ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 43 (2017), No. 6, pp. 1699–1707

Title:

Properties of matrices with numerical ranges in a sector

Author(s):

D. Zhang, L. Hou and L. Ma

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 6, pp. 1699–1707
Online ISSN: 1735-8515

PROPERTIES OF MATRICES WITH NUMERICAL RANGES IN A SECTOR

D. ZHANG*, L. HOU AND L. MA

(Communicated by Abbas Salemi)

Abstract. Let A be a complex $n \times n$ matrix and assume that the numerical range of A lies in the set of a sector of half angle α denoted by S_α. We prove the numerical ranges of the conjugate, inverse and Schur complement of any order of A are in the same S_α. The eigenvalues of some kinds of matrix product and numerical ranges of hadmard product, star-congruent matrix and unitary matrix of polar decompostion are also included in the same sector. Furthermore, we extend some inequalities about eigenvalues and singular values and the linear fractional maps to this class of matrices.

Keywords: Numerical ranges, sector, positive definite, Toeplitz decomposition.
MSC(2010): Primary: 15A48; Secondary: 15A45, 15A57.

1. **Introduction**

Let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. In the present analysis, we use the Toeplitz (sometimes also called the Hermitian) decomposition of A:

$$A = B + iC, \tag{1.1}$$

where

$$B = \frac{1}{2}(A + A^*),\ C = \frac{1}{2i}(A - A^*).$$

If $B > 0$ and $C > 0$, then A is said to be accretive-dissipative. And this kind of matrices of order n will be denoted by $\mathbb{M}_n^{++}$.

Recall that the numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax :\ x \in \mathbb{C}^n,\ x^*x = 1\}.$$

Article electronically published on 30 November, 2017.
Received: 25 July 2015, Accepted: 21 September 2016.
*Corresponding author.

Also, we define a sector on the complex plane

$$S_\alpha = \{z \in \mathbb{C} : \mathcal{R}z > 0, |\, \mathcal{I}z \,| \leq \mathcal{R}z \tan(\alpha)\}, \ \alpha \in [0, \frac{\pi}{2}). \tag{1.2}$$

Besides, $\mathbb{P}_n$ denotes the set of positive semidefinite $n \times n$ matrices.

A comprehensive survey on the properties of $\mathbb{M}_n^{++}$ can be found in [7]. Meanwhile, according to [7], we know that accretive-dissipative matrices first appeared in [9] for the special case in which both B and C in (1.1) are real symmetric matrices, which results in a complex symmetric A. It turned out that these matrices possess remarkable properties with respect to Gaussian elimination, very similar to those of ordinary positive-definite matrices [9] (also, see [8, 14]). Determinantal inequalities of Fisher type for accretive-dissipative matrices were proved and improved in [5, 12, 13, 16, 17]. Moreover, sectorial matrices were examined in [1] which focused on discussing the criteria for a matrix to be sectorial.

Later, many interesting inequalities and properties for matrices with numerical ranges in a sector are investigated by several authors. For example, in [4, 15] the authors discussed the determinant, eigenvalue and singular value inequalities. In [18], Lin extended the result of Haynsworth and Hartfiel and Zhang also made an extension of Matic's results to a sector in [22]. Rotfel'd inequality for partitioned matrices was as well discussed in such condition in [6].

In [7] some remarkable characteristics are proved, and we see that many properties of matrices $\mathbb{M}_n^{++}$ are natural extension of the corresponding properties of positive-definite matrices. In this article, we will extend the properties to matrices with numerical ranges in a sector.

2. Main results

In this section, we prove various aspects of properties of matrices with numerical ranges in a sector.

Here we assume that A has the form in (1.1) and a subset of $\mathbb{C}$ as a sector of half angle α if it is of the form $\{e^{i\varphi}z : z \in S_\alpha\}$ for some $\varphi \in [0, 2\pi)$ and S_α defined in (1.2).

We begin with several lemmas.

