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ON NUCLEI OF SUP-Σ-ALGEBRAS

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Abstract. In this paper, algebraic investigations on sup-Σ-algebras are presented. A representation theorem for sup-Σ-algebras in terms of nuclei and quotients is obtained. Consequently, the relationship between the congruence lattice of a sup-Σ-algebra and the lattice of its nuclei is fully developed.

Keywords: Sup-Σ-algebra, nucleus, congruence, quotient.
MSC(2010): Primary: 06F99; Secondary: 08C05.

1. **Preliminaries**

Difference quantale-like structures, such as quantales, locales, quantale modules, quantale algebras, S-quantales, etc., have been studied in the recent decades (see [5–8, 10]), and they have been widely applied in algebra, logic, and computer science [4, 9]. The algebraic approach of kinds of quantale-like structures has also been investigated [3–5]. The destination of this work is to consider such approach in sup-Σ-algebras. We will generalize the results concerning quotients and also a well-known representation theorem of quantales into sup-Σ-algebras.

Throughout the paper, $\Sigma = \langle S, O\rangle$ will be a fixed but arbitrary signature, where S is a set of sorts, O is a family of operation symbols. A Σ-*algebra* A is an S-indexed family of sets A_s, $s \in S$, equipped with operations $o_A : A_{s_1} \times \ldots \times A_{s_n} \to A_s$ for each operation symbol o of rank $s_1 \ldots s_n \to s$, $n \in \mathbb{N}$.

A homomorphism $h : A \to B$ of Σ-algebras A and B is an S-indexed family of mappings $h_s : A_s \to B_s$, $s \in S$, such that

$$h_s(o_A(x_1, \ldots, x_n)) = o_B(h_{s_1}(x_1), \ldots, h_{s_n}(x_n)),$$

for any $o \in O$ with rank $s_1 \ldots s_n \to s$, $n \in \mathbb{N}$, $x_i \in A_{s_i}$, $i = 1, \ldots, n$.

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A Σ-algebra A is said to be ordered if for each sort $s \in S$, A_s is a poset, and $o_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$ preserves ordering for each $o \in O$ with rank $s_1 \ldots s_n \to s$, $n \in \mathbb{N}$.

Definition 1.1. Let A, B be ordered Σ-algebras, $h : A \to B$ an S-indexed family of mappings $h_s : A_s \to B_s$, $s \in S$. We say that h is a subhomomorphism if each h_s is monotone and

$$o_B(h_{s_1}(x_1), \ldots, h_{s_n}(x_n)) \leq h_s(o_A(x_1, \ldots, x_n)),$$

for $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \cdots s_n \to s$, $x_i \in A_{s_i}$, $i = 1, \ldots, n$.

Homomorphisms between ordered Σ-algebras are defined in the usual way.

By a sup-lattice-ordered Σ-algebra, simply, a sup-Σ-algebra, we mean an ordered Σ-algebra whose carriers are also sup-lattices and whose operations are sup-lattice homomorphisms in each variable separately. A homomorphism of sup-Σ-algebras is a homomorphism of ordered Σ-algebras whose components are sup-lattice homomorphisms.

2. Mappings and homomorphisms

In a natural way, every poset can be considered as a category, and monotone mappings between posets can be considered as functors. In such a category coproducts are joins.

Let A be a sup-Σ-algebra, $a_{s_1}, \ldots, a_{s_{i-1}}, a_{s_{i+1}}, \ldots, a_{s_n}$ some elements of A_{s_j}, for $n \in \mathbb{N}$, $j \in \{1, \ldots, n\} \setminus \{i\}$, $i = 1, \ldots, n$. Then for $o \in O$ with rank $s_1 \ldots s_n \to s$, the *elementary translation* $o_A(a_{s_1}, \ldots, a_{s_{i-1}}, \_, a_{s_{i+1}}, \ldots, a_{s_n}) : A_{s_i} \to A_s$ is a mapping, which we write as o_i with dependence a_{s_j} from A_{s_j}, for $j \in \{1, \ldots, n\} \setminus \{i\}$. Since an elementary translation o_i preserves joins, it has a right adjoint denoted by $o_i^* : A_s \to A_{s_i}$, satisfying

$$o_i(x_{s_i}) \leq a_s \iff x_{s_i} \leq o_i^*(a_s), \tag{2.1}$$

for all $x_{s_i} \in A_{s_i}, a_s \in A_s$, and also

$$x_{s_i} \leq o_i^*(o_i(x_{s_i})),\ \ o_i(o_i^*(a_s)) \leq a_s. \tag{2.2}$$

The following proposition can be easily obtained from (2.1) and (2.2).

