t_n v_1 - \cdots - t_n^{m-1}v_{m-1}}{t_n^m} \to y,$$

$$\frac{z_n - z_0 - t_n w_1 - \cdots - t_n^{m-1}w_{m-1}}{t_n^m} \to z.$$

Since $(v_i, w_i) \in (-C)\times(-D)$, it follows from (3.9) that

$$\left\langle c^*, \frac{y_n - y_0 - t_n v_1 - \cdots - t_n^{m-1}v_{m-1}}{t_n^m}\right\rangle + \left\langle d^*, \frac{z_n - z_0 - t_n w_1 - \cdots - t_n^{m-1}w_{m-1}}{t_n^m}\right\rangle \geq 0.$$

Taking $n \to +\infty$, we get

$$\langle d^*, y\rangle + \langle d^*, z\rangle \geq 0.$$

If, additionally, there exists $\overline{x} \in S$: $\langle d^*, G(\overline{x})\rangle \cap -\text{int}\mathbb{R}_+ \neq \emptyset$, we prove that $c^* \neq 0$. Suppose to the contrary, i.e., $c^* = 0$, so $d^* \neq 0$. By the assumption, there exists $\overline{z} \in G(\overline{x})$ with $\langle d^*, \overline{z}\rangle < 0$. However, it follows from (3.9) that $\langle d^*, \overline{z}\rangle \geq 0$, which is a contradiction.

(ii) It is similar to the proof of Theorem 3.5. □

Remark 3.7. (i) A sufficient condition of (3.8) was given in [28, Proposition 3.2].

(ii) Theorem 3.6 is an extension of several existing results in the literature concering equilibrium conditions for (VEP) in terms of generalized derivatives, such as [19, Theorem 3.1], [29, Theorem 3.1], [30, Propositions 4.1, 5.1, 5.2], [32, Theorem 4.5], [33, Theorem 3.2], to the case that the ordering cone in both the objective space and the constraint space have empty interior. Moreover, the convexity condition in the paper is weaker than that in the above-mentioned papers.

(iii) If $\text{int}C \neq \emptyset$, Theorems 3.4-3.6 give us strong duality, converse duality, and equilibrium conditions for weakly efficient solutions of (VEP), respectively, in the case of constraint cone D having possibly empty interior. In this case, the conditions that (3.1) (for Theorems 3.4, 3.5) and (3.6) (for Theorem 3.6) are strict inequalities can be omitted. The reaseon is that intC is open, while qriC is not. Results for some kinds of solutions for (VEP), e.g., Pareto efficient solution, Henig efficient solutions, and other types of proper efficient solutions, see [5, 22], can be implied similarly when $\text{int}D = \emptyset$.

To illustrate Remark 3.7(iii), we consider the following examples.

Example 3.8. Let $X = Z = l^2$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $S = X$, and $D = l^2_+$. We define the mappings $F : X \times X \to Y$ and $G : X \to Z$ by, for $c = \{c_n\}_{n\in\mathbb{N}} \in l^2_+$, $c_n = 1/n^2$ for all $n \in \mathbb{N}$, $x_1, x_2, x \in X$,

$$F(x_1, x_2) := \{y \in Y | y \geq \langle c, x_1 + x_2\rangle\}, \quad G(x) := -x.$$

Then, $\Omega := \{x \in S | G(x) \cap -D \neq \emptyset\} = l^2_+$. Let $x_0 = 0_{l^2} \in \Omega$, $y_0 = 0 \in F_{x_0}(x_0)$, $z_0 = 0_{l^2}$. By calculating, we get

$$D(F_{x_0}, G)_+(x_0, y_0, z_0)(u) = \{(\langle c, u\rangle, -u)\} + C \times D.$$

Then, there exists $(c^*, d^*) = (1, c) \in (C^* \times D^*) \setminus \{(0, 0)\}$ such that (3.6) and (3.7) hold. It follows from Theorem 3.6(ii) that x_0 is a weakly efficient solution of (VEP).

Example 3.9. Let $X = Y = \mathbb{R}$, $Z = l^2$, $S = C = \mathbb{R}_+$, and $D = l^2_+$. For given $\lambda = \{\lambda_n\}_{n\in\mathbb{N}} \in l^2_+$, $\lambda_n = 1/n$ for all $n \in \mathbb{N}$, we consider the mapping $F : S \times S \to Y$ and $G : S \to Z$ defined by

$$F(x_1, x_2) := \{y \in Y | y \geq -\sqrt{x_1} - \sqrt{x_2}\}, \quad G(x) := -x\lambda.$$

Then, $\Omega := \{x \in S | G(x) \cap -D \neq \emptyset\} = S$.