Lemma 2.1 ([3, Corollary 2.4])**.** *Let $0 \leq \alpha < \frac{\pi}{2}$, $0 < \gamma < 1$ and T be a complex square matrix with $W(T) \subseteq S_\alpha$. Then $W(T^\gamma) \subseteq S_{\alpha\gamma}$.*

Lemma 2.2 ([20, Lemma 2.1])**.** *Let A has positive definite Hermitian part and Let $H = H(A)$ and $S = S(A)$. Then A is invertible and A^{-1} has positive definite Hermitian part given by*

$$H(A^{-1}) = \frac{A^{-1} + A^{-*}}{2} = (H + S^* H^{-1} S)^{-1}$$

Lemma 2.3 ([10, p. 10]). *Spectral containment: for all $A \in \mathbb{M}_n$,*

$$\sigma(A) \subset W(A),$$

where $\sigma(A)$ denotes the spectrum of A.

2.1. The set S_α.

Property 2.4. If $W(A) \subseteq S_\alpha,\ A \in \mathbb{M}_n$, then:
(a) $W(A^T) \subseteq S_\alpha$;
(b) $W(\overline{A}) \subseteq S_\alpha$;
(c) $W(A^{-1}) \subseteq S_\alpha$.

Property 2.5. Let $W(A) \subseteq S_\alpha,\ A \in \mathbb{M}_n$. Then there exists a square root R of A, i.e., $R^2 = A$, such that $W(R) \subseteq S_{\frac{\alpha}{2}}$.
If

$$R = S + iT$$

is the Toeplitz decomposition of R, then

$$0 \leq T \leq S \tan \frac{\alpha}{2}.$$

Especially, let $0 < \gamma < 1$, then $W(A^\gamma) \subseteq S_{\gamma\alpha}$.

Property 2.6. Let $W(A) \subseteq S_\alpha,\ A \in \mathbb{M}_n$ and Q be an arbitrary nonsingular matrix in $\mathbb{M}_n$. Then $W(Q^*AQ) \subseteq S_\alpha$. In particular, the numerical range of any matrix $\widehat{A}$, obtained by symmetric reordering of rows and columns in A, would be included in S_α.

Property 2.7. If $W(A_1), W(A_2) \subseteq S_\alpha,\ A_1, A_2 \in \mathbb{M}_n$ and λ_1, λ_2 are any positive number, then $W(\lambda_1 A_1 + \lambda_2 A_2) \subseteq S_\alpha$. In other words, S_α is a convex cone.

2.2. Entries, submatrices, numerical range, eigenvalues, and singular values.

Property 2.8. The diagonal entries of a matrix $A \in \mathbb{M}_n,\ W(A) \subseteq S_\alpha$, are complex numbers of the form $\beta + i\gamma$ and $\mid \gamma \mid \leq \beta \tan \alpha$, where $\beta > 0$ and $\gamma \in \mathbb{R}$. More generally, the numerical range of a principal submatrix of A of any order k, $1 \leq k \leq n$, would be included in S_α.

Property 2.9. Let $W(A) \subseteq S_\alpha,\ A \in \mathbb{M}_n$, and let A_k be the order k leading principal submatrix of A. Then the numerical range of Schur complement A/A_k is contained in S_α.

Property 2.10. Let $0 < \alpha < \frac{\pi}{2}$ and let $W(A) \subseteq S_\alpha,\ A \in \mathbb{M}_n$. Then we have

$$\max_{l,k} |a_{lk}| \leq \sec \alpha \max_j |a_{jj}|,$$

$$|a_{jj}^{(k)}| \leq \sec^2 \alpha \max_j |a_{jj}|,\ 1 \leq k \leq n-1.$$

Property 2.11. For any matrix $A \in \mathbb{M}_n$, $W(A) \subseteq S_\alpha$, its eigenvalues are complex numbers of the form $\beta + i\gamma$ and $|\gamma| \leq \beta \tan\alpha$, where $\beta > 0$ and $\gamma \in \mathbb{R}$.