Proposition 2.1. *Let A be a sup-Σ-algebra. Then for $n \in \mathbb{N}, o \in O$ with rank $s_1 \ldots s_n \to s$, o_i with dependence a_{s_j} from A_{s_j}, $j \in \{1, \ldots, n\} \setminus \{i\}$, $i \in \{1, \ldots, n\}$, the following conditions hold:*

(1) $\forall\, b \in A_s$, $o_i(o_i^*(b)) = b \iff (\exists\, c \in A_{s_i})$ *such that* $o_i(c) = b$,

(2) $\forall\, c \in A_{s_i}$, $o_i^*(o_i(c)) = c \iff (\exists\, b \in A_s)$ *such that* $o_i^*(b) = c$.

Proof. We only show the sufficiency of (1). Assume that $o_i(c) = b$, for some $c \in A_{s_i}$. Then $c \leq o_i^*(b)$ by (2.1), which implies that

$$b = o_i(c) \leq o_i(o_i^*(b)) \leq b,$$

by (2.2). □

Note that for any sup-Σ-algebra homomorphism $h : A \to B$ with an S-indexed family of mappings $h_s : A_s \to B_s,\ s \in S$, since h_s preserves arbitrary joins, it has a right adjoint, denoted by $h_s^* : B_s \to A_s$. Similar to (2.1) and (2.2), we have

$$h_s(a_s) \le b_s \iff a_s \le h_s^*(b_s), \tag{2.3}$$

$$a_s \le h_s^*(h_s(a_s)), \quad h_s(h_s^*(b_s)) \le b_s, \tag{2.4}$$

for every $s \in S$, $a_s \in A_s, b_s \in B_s$.

For a subhomomorphism $h : A \to B$ between sup-Σ-algebras, which is an S-indexed family of mappings $h_s : A_s \to B_s,\ s \in S$, and for $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \dots s_n \to s$, o_i with dependence a_{s_j} from A_{s_j}, $j \in \{1, \dots, n\} \setminus \{i\},\ i = 1, \dots, n$, the elementary translation $o_i(h)$ on B induced by h has the form

$$o_i(h) := o_B(h_{s_1}(a_{s_1}), \dots, h_{s_{i-1}}(a_{s_{i-1}}), \_, h_{s_{i+1}}(a_{s_{i+1}}), \dots, h_{s_n}(a_{s_n})) : B_{s_i} \to B_s.$$

Furthermore, if $h : A \to B$ is a homomorphism of sup-Σ-algebras, then $o_i(h)$ has right adjoint $o_i(h)^* : B_s \to B_{s_i}$. Therefore,

$$o_i(h)(x_{s_i}) \le b_s \iff x_{s_i} \le o_i(h)^*(b_s), \tag{2.5}$$

for every $x_{s_i} \in B_{s_i}$, $i \in \{1, \dots, n\}$, $b \in B_s$, and

$$x_{s_i} \le o_i(h)^*(o_i(h)(x_{s_i})),\ o_i(h)(o_i(h)^*(b_s)) \le b_s. \tag{2.6}$$

Proposition 2.2. *Let A and B be sup-Σ-algebras and let $h : A \to B$ be a homomorphism of sup-Σ-algebras with S-indexed family of mappings $h_s : A_s \to B_s,\ s \in S$. Then for all $b \in B_s$, $n \in \mathbb{N}$, $i \in \{1, \dots, n\}$, $o \in O$ with rank $s_1 \dots s_n \to s$, o_i with dependence a_{s_j} from A_{s_j}, $j \in \{1, \dots, n\} \setminus \{i\}$, we have*

$$h_{s_i}^*(o_i(h)^*(b)) = o_i^*(h_s^*(b)).$$

Proof. Since h is a homomorphism of sup-Σ-algebras, by (2.1), (2.3), one has that

$$\begin{aligned} h_{s_i}^*(o_i(h)^*(b)) \le o_i^*(h_s^*(b)) &\iff o_i(h_{s_i}^*(o_i(h)^*(b))) \le h_s^*(b) \\ &\iff h_s(o_i(h_{s_i}^*(o_i(h)^*(b)))) \le b \\ &\iff o_i(h)(h_{s_i}((h_{s_i}^*(o_i(h)^*(b))))) \le b \\ &\iff h_{s_i}(h_{s_i}^*(o_i(h)^*(b))) \le o_i(h)^*(b), \end{aligned}$$

