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AN UPPER BOUND FOR THE REGULARITY OF POWERS OF EDGE IDEALS

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Abstract. A recent result due to Hà and Van Tuyl proved that the Castelnuovo-Mumford regularity of the quotient ring $R/I(G)$ is at most matching number of G, denoted by $\mathrm{match}(G)$. In this paper, we provide a generalization of this result for powers of edge ideals. More precisely, we show that for every graph G and every $s \geq 1$,

$$\mathrm{reg}(R/I(G)^s) \leq (2s-1)|E(G)|^{s-1}\mathrm{match}(G).$$

Keywords: Edge ideal, Castelnuovo-Mumford regularity, matching number.
MSC(2010): Primary: 13D02; Secondary: 05E40, 05C38.

1. Introduction

For any homogeneous ideal I of a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$, there exists a graded minimal free resolution

$$0 \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{1,j}(I)} \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}(I)} \longrightarrow I \longrightarrow 0.$$

The Castelnuovo-Mumford regularity (or simply, regularity) of I, denote by $\mathrm{reg}(I)$, is defined by

$$\mathrm{reg}(I) = \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

The regularity of I is an important invariant in commutative algebra.

While the regularity is linear for large powers, (see [2,6]), the study of initial behavior of regularity is often quite mysterious. For this reason, computing the exact values or finding bounds for the regularity of powers of I is a difficult problem.

There is a one-to-one correspondence between square-free monomial ideals generated in degree two in $R = \mathbb{K}[x_1, \ldots, x_n]$ and finite simple graphs with

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n vertices. This correspondence is realized by forming an ideal $I(G)$, called the edge ideal, where $x_i x_j$ is a generator of $I(G)$ if and only if x_i and x_j are connected by an edge of G. By abusing the notation, we identify the generators of $I(G)$ by the edges of G.

Finding connection between algebraic properties of an edge ideal and invariants of graph is of great interest. One question in this area is to explain the regularity of edge ideals and their powers by some information from combinatorial invariants of the associated graph. Many questions have been verified only in very special cases. For some classes of graphs, for example, chordal graphs and well-covered bipartite graphs, it was shown that the regularity of the quotient ring $R/I(G)$ is equal to the maximum cardinality of the induced matching of G, denoted by $\mathrm{indmatch}(G)$. (see [3, Corollary 6.9] and [7, Theorem 1.1]).
Also, in [5, Lemma 2.2], it was shown that for any graph G, the inequality

$$\mathrm{reg}(R/I(G) \geq \mathrm{indmatch}(G)$$

holds.

Recently, Beyarslan, Hà and Trung found a lower bound for the regularity of powers of edge ideals in terms of the induced matching number of G. They proved that for every graph G and every integer $s \geq 1$, one has

$$\mathrm{reg}(I(G)^s) \geq 2s + \mathrm{indmatch}(G) - 1.$$

(see [1, Theorem 4.5]). In the same paper, they proved the equality for every $s \geq 1$, if G is a forest and for every $s \geq 2$, if G is a cycle. (see [1, Theorems 4.7 and 5.2]).

Also, in [8, Theorem 2.5], it was shown that if G is a whiskered cycle graph, then for all $s \geq 1$

$$\mathrm{reg}(I(G)^s) = 2s + \mathrm{indmatch}(G) - 1.$$

On the other hand, for every graph G, Hà and Van Tuyl prove that the regularity of $R/I(G)$ is bounded above by matching number of G, denoted by $\mathrm{match}(G)$, (see [3, Theorem 6.7]).

Our goal is to extend the matching upper bound of Hà and Van Tuyl to powers of edge ideals. More precisely, in Theorem 2.1, we show that for every graph G and every integer $s \geq 1$,

$$\mathrm{reg}(R/I(G)^s) \leq (2s-1)|E(G)|^{s-1}\mathrm{match}(G).$$

2. The main result

A matching in a graph G is a set of edges no two of which have a common endpoint. The largest size of a matching in G is called its matching number and is denoted by $\mathrm{match}(G)$.