Let $x_0 = 0 \in \Omega$, $y_0 = 0 \in F_{x_0}(x_0)$, and $z_0 = 0_{l^2}$. It is easy to check that $(F_{x_0} - y_0, G)_+$ is generalized $(C \times D)$-subconvexlike on S. Moreover, $\text{qri}(\text{cone}((F_{x_0} - y_0, G)_+(S)) + \text{int}C \times \text{qri}D)) \neq \emptyset$ and for all $u \geq 0$,

$$D(F_{x_0}, G)_+(x_0, y_0, z_0)(u) = \mathbb{R} \times (\{-u\lambda\} + D).$$

Suppose that there exists $(c^*, d^*) \in C^* \times D^*$ such that (3.6) and (3.7) hold. Then for all $x \geq 0$, $u \in \mathbb{R}$, we have

$$c^*(u) + \langle d^*, -x\lambda\rangle \geq \langle c^*, y_0\rangle = 0.$$

With $u = -1$, one gets

$$c^* + \sqrt{x}\langle d^*, \lambda\rangle \leq 0,$$

which implies that $c^* = 0$ and $d^* = 0_{l^2}$. By Theorem 3.6(i), x_0 is not a weakly efficient solution of (VEP)

In the rest of this section, we consider the following constrained set-valued optimization problem. Let X, Y, Z, S, C, D and G be as for (VEP), and $H : X \to 2^Y$, our problem is

$$\text{(SOP)} \;\; \text{Minimize}\, H(u) \quad \text{subject to} \quad u \in \Omega,$$

where $\Omega := \{u \in S | G(u) \cap -D \neq \emptyset\}$.

A point $(u_0, v_0) \in \text{gr}H$ is said to be a quasi-relative efficient solution of (SOP) if $u_0 \in \Omega$ and $(H(\Omega) - v_0) \cap -\text{qri}C = \emptyset$.

By setting $F(x, u) := H(u) - v_0$ for $x, u \in S$, (SOP) become a special case of (VEP).

The Mond-Weir dual of Lagrange type (D_{MW}SOP) of (SOP) is defined by

$$\text{maximize } v$$

$$\inf_{(k,z)\in(H,G)_+(S)} \{\langle c^*, k\rangle + \langle d^*, z\rangle\} \geq \langle c^*, v\rangle, \tag{3.10}$$

$$(c^*, d^*) \in (C^* \times D^*) \setminus \{(0, 0)\}, \text{ and } v \in F(u_0, u),\ u \in \Omega. \tag{3.11}$$

A feasible point (c^*, d^*, v_0) is said to be a quasi-relative efficient solution of (D_{MW}SOP) if $(\Delta' - v_0) \cap \text{qri}C = \emptyset$, where

$$\Delta' := \{v \in Y | \exists (c^*, d^*) \in (C^* \times D^*) \setminus \{(0, 0)\} \text{ such that}$$
$$(c^*, d^*, v) \text{ satisfies the constraints of } (\text{D}_{\text{MW}}\text{SOP})\}.$$

Remark 3.10. $(c^*, d^*, \overline{v})$ is a feasible point of (D_{MW}SOP) if and only if $(c^*, d^*, 0)$ is a feasible point of (D_{MW}VEP) with respect to u_0, for some $u_0 \in S$ and $F_{u_0}(u) := H(u) - \overline{v}$.

Duality theorems for (SOP)-(D_{MW}SOP) and optimality conditions for (SOP) can be obtained directly from corresponding results of (VEP)-(D_{MW}VEP) as follows.

Theorem 3.11. (i) (*Weak duality*) *If* $(\hat{u},\hat{v}) \in \mathrm{gr}H$ *is a feasible point of* (SOP) *and* $(c^*, d^*, \overline{v})$ *is a feasible point* ($\mathrm{D_{MW}SOP}$), *then* $\langle c^*, \overline{v}\rangle \leq \langle c^*, \hat{v}\rangle$.

(ii) (*Strong duality*) *Let* $u_0 \in \Omega$, $v_0 \in H(u_0)$, *and* $z_0 \in G(u_0) \cap -D$. *Suppose that* u_0 *is a quasi-relative efficient solution of* (SOP), $(H - v_0, G)_+$ *is generalized* $(C \times D)$*-subconvexlike on* S, *and* $\mathrm{qri}(\mathrm{cone}_+((H - v_0, G)_+(S)) + \mathrm{qri}(C \times D)) \neq \emptyset$. *Then, there exists* $(c^*, d^*) \in (C^* \times D^*) \setminus \{(0,0)\}$ *such that* (c^*, d^*, v_0) *is a feasible point of* ($\mathrm{D_{MW}SOP}$). *If, additionally, the inequality of* (3.10) *is strict for all feasible points, then* (c^*, d^*, v_0) *is a quasi-relative efficient solution.*

(iii) (*Converse duality*) *Let* $u_0 \in \Omega$ *and* $v_0 \in H(u_0)$. *Suppose that* (c^*, d^*, v_0) *is a feasible point of* ($\mathrm{D_{MW}SOP}$) *such that* (3.10) *is a strict inequality. Then,* u_0 *is a quasi-relative efficient solution of* (SOP).