and the last inequality natural holds by (2.4). Conversely,

$$\begin{aligned} o_i^*(h_s^*(b)) \le h_{s_i}^*(o_i(h)^*(b)) &\iff h_{s_i}(o_i^*(h_s^*(b))) \le o_i(h)^*(b) \\ &\iff o_i(h)h_{s_i}((o_i^*(h_s^*(b)))) \le b \\ &\iff h_s(o_i(o_i^*(h_s^*(b))) \le b \\ &\iff o_i(o_i^*(h_s^*(b))) \le h_s^*(b), \end{aligned}$$

and the last inequality holds by (2.2). $\square$

Proposition 2.3. *Let A and B be sup-Σ-algebras and let $h : A \to B$ be a homomorphism of sup-Σ-algebras with S-indexed family of mappings $h_s : A_s \to B_s$, $s \in S$. Then $h^* : B \to A$ is a sup-Σ-algebra subhomomorphism.*

Proof. Clearly, h_s^* is an order preserving mapping for each sort $s \in S$. For every $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \ldots s_n \to s$, $b_{s_i} \in B_{s_i}, i \in \{1, \ldots, n\}$, since

$$\begin{aligned} &o_A(h_{s_1}^*(b_{s_1}), \ldots, h_{s_n}^*(b_{s_n})) \leq h_s^*(o_B(b_{s_1}, \ldots, b_{s_n})) \\ &\qquad \iff h_s(o_A(h_{s_1}^*(b_{s_1}), \ldots, h_{s_n}^*(b_{s_n}))) \leq o_B(b_{s_1}, \ldots, b_{s_n}) \\ &\qquad \iff o_B(h_{s_1}(h_{s_1}^*(b_{s_1})), \ldots, h_{s_n}(h_{s_n}^*(b_{s_n}))) \leq o_B(b_{s_1}, \ldots, b_{s_n}), \end{aligned}$$

and the operation o_B preserves ordering, it follows that h^* is a subhomomorphism. □

3. Nuclei

Nuclei play an important role in the study of quotients of various quantale-like structures. In this section, we study properties of nuclei and prenuclei on sup-Σ-algebras.

Recall that an order preserving mapping j on a poset P is called a *closure operator* if it is increasing and idempotent. Let A be a sup-Σ-algebra, a *nucleus* j on A is an S-indexed family of closure operators which is subhomomorphic [4].

Similar to [2] Definition 2.3.3, we define a prenucleus on a sup-Σ-algebra.

Definition 3.1. We call a subhomomorphism j on a sup-Σ-algebra A a *prenucleus* if it is an S-indexed family of mappings j_s on A_s, where j_s is monotone, and increasing for each $s \in S$.

Clearly, a nucleus is an idempotent prenucleus. For a prenucleus j on a sup-Σ-algebra A, we write A_j as an S-indexed family of $A_{s_{j_s}}$, where $A_{s_{j_s}}$=$\{a \in A_s | j_s(a) = a\}$ for any $s \in S$. Denote by $v(j) : A \to A$ an S-indexed family of mappings $v(j)_s : A_s \to A_s$, $s \in S$, defined by

$$v(j)_s(a) := \bigwedge\{b \in A_{s_{j_s}} \mid a \leq b\},$$

for any $a \in A_s$.

Proposition 3.2. *Let A be a sup-Σ-algebra, j a prenucleus on A with the S-indexed family of mappings j_s, $s \in S$. Then the mapping $v(j) : A \to A$ is a nucleus on A.*

Proof. It is routine to check that for each sort $s \in S$, $v(j)_s$ is a closure operator, and $v(j)_s \circ j_s = v(j)_s$. For $n \in \mathbb{N}$, $s_i \in S$, $i \in \{1, \ldots, n\}$, define a mapping $f_{s_i} : A_{s_i} \to A_{s_{j_s}}$ by

$$f_{s_i}(x) := v(j)_s(o_i(x)),$$

for every $x \in A_{s_i}$, $o \in O$ with rank $s_1 \dots s_n \to s$, o_i with dependence a_{s_j} from A_{s_j}, $j \in \{1, \dots, n\} \setminus \{i\}$. Since j_{s_i} is increasing for each $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$, it follows that

$$\begin{aligned} f_{s_i}(j_{s_i}(x)) &= v(j)_s(o_i(j_{s_i}(x))) \\ &\leq v(j)_s(o_i(j)(j_{s_i}(x))) \\ &\leq v(j)_s \circ j_s(o_i(x)) \\ &= v(j)_s(o_i(x)) \\ &= f_{s_i}(x) \\ &\leq f_{s_i}(j_{s_i}(x)). \end{aligned}$$