Especially, from (1.1), we get

$$|\lambda(B^{-1}C)| \leq \tan\alpha.$$

Property 2.12. (Sectoral decomposition) If $A \in \mathbb{M}_n$, $W(A) \subseteq S_\alpha$, then there exits an invertible matrix X and a unitary diagonal matrix $Z = diag(e^{i\theta_1}, \ldots, e^{i\theta_n})$ with all $|\theta_j| \leq \alpha$ such that $A = XZX^*$. Moreover, such a matrix Z is unique up to permutation.

Property 2.13. Let $A \in \mathbb{M}_n$, $W(A) \subseteq S_\alpha$ and $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{22} is $q \times q$, $q \leq [n/2]$. Then

$$\sigma_j(A/A_{11}) \leq \sec^2(\alpha)\sigma_j(A_{22}),\ j = 1, \ldots, q,$$

$$\sigma_j(A) \leq \sec^2(\alpha)\lambda_j(\Re A),\ j = 1, \ldots, n,$$

$$\Re(A/A_{11}) \leq \sec^2(\alpha)\Re A_{22},$$

where $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$, $\sigma_j(\cdot)$ means the j-th largest singular value and $\lambda_j(\cdot)$ denotes the j-th largest eigenvalue.

Property 2.14. Let $A \in \mathbb{M}_n$, $W(A) \subseteq S_\alpha$. $s_1 \geq s_2 \geq \cdots \geq s_n$ are the singular values of A, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ are the eigenvalues of B and C in (1.1).

Then the n-tuple $\{s_1^2, s_2^2, \ldots, s_n^2\}$ is majorized by the n-tuple $\{\alpha_1^2 + \beta_1^2, \alpha_2^2 + \beta_2^2, \ldots, \alpha_n^2 + \beta_n^2\}$, i.e.,

$$\{s_j^2\} \prec \{\alpha_j^2 + \beta_j^2\}, \tag{2.1}$$

$$\{s_j^2\} \prec_w \{\alpha_j^2 \sec^2\alpha\}. \tag{2.2}$$

Equality (2.1) can also be rewritten as

$$\|A\|_F^2 = \|B\|_F^2 + \|C\|_F^2. \tag{2.3}$$

Note: There will be a conjecture, whether $\{\beta_j^2 \csc^2\alpha\} \prec_w \{s_j^2\}$ is true. The answer is no. We give a counter-example:

$$A = B + iC = \begin{pmatrix} 1 + 1.6294i & 1 + 1.0328i \\ 1 + 1.0328i & 2 + 1.8268i \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 1.6294 & 1.0328 \\ 1.0328 & 1.8268 \end{pmatrix},$$

then $\tan\alpha = 2.1184$ determined by [1, Lemma 2], $s_1^2(A) = 14.4364$ and $\csc^2(\alpha)\beta_1^2 = 9.3529 \Rightarrow \csc^2(\alpha)\beta_1^2 - s_1^2(A) = -5.0835$.

2.3. Operations on matrices with numerical ranges contained in S_α.

Property 2.15. Let $W(A) \subseteq S_\alpha,\ A \in \mathbb{M}_n$ and $S \in \mathbb{P}_n$. Then the eigenvalues of both AS and SA are complex numbers of the form $\beta + i\gamma$, $|\gamma| \leq \beta \tan\alpha$, where $\beta > 0$ and $\gamma \in \mathbb{R}$.

Property 2.16. Let $W(A) \subseteq S_\alpha,\ A \in \mathbb{M}_n$ and $S \in \mathbb{P}_n$. Then the numerical range of Hadmard product $A \circ S$ still belongs to S_α.

2.4. Miscellaneous.

Property 2.17. Let $W(A) \subseteq S_\alpha,\ A \in \mathbb{M}_n$ and

$$A = HU$$

be the polar decomposition of A, where U is the unitary factor. Then $W(U) \subseteq S_\alpha$

The linear fractional function

$$v = \frac{z-1}{z+1}$$

maps the right half complex plane $x > 0,\ y \in \mathbb{R}$ onto the unit disc.