Proof. (i) Set $F_{\hat{u}}(u) := H(u) - \overline{v}$, it follows from Remark 3.10 that $(c^*, d^*, 0)$ is a feasible point of ($\mathrm{D_{MW}VEP}$) with respect to $\hat{u}$. Since $\hat{u}$ is also a feasible point of (VEP), from Theorem 3.3, we get for all $v \in F_{\hat{u}}(\hat{u})$,

$$\langle c^*, 0\rangle \leq \langle c^*, v\rangle,$$

equivalently, for all $v \in H(\hat{u})$,

$$0 \leq \langle c^*, v - \overline{v}\rangle,$$

which implies that $\langle c^*, \overline{v}\rangle \leq \langle c^*, \hat{v}\rangle$.

(ii) and (iii) follow immediately from Theorems 3.4, 3.5 with $x_0 := u_0$, $F_{x_0}(u) := H(u) - v_0$, $y_0 := 0$ ($y_0 \in F_{x_0}(x_0)$). □

Theorem 3.12. *Let* $u_0 \in \Omega$, $v_0 \in H(u_0)$, $z_0 \in G(x_0) \cap -D$ *and* $(u_i, v_i, w_i) \in X \times Y \times Z$.

(i) (*Necessary condition*) *Suppose that* $(v_1, w_1) \in (-C) \times (-D)$, $i = 1 \ldots m-1$, u_0 *is a quasi-relative efficient solution of* (SOP), $(H - v_0, G)_+$ *is generalized* $(C \times D)$*-subconvexlike on* S, *and* $\mathrm{qri}(\mathrm{cone}((H - v_0, G)_+(S)) + \mathrm{qri}(C \times D)) \neq \emptyset$. *Then, there exists* $(c^*, d^*) \in (C^* \times D^*) \setminus \{(0,0)\}$ *such that for all* $(v,w) \in D^m(H,G)_+(x_0, y_0, z_0, u_1, v_1, w_1 \ldots u_{m-1}, v_{m-1}, w_{m-1})(\Lambda)$, *where* $\Lambda := \mathrm{dom}D^m(H,G)_+(x_0, y_0, z_0, u_1, v_1, w_1 \ldots u_{m-1}, v_{m-1}, w_{m-1})$

$$\langle c^*, v\rangle + \langle d^*, w\rangle \geq 0, \tag{3.12}$$

and

$$\langle d^*, z_0\rangle = 0. \tag{3.13}$$

If, additionally, there exists $\overline{x} \in S$: $\langle d^*, G(\overline{x})\rangle \cap -\mathrm{int}\mathbb{R}_+ \neq \emptyset$, *then* $c^* \neq 0$.

(ii) (*Sufficient condition*) *Assume that the condition* (3.8) *is fulfilled for* (H, G) *and there is* $(c^*, d^*) \in (C^* \times D^*) \setminus \{(0,0)\}$ *such that* (3.12) (*being a strict inequality*) *and* (3.13) *hold. Then,* u_0 *is a quasi-relative efficient solution of* (SOP).

Proof. It follows from Theorem 3.6 with $x_0 := u_0$, $F_{x_0}(u) := H(u) - v_0$, $y_0 := 0$ ($y_0 \in F_{x_0}(x_0)$). □

Remark 3.13. (i) Remarks 3.1 and 3.7(iii) are still valid for (SOP) and results similar to Theorem 3.12 in terms of other generalized derivatives can be established for (SOP).

(ii) If the mapping F and G are single-valued and intC $\neq \emptyset$, (SOP)-($\mathrm{D_{MW}}$SOP) reduce to ($\mathrm{PV^C}$)-($\mathrm{DV^{C_L}}$) in [21] with respect to weakly efficient solutions. Thus, Theorems 3.11, 3.12 can be considered as extended results on duality and optimality conditions of ($\mathrm{PV^C}$) and ($\mathrm{DV^{C_L}}$) from a single-valued optimization problem with ordering cones having nonempty interior to a set-valued optimization problem relative to nonsolid cones.

4. Conclusions

The paper has been devoted to duality for vector equilibrium problems (VEP) with constraints relative to set-valued mappings. The new results have been expressed in terms of quasi-relative efficient solutions. They are extensions of weakly efficient solutions when the ordering cone in the objective space has empty interior. Their applications to optimality conditions for (VEP) have been established with respect to nonsolid cones. Since (SOP) is a particular case of (VEP), one has obtained results on duality and optimality conditions of (SOP) from the corresponding ones of (VEP) immediately. They are extensions of ($\mathrm{PV^C}$) and ($\mathrm{DV^{C_L}}$) in [21] to the case of nonsolid cones and set-valued mappings. We have also provided several examples to illustate our results and to ensure that our assumptions are essential.

In [21], duality theorems of ($\mathrm{PV^C}$) and ($\mathrm{DV^{C_L}}$) were established in Theorem 14 under some regularity conditions. Thus, for possible developments of our work, we intend to extend these conditions to the case of set-valued optimization with ordering cones having empty interior. On the other hand, for an equilibrium problem of the Stampacchia type (like the one under consideration here), the corresponding Minty problem is usually taken for its dual, see [2, 17, 25]. This dual problem is relative to the dual relationship between minsup-points and maxinf-points. A natural question arises: is there any connection between such dual scheme and the dual problem proposed in the paper? Finding the answer for this question is a promising study.

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