We note that in above proof, due to j being a subhomomorphism, we get the first inequality. The next two inequalities follow from the fact that $v(j)_s$ being idempotent, and f_{s_i} being monotone, correspondingly. So $f_{s_i} \circ j_{s_i} = f_{s_i}$, and thus $f_{s_i} \circ v(j)_{s_i} = f_{s_i}$ by [2, Lemma 2.3.2]. Therefore, for every $b_{s_i} \in A_{s_i}$,

$$\begin{aligned} o_i(v(j)_{s_i}(b_{s_i})) &\leq v(j)_s(o_i(v(j)_{s_i}(b_{s_i}))) \\ &= f_{s_i}(v(j)_{s_i}(b_{s_i})) \\ &= f_{s_i}(b_{s_i}) \\ &= v(j)_s(o_i(b_{s_i})). \end{aligned}$$

Applying this fact n times and by the fact that $v(j)_s$ is idempotent, it follows that for any $n \in \mathbb{N}$, $a_{s_i} \in A_{s_i}, i \in \{1, \dots, n\}$,

$$o_A(v(j)_{s_1}(a_{s_1}), \dots, v(j)_{s_n}(a_{s_n})) \leq v(j)_s(o_A(a_{s_1}, \dots, a_{s_n})),$$

which indicates that $v(j)$ is a nucleus on A. □

Lemma 3.3. *If j is a nucleus on a sup-Σ-algebra A with an S-indexed family of closure operators j_s, $s \in S$, then for any $a_i \in A_s$, $i \in I$,*

$$j_s(\bigvee_{i \in I} j_s(a_i)) = j_s(\bigvee_{i \in I} a_i).$$

Proof. The inequality $j_s(\bigvee_{i \in I} a_i) \leq j_s(\bigvee_{i \in I} j_s(a_i))$ follows by j_s being increasing and monotone. Conversely, since $j_s(a_i) \leq j_s(\bigvee_{i \in I} a_i)$ for each $i \in I$, we have $\bigvee_{i \in I} j_s(a_i) \leq j_s(\bigvee_{i \in I} a_i)$. Hence $j_s(\bigvee_{i \in I} j_s(a_i)) \leq j_s(j_s(\bigvee_{i \in I} a_i)) = j_s(\bigvee_{i \in I} a_i)$ by the fact that j_s is idempotent. □

Lemma 3.4 ([4, Lemma 2.2.6])**.** *Let j be a nucleus on a sup-Σ-algebra A with an S-indexed family of closure operators j_s, $s \in S$. If for $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \dots s_n \to s$, $x_i, x_i^{'} \in A_{s_i}$, we have $x_i \leq x_i^{'} \leq j_{s_i}(x_i)$, for any $i \in \{1, \dots, n\}$, then*

$$j_s(o_A(x_1^{'}, \dots, x_n^{'})) = j_s(o_A(x_1, \dots, x_n)).$$

Lemma 3.5 ([4, Proposition 2.2.9])**.** *Let $h : A \to B$ be a homomorphism of sup-Σ-algebras with an S-indexed family of mappings $h_s, s \in S$, and let h^* denote the S-indexed family of right adjoints h_s^*. Then $j = h^* \circ h$ is a nucleus on A.*

Lemma 3.6. *Let A be a sup-Σ-algebra, j a prenucleus on A with an S-indexed family of closure operators j_s, $s \in S$. Then for $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \dots s_n \to s$, $i = 1, \dots, n$, o_i with dependence a_{s_j} from A_{s_j}, $j \in \{1, \dots, n\} \backslash \{i\}$, we have*

$$j_{s_i}(o_i^*(b)) \leq o_i^*(j_s(b))$$

for any $b \in A_s$.