When $x \geq 1$

$$|v| < 1,\ 1 > u = \mathcal{R}(v) \geq 0. \tag{2.4}$$

The corresponding property of matrices with numerical ranges contained in S_α is as follows.

Property 2.18. Let $W(A) \subseteq S_\alpha,\ A \in \mathbb{M}_n$, having the decomposition form (1.1) and let

$$V = (A-I)(A+I)^{-1}.$$

When $B \geq I$, then the field of values $W(V)$ belongs to half disc (2.4).

Remark 2.19. Let $W(A_1),\ W(A_2) \subseteq S_\alpha,\ A_1,\ A_2 \in \mathbb{M}_n$, $A_1 = B_1 + iC_1$ and $A_2 = B_2 + iC_2$. If

$$B_2 \geq B_1 \textit{ and } C_2 \geq C_1,$$

then either

$$|\det A_2| \geq |\det A_1| \tag{2.5}$$

or

$$|\det A_1| \geq |\det A_2|, \tag{2.6}$$

is possible.

We give two examples:

a) When A is accretive-dissipative, (2.5) is apparent.

b) We assume

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

$|\det A_1| = 13.0384 > |\det A_2| = 3.1623$, which testifies (2.6);

Especially, we let

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} -1-\epsilon & 0 \\ 0 & -2-\epsilon \end{pmatrix},$$

and could find an appropriate ϵ, which makes the numerical ranges of A_1 and A_2 in the same S_α.

3. **Proofs and comments**

1. Clearly,

$$x^* A^T x = [x^* A^T x]^T = x^T A \overline{x} = y^* A y, \tag{3.1}$$

where $y = \overline{x}$.

$$\overline{x^* \overline{A} x} = x^T A \overline{x} = z^* A z, \tag{3.2}$$

where $z = \overline{x}$.

So, from (3.1), (3.2), we get (a),(b) of Property 2.4.

And we could conclude $W(A^*) \subseteq S_\alpha$. Also, for any nonsingular $X \in \mathbb{M}_n$, $W(A) = W(XAX^*)$. Therefore, $W(A^{-1}) = W(AA^{-1}A^*) = W(A^*) \subseteq S_\alpha$.

2. From Lemma 2.1, we can see that, $W(R) \subseteq S_{\frac{\alpha}{2}}$, so

$$0 \le T \le S \tan \frac{\alpha}{2}.$$

And when $0 < \gamma < 1$, the conclusion $W(A^\gamma) \subseteq S_{\gamma\alpha}$ is trivial.

3. For

$$x^* Q^* A Q x = y^* A y \subseteq W(A),$$

where $x \in \mathbb{C}^n$, $x^* Q^* Q x = 1$, $y = Qx$. Property 2.6 is obvious.

4. Following (1.1), $A_1 = B_1 + iC_1, A_2 = B_2 + iC_2$,

$$\frac{|\, x^* C_1 x \,|}{x^* B_1 x} \le \tan\alpha, \frac{|\, x^* C_2 x \,|}{x^* B_2 x} \le \tan\alpha, \tag{3.3}$$

so we get

$$\frac{|\, \lambda_1 x^* C_1 x \,| + |\, \lambda_2 x^* C_2 x \,|}{\lambda_1 x^* B_1 x + \lambda_2 x^* B_2 x} \le \tan\alpha, \tag{3.4}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $x^* x = 1$, $x \in \mathbb{C}^n$. Obviously, $W(\lambda_1 A_1 + \lambda_2 A_2) \subseteq S_\alpha$.

5. Since $B = [b_{ij}] > 0$, C is Hermitian in (1.1), we have $b_{ii} > 0$, $c_{ii} \in \mathbb{R}$. Now, we take $x = e_i \in \mathbb{C}^n$, the ith element is 1, others are 0.

$$x^* A x = b_{ii} + i c_{ii} \subseteq S_\alpha,$$

so Property 2.8 is apparent.