Let $V = \{x_1, \ldots, x_n\}$ be a finite set, and let $E = \{e_1, \ldots, e_s\}$ be a family of distinct subset of V. The pair $\mathcal{H} = (V, E)$ is called a hypergraph if $e_i \neq \varnothing$ for each i. The elements of V are called vertices and the elements of E are called edges of the hypergraph.

A hypergraph $\mathcal{H}$ is simple if

(1) $\mathcal{H}$ has no loops, i.e., $|e_i| \geq 2$ for all $e_i \in E$, and
(2) $\mathcal{H}$ has no multiple edges, i.e., if $e_i, e_j \in E$ and $e_i \subseteq e_j$, then $i = j$.

A hypergraph $\mathcal{H}$ is said to be $d-$uniform if $|e_i| = d$ for every edge $e_i \in E$.
We can associate to every simple hypergraph $\mathcal{H} = (V, E)$ a square-free monomial ideal

$$\mathcal{I}(\mathcal{H}) = \langle \{x^{e_i} = \prod_{x \in e_i} x \mid e_i \in E\} \rangle \subseteq R = \mathbb{K}[x_1, \ldots, x_n].$$

We call the ideal $\mathcal{I}(\mathcal{H})$ the edge ideal of $\mathcal{H}$.

If $\mathcal{H} = (V, E)$ is a hypergraph, then a $2-$collage for $\mathcal{H}$ is a subset $\mathcal{C}$ of the edges with the property that for each $e \in E$ there exists a vertex v such that $e \setminus \{v\}$ is contained in at least one edge of $\mathcal{C}$.

We are now ready to prove the main result of this paper.

Theorem 2.1. *Let $G = (V(G), E(G))$ be a graph and $I = I(G)$ be its edge ideal. Then for all $s \geq 1$, we have*

$$\operatorname{reg}(R/I^s) \leq (2s-1)|E(G)|^{s-1}\operatorname{match}(G).$$

Proof. For $s = 1$ the inequality is known. (see [3, Theorem 6.7]).
Now assume that $s \geq 2$. The ideal I^s is generated by monomials of degree $2s$. Let J be the polarization of I^s, which is considered in a new polynomial ring, say T. Then J is the edge ideal of a $2s$-uniform hypergraph $\mathcal{H}$. By [4, Theorem 1.1], we know that

$$\operatorname{reg}(T/J) \leq (2s-1)c,$$

where, c is the minimum size of a $2-$collage in $\mathcal{H}$.

Note that polarization does not change the regularity. Hence, it is enough to show that

$$c \leq |E(G)|^{s-1}\operatorname{match}(G).$$

For every generator u of I^s, we denote its polarization by $u^{pol} \in J$.

Let $M = \{e_{i_1}, \ldots, e_{i_m}\}$ be a maximal matching of G with $\operatorname{match}(G) = m$. It is clear that every generator of I^s is of the form $u = u_1 \cdots u_s$, where for every k, the monomial u_k is an edge of G. We claim that the edges of $\mathcal{H}$ which correspond to the set

$$\mathcal{A} = \{(u_1 \cdots u_s)^{pol} \mid \exists k : u_k \in M\},$$

is a $2-$collage of $\mathcal{H}$, and the claim completes the proof.

To prove the claim assume that $(u_1 \cdots u_s)^{pol}$ is a generator of J which does not belong to $\mathcal{A}$. Then $u_1, \ldots, u_s \notin M$. Therefore, there exists an integer

$1 \leq r \leq m$, such that $e_{i_r} \cap u_1 \neq \emptyset$. Because, otherwise we can add u_1 to M which is contradiction by the maximality of M.

Assume now that $(e_{i_r} \cap u_1) = \{y\}$. Let x be the other vertex of u_1. Assume that x is the vertex of exactly ℓ edges among $u_1, \ldots, u_s$. Suppose that x_ℓ is the vertex of $\mathcal{H}$ which corresponds to the ℓth power of x. If we remove x_ℓ from the edge which is associated to $(u_1 \cdots u_s)^{pol}$, then the resulting set is a subset of the edge associated to $(e_{i_r} u_2 \cdots u_s)^{pol}$. This proves the claim and completes the proof of the theorem. □

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