Proof. To prove $j_{s_i}(o_i^*(b)) \leq o_i^*(j_s(b))$, it is sufficient to show that $o_i(j_{s_i}(o_i^*(b))) \leq j_s(b)$ by (2.1). The latter inequality holds because

$$\begin{aligned} o_i(j_{s_i}(o_i^*(b))) &= o_A(a_{s_1}, \dots, j_{s_i}(o_i^*(b)), \dots, a_{s_n}) \\ &\leq o_A(j_{s_1}(a_{s_1}), \dots, j_{s_i}(o_i^*(b)), \dots, j_{s_n}(a_{s_n})) \\ &\leq j_s(o_A(a_{s_1}, \dots, o_i^*(b), \dots, a_{s_n})) \\ &= j_s(o_i(o_i^*(b))) \\ &\leq j_s(b), \end{aligned}$$

for $n \in \mathbb{N}$, $a_{s_j} \in A_{s_j}$, $j \in \{1, \dots, n\} \setminus \{i\}$. □

4. A representation theorem

In this section, we will generalize a well-known representation theorem of quantales into sup-Σ-algebras. Let A be a sup-Σ-algebra, j a nucleus on A with an S-indexed family of closure operators j_s, $s \in S$. Then A_j, which is an S-indexed family of $A_{s_{j_s}}$, is a sup-Σ-algebra under the operation induced from A :

$$o_{A_j}(a_{s_1}, \dots, a_{s_n}) = j_s(o_A(a_{s_1}, \dots, a_{s_n})),$$

where $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \dots s_n \to s$, $a_{s_i} \in A_{s_{i_{j_{s_i}}}}$, $i \in \{1, \dots, n\}$, and by the fact that $A_{s_{j_s}}$ is a complete lattice under joins

$$\bigvee M = j_s(\bigvee M),$$

for every $M \subseteq A_{s_{j_s}}$, $s \in S$.

Proposition 4.1. *Let A be a sup-Σ-algebra, and let B be an S-indexed family of subsets B_s of A_s, $s \in S$. Then $B = A_j$ for some nucleus j on A if only if B_s is closed under meets and $o_i^*(b_s) \in B_{s_i}$ for $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \dots s_n \to s$, $b_s \in B_s$, $i \in \{1, \dots, n\}$, o_i with dependence a_{s_j} from A_{s_j}, $j \in \{1, \dots, n\} \setminus \{i\}$.*

Proof. Necessity. Suppose that j is a nucleus on A with the S-indexed family of closure operators j_s, $s \in S$, and $B = A_j$. Then $B_s = A_{s_{j_s}}$, for each $s \in S$. Thus B_s is closed under meets. For $n \in \mathbb{N}$, $i \in \{1, \dots, n\}$, $o \in O$ with rank

$s_1 \dots s_n \to s$, o_i with dependence a_{s_j} from A_{s_j}, $j \in \{1, \dots, n\} \setminus \{i\}$, $b_s \in B_s$, $s \in S$, since $o_i^*(b_s) \le j_{s_i}(o_i^*(b_s)) \le o_i^*(j_s(b_s)) = o_i^*(b_s)$ by Lemma 3.6, it follows that $o_i^*(b_s) \in B_{s_i}$.

Sufficiency. Assume that for every $s \in S$, B_s is closed under meets and $o_i^*(b_s) \in B_{s_i}$, for every $b_s \in B_s$, $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$. Define a mapping j_s on A_s, $s \in S$, by

$$j_s(a) := \bigwedge \{b \in B_s \mid a \le b\},$$

for any $a \in A_s$. It is routine to check that j_s is a closure operator and $B_s = A_{s_{j_s}}$. Let j be the S-indexed family of closure operators j_s, $s \in S$. We next show that j is a subhomomorphism on A.

To prove $o_A(j_{s_1}(a_{s_1}), \dots, j_{s_n}(a_{s_n})) \le j_s(o_A(a_{s_1}, \dots, a_{s_n}))$, for any $o \in O$ with rank $s_1 \dots s_n \to s$, $n \in \mathbb{N}$, it is sufficient to show that for each $x \in B_s$, with $o_A(a_{s_1}, \dots, a_{s_n}) \le x$, $o_A(j_{s_1}(a_{s_1}), \dots, j_{s_n}(a_{s_n})) \le x$.