6. Thanks to $(A/A_k)^{-1} = A^{-1}[k+1, k+2, \ldots, n]$, Property 2.9 follows from Property 2.4.

7. Property 2.10 has already been proved in [5, Proposition 3.1, 3.2].

8. According to spectral containment in Lemma 2.3, namely,

$$\delta(A) \subseteq W(A),$$

Property 2.11 is clear.

From (1.1) and Property 2.6, we know

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = I + iB^{-\frac{1}{2}}CB^{-\frac{1}{2}}.$$

Then

$$|\lambda(B^{-1}C)| \leq \tan\alpha.$$

9. Property 2.12 has already been proved in [21, Theorem 2.1]. And in Property 2.13, the first two inequalities are proved in [4, Theorem 1.1, 3.1] and the third inequality is proved in [19, Theorem 2]

10. A similar result is proved in the recent paper [2] for (2.1). Equality (2.3) is valid for an arbitrary matrix $A \in \mathbb{M}_n$, which easily follows from the relations

$$A^*A = B^2 + C^2 + i(BC - CB).$$

For (2.2),

$$\pm C \leq B\tan\alpha \Rightarrow |\beta_j| \leq \alpha_j \tan\alpha,$$

then

$$\sum_{j=1}^{k} \alpha_j^2 + \beta_j^2 \leq \sum_{j=1}^{k} \alpha_j^2 + \alpha_j^2 \tan^2\alpha = \sum_{j=1}^{k} \alpha_j^2 \sec^2\alpha,$$

so

$$\sum_{j=1}^{k} s_j^2 \leq \sum_{j=1}^{k} \alpha_j^2 \sec^2\alpha,\ k = 1, \cdots\cdots, n.$$

11. Turning to Property 2.15, we first note that matrices AS and SA always have the same eigenvalues and assume that S is positive definite. Denoted by Z the (unique) positive-definite square root of S. Then AS is similar to the matrix:

$$Z(AS)Z^{-1} = ZAZ = ZBZ + iZCZ,$$

whose numerical range obviously belongs to S_α. Now, the required assertion follows from Property 2.11. The case for singular S follows from a continuity argument.

12. We know that

$$x^*(A \circ S)x = tr((diag\overline{x})A(diag(x))S^T) = tr(K^{\frac{1}{2}}(diag\overline{x})A(diag(x))K^{\frac{1}{2}}),$$

where $K = S^T$, $x \in \mathbb{C}^n$, $x^*x = 1$.

From Property 2.6, 2.7, (3.3) and (3.4), Property 2.16 is apparent.

There is also a second proof:

Since $W(A) \subseteq S_\alpha$, $A \in \mathbb{M}_n$ and $S \in \mathbb{P}_n$, so

$$C \leq B\tan\alpha.$$

Then

$$C \circ S \leq B \circ S \tan \alpha.$$

13. To prove Property 2.17, we note that all eigenvalues of the matrix

$$U = H^{-1}A$$

belong to $W(A)$ (see Property 2.15). Since the field of values of a unitary (in fact, even a normal) matrix is the convex hull of its eigenvalues (see equation (3.3), (3.4)), we conclude that $W(U) \subseteq S_\alpha$.

14. Turning to Property 2.18, we can easily verify the relation

$$(A-I)^*(A-I) - (A+I)^*(A+I) = -4B.$$

This relation combined with the fact that the matrix on the right-hand side is negative-definite implies that

$$\| (A-I)y \| \leq \| (A+I)y \|$$

for any vectors y, so $\| V \|_2 \leq 1$.

Thus, $W(V)$ is a subset of the unit disc. Now, observe that

$$V = (A-I)(A+I)^{-1} = I - 2(A+I)^{-1},$$

$$(A+I)^{-1} = (B+I+iC)^{-1}.$$

By Lemma 2.2, the Hermitian part of $(A+I)^{-1}$, $H = [B+I+C(B+I)^{-1}C]^{-1}$.

And the Hermitian part of V,

$$H(V) = I - 2[B+I+C(B+I)^{-1}C]^{-1}.$$

So, when we assume $B > I$, $I > H(V) \geq 0$ is sharp, then the field of values $W(V)$ belongs to half disc (2.4).