Since

$$o_A(a_{s_1}, \dots, a_{s_n}) \le x \iff a_{s_1} \le o_1^*(x) \in B_{s_1} = A_{s_1 j_{s_1}},$$

it follows that $j_{s_1}(a_{s_1}) \le o_1^*(x)$, and then

$$j_{s_1}(a_{s_1}) \le o_1^*(x) \iff o_A(j_{s_1}(a_{s_1}), a_{s_2}, \dots, a_{s_n}) \le x \iff a_{s_2} \le o_2^*(x).$$

Similarly, the inequality $j_{s_2}(a_{s_2}) \le o_2^*(x)$ turns out that

$$o_A(j_{s_1}(a_{s_1}), j_{s_2}(a_{s_2}), a_{s_3}, \dots, a_{s_n}) \le x.$$

Finally, we achieve that

$$o_A(j_{s_1}(a_{s_1}), j_{s_2}(a_{s_2}), \dots, j_{s_n}(a_{s_n})) \le x,$$

as required. □

For a sup-Σ-algebra A, let $\mathcal{P}(A_s)$ be the powerset of A_s, $s \in S$, and $\mathcal{P}(A)$ the S-indexed family of powersets $\mathcal{P}(A_s)$. Then $\mathcal{P}(A_s)$ is a complete lattice under the inclusion as a partial order. Furthermore, $\mathcal{P}(A)$ with the pointwise operations induced from A :

$$o_{\mathcal{P}(A)}(X_{s_1}, \dots, X_{s_n}) = \{o_A(x_{s_1}, \dots, x_{s_n}) \mid x_{s_1} \in X_{s_1}, \dots, x_{s_n} \in X_{s_n}\},$$

for any $o \in O$, with rank $s_1 \dots s_n \to s$, $n \in \mathbb{N}$, becomes a sup-Σ-algebra.

The following result gives a representation for sup-Σ-algebras in terms of nuclei and quotients.

Theorem 4.2 (Representation Theorem). *Let A be a sup-Σ-algebra. Then there is a nucleus $j : \mathcal{P}(A) \to \mathcal{P}(A)$, such that $A \cong \mathcal{P}(A)_j$.*

Proof. Let $j : \mathcal{P}(A) \to \mathcal{P}(A)$ be an S-indexed family of mapping j_s on $\mathcal{P}(A_s)$, $s \in S$, where j_s is defined by

$$j_s(X_s) := (\bigvee X_s){\downarrow}, \ \ \forall X_s \in \mathcal{P}(A_s).$$

Clearly, j_s is a closure operator. Next we show that j is subhomomorphic, that is

$$o_{\mathcal{P}(A)}(j_{s_1}(X_{s_1}), \ldots, j_{s_n}(X_{s_n})) \subseteq j_s(o_{\mathcal{P}(A)}(X_{s_1}, \ldots, X_{s_n})),$$

for each $X_{s_i} \in \mathcal{P}(A_{s_i})$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$, and $o \in O$ with rank $s_1 \ldots s_n \to s$.

Take $o_A(d_{s_1}, \ldots, d_{s_n}) \in o_{\mathcal{P}(A)}(j_{s_1}(X_{s_1}), \ldots, j_{s_n}(X_{s_n}))$, where $d_{s_i} \in j_{s_i}(X_{s_i}) = (\bigvee X_{s_i})\downarrow$, $X_{s_i} \subseteq A_{s_i}$, $s_i \in S$, $i \in \{1, \ldots, n\}, n \in \mathbb{N}$. Since

$$\begin{aligned} o_A(d_{s_1}, \ldots, d_{s_n}) &\leq o_A(\bigvee X_{s_1}, \ldots, \bigvee X_{s_n}) \\ &= \bigvee\{o_A(a_{s_1}, \ldots, a_{s_n}) \mid a_{s_i} \in X_{s_i}, i = 1, \ldots, n\} \\ &= \bigvee o_{\mathcal{P}(A)}(X_{s_1}, \ldots, X_{s_n}), \end{aligned}$$

and

$$\begin{aligned} j_s(o_{\mathcal{P}(A)}(X_{s_1}, \ldots, X_{s_n})) &= (\bigvee o_{\mathcal{P}(A)}(X_{s_1}, \ldots, X_{s_n}))\downarrow \\ &= \{x \in A_s \mid x \leq \bigvee o_{\mathcal{P}(A)}(X_{s_1}, \ldots, X_{s_n})\}, \end{aligned}$$