Acknowledgement

The authors would like to thank the reviewer(s) very much for their careful readings, valuable suggestions and comments, which helped to improve this article more effectively. The work was supported by Science Foundation of China (11271247).

References

[1] Yu.M. Arlinskii and A.B. Popov, On sectorial matrices, *Linear Algebra Appl.* **370** (2003) 133–146.

[2] R. Bhatia and X.Z. Zhan, Compact operator whose real and imaginary parts are positive, *Proc. Amer. Math. Soc.* **129** (2001) 2277–2281.

[3] S. Drury, Principal powers of matrices with positive definite real part, *Linear Multilinear Algebra* **63** (2015) 296–301.

[4] S. Drury and M. Lin, Singular value inequalities for matrices with numerical ranges in a sector, *Oper. Matrices* **8** (2014) 1143–1148.

[5] S.W. Drury, Fischer determinantal inequalities and Higham's Conjecture, *Linear Algebra Appl.* **439** (2013) 3129–3133.

[6] X.H. Fu and Y. Liu, Rotfel'd inequality for partitioned matrices with numerical ranges in a sector, *Linear Multilinear Algebra* **64** (2016) 105–109.
[7] A. George and Kh. D. IKramov, On the properties of Accretive-Dissipative Matrices, *Math. Notes* **77** (2005) 767–776.
[8] A. George, Kh.D. Ikramov and A.B. Kucherov, On the growth factor in Gaussian elimination for generalized Higham matrices, *Numer. Linear Algebra Appl.* **9** (2002) 107–114.
[9] N.J. Higham, Factorizing complex sysmmetric matrices with positive real and imaginary parts, *Math. Comp.* **67** (1998) 1591–1599.
[10] R.A. Horn and C.R. Johnson, Topics in Matrix Analysis, Cambridge Univ. Press, Cambridge, 1994.
[11] R.A. Horn and C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, 1985; Russian translation; Mir, Moscow, 1990.
[12] Kh.D. Ikramov, Determinantal Inequalities for Accretive-Dissipative Matrices, *J. Math. Sci.* **121** (2004) 2458–2464
[13] Kh.D. Ikramov and V.N. Chugunov, Inequalities of Fisher and Hadamard types for accretive-dissipative matrices, *Dokl. Ross. Akad. Nauk* **384** (2002) 585–586
[14] Kh.D. Ikramov and A.B. Kucherov, Bounding the growth factor in Gaussian elimination for Buchley's class of complex symmetric matrices, *Numer. Linear Algebra Appl.* **7** (2000) 269–274.
[15] C.K. Li and N.S. Sze, Determinantal and eigenvalue inequalities for matrices with numerical ranges in a sector, *J. Math. Anal. Appl.* **410** (2014) 487–491.
[16] M. Lin, Reversed determinantal inequalities for accretive-dissipative matrices, *Math. Inequal. Appl.* **15** (2012) 955–958.
[17] M. Lin, Fischer type determinantal inequalities for accretive-dissipative matrices, *Linear Algebra Appl.* **438** (2013) 2808–2812.
[18] M. Lin, Extension of a result of Haynsworth and Hartfiel, *Arch. Math.* **104** (2015) 93–100.
[19] M. Lin, Some inequalities for sector matrices, *Oper. Matrices* **10** (2016), no. 4, 915–921.
[20] R. Mathias, Matrices with positive-definite Hermitian part: Inequalities and linear systems, *SIAM J. Matrix Anal. Appl.* **13** (1992) 640–654.
[21] F. Zhang, A matrix decomposition and its application, *Linear Multilinear Algebra* **63** (2015) 2033–2042.
[22] P.P. Zhang, Extension of Matic's results, *Linear Algebra Appl.* **486** (2015) 328–334.

(Dengpeng Zhang) Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail address: `zhangdengpeng@sina.cn`

(Lei Hou) Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail address: `houlei@shu.edu.cn`

(Lulu Ma) Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail address: `amandalulu918@163.com`