we obtain that $o_{\mathcal{P}(A)}(j_{s_1}(X_{s_1}), \ldots, j_{s_n}(X_{s_n})) \subseteq j_s(o_{\mathcal{P}(A)}(X_{s_1}, \ldots, X_{s_n}))$, as needed. Note that, for any $s \in S$, $D \subseteq A_s$, $j_s(D) = D$ if only if $D = d_s\downarrow$ for some $d_s \in A_s$. So

$$\mathcal{P}(A_s)_{j_s} = \{D \in \mathcal{P}(A_s) \mid j_s(D) = D\} = \{D \subseteq A_s \mid D = d_s\downarrow, \text{ for some } d_s \in A_s\},$$

for each $s \in S$. Let $\psi : A \to \mathcal{P}(A)_j$ be an S-indexed family of $\psi_s : A_s \to \mathcal{P}(A_s)_{j_s}$, $s \in S$, which defined by

$$\psi_s(a) := a\downarrow, \forall a \in A_s.$$

We need to prove that ψ is a bijective sup-Σ-algebra homomorphism. Clearly, for any $s \in S$, ψ_s is a bijective join-preserving mapping. Moreover, for any $o \in O$, with rank $s_1 \ldots s_n \to s$, $n \in \mathbb{N}$, one has that

$$\begin{aligned} o_{\mathcal{P}(A)_j}(\psi_{s_1}(a_{s_1}), \ldots, \psi_{s_n}(a_{s_n}))) &= o_{\mathcal{P}(A)_j}(a_{s_1}\downarrow, \ldots, a_{s_n}\downarrow) \\ &= j_s(o_{\mathcal{P}(A)}(a_{s_1}\downarrow, \ldots, a_{s_n}\downarrow)) \\ &= (\bigvee o_{\mathcal{P}(A)}(a_{s_1}\downarrow, \ldots, a_{s_n}\downarrow))\downarrow \\ &= \{x \in A_s \mid x \leq \bigvee o_{\mathcal{P}(A)}(a_{s_1}\downarrow, \ldots, a_{s_n}\downarrow)\}, \end{aligned}$$

and $\psi_s(o_A(a_{s_1}, \ldots, a_{s_n})) = (o_A(a_{s_1}, \ldots, a_{s_n}))\downarrow$. However, straightforward checking shows that $o_A(a_{s_1}, \ldots, a_{s_n})$ is the sup of $o_{\mathcal{P}(A)}(a_{s_1}\downarrow, \ldots, a_{s_n}\downarrow)$, i.e.,

$$o_A(a_{s_1}, \ldots, a_{s_n}) = \bigvee o_{\mathcal{P}(A)}(a_{s_1}\downarrow, \ldots, a_{s_n}\downarrow).$$

Consequently, we get that $\psi_s(o_A(a_{s_1}, \ldots, a_{s_n})) = o_{\mathcal{P}(A)_j}(\psi_{s_1}(a_{s_1}), \ldots, \psi_{s_n}(a_{s_n}))$. □

5. **Quotients of sup-Σ-algebras**

We write the set of all nuclei on a sup-Σ-algebra A by Nuc(A). Define a relation on Nuc(A) by

$$j \leq j^{'} \iff j_s \leq j^{'}_s, \forall s \in S,$$

where $j\,(j^{'})$ is an S-indexed family of mappings of $j_s\,(j^{'}_s$, respectively), and $j_s \leq j^{'}_s$ is under pointwise ordering.

Lemma 5.1 ([4, Proposition 2.2.8])**.** *Let A be a sup-Σ-algebra, j, $j^{'} \in \mathrm{Nuc}(A)$. Then the following conditions hold.*

(1) Nuc(A) *is a complete lattice,*

(2) $j \leq j^{'}$ *if and only if* $A_{s_{j^{'}_s}} \subseteq A_{s_{j_s}}$*, for all sorts* $s \in S$*,*

(3) $j \leq j^{'}$ *if and only if for every sort* $s \in S$*,* $x, y \in A_s$*,* $j_s(x) = j_s(y)$ *implies that* $j^{'}_s(x) = j^{'}_s(y)$*.*

The final section is devoted to find out the relation between Nuc(A) and Con(A) for a sup-Σ-algebra.

Recall that a congruence ρ on a sup-Σ-algebra A is an S-indexed family of equivalence relations ρ_s on A_s, $s \in S$, which are compatible with arbitrary joins and operations. The set of all congruences on A is denoted by Con(A).

Let ρ be a congruence on a sup-Σ-algebra A, we write A/ρ as the S-indexed family of A_s/ρ_s, $s \in S$, where $A_s/\rho_s = \{[a]_{\rho_s} \mid a \in A_s\}$. Define joins on the quotient set A_s/ρ_s, $s \in S$, by

$$\bigvee_{k \in I} [a_k]_{\rho_s} := [\bigvee_{k \in I} a_k]_{\rho_s}.$$

Clearly, A_s/ρ_s is a complete lattice. Define the operation on A/ρ by

$$o_{A/\rho}([a_1]_{\rho_{s_1}}, \ldots, [a_n]_{\rho_{s_n}}) := [o_A(a_1, \ldots, a_n)]_{\rho_s},$$

for $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \ldots s_n \to s$, $a_i \in A_{s_i}$, $i \in \{1, \ldots, n\}$. Then the operation is well defined. Moreover, for arbitrary $x_k \in A_{s_k}, k \in I$, and

$a_j \in A_{s_j}$, $s_j \in S$, $j \in \{1, \ldots, n\} \setminus \{i\}$, the equalities

$$\begin{aligned} & o_{A/\rho}([a_1]_{\rho_{s_1}}, \ldots, \bigvee_{k \in I} [x_k]_{\rho_{s_i}}, \ldots, [a_n]_{\rho_{s_n}}) \\ &= o_{A/\rho}([a_1]_{\rho_{s_1}}, \ldots, [\bigvee_{k \in I} x_k]_{\rho_{s_i}}, \ldots, [a_n]_{\rho_{s_n}}) \\ &= [o_A(a_1, \ldots, \bigvee_{k \in I} x_k, \ldots, a_n)]_{\rho_s} \\ &= [\bigvee_{k \in I} o_A(a_1, \ldots, x_k, \ldots, a_n)]_{\rho_s} \\ &= \bigvee_{k \in I} [o_A(a_1, \ldots, x_k, \ldots, a_n)]_{\rho_s}, \end{aligned}$$

indicate that A/ρ is a sup-Σ-algebra.

Let the surjective mapping $\pi : A \to A/\rho$ be an S-indexed family of mappings $\pi_s : A_s \to A_s/\rho_s$, $s \in S$, which defined by

$$\pi_s(a) := [a]_{\rho_s}, \ \forall a \in A_s.$$

It is easy to see that π is a homomorphism of sup-Σ-algebras, and its right adjoint is denoted by π^*.

Lemma 5.2. *Let A be a sup-Σ-algebra, π is the mapping mentioned above. Then $\pi^*\pi$ is a nucleus on A.*

Proof. It is a consequence of Lemma 3.5. □

Lemma 5.3 ([4, Proposition 2.2.11])**.** *Let $h : A \to B$ be a homomorphism of sup-Σ-algebras. For any sort $s \in S$, $x, y \in A_s$, one has that*

$$h_s(x) = h_s(y) \Longleftrightarrow h_s^* h_s(x) = h_s^* h_s(y).$$

As usual, for any mapping $j : A \to B$, which is an S-indexed family of mappings $j_s, s \in S$, between two sup-Σ-algebras, we denote ker j as the S-indexed family of equivalence relations kerj_s, $s \in S$, where ker$j_s = \{(a, b) \in A_s \times A_s \mid j_s(a) = j_s(b)\}$.

Lemma 5.4. *Let A be a sup-Σ-algebra, j a nucleus on A with the S-indexed family of closure operators j_s, $s \in S$. Then* ker j *is a congruence on A.*

Proof. For each $(a_k, a_k^{'}) \in$ kerj_s, $k \in I$, since $j_s(a_k) = j_s(a_k^{'})$, and $j_s(\bigvee_{k \in I} j_s(a_k)) = j_s(\bigvee_{k \in I} j_s(a_k^{'}))$, it follows that ker$j_s$ is compatible with joins by Lemma 3.3.

For $n \in \mathbb{N}$, given $(a_{s_i}, b_{s_i}) \in \ker j_{s_i}$, $i \in \{1, \ldots, n\}$, then for any $o \in O$ with rank $s_1 \ldots s_n \to s$, by Lemma 3.4, one has that

$$\begin{aligned} j_s(o_A(a_{s_1}, \ldots, a_{s_n})) &= j_s(o_A(j_{s_1}(a_{s_1}), \ldots, j_{s_1}(a_{s_n}))) \\ &= j_s(o_A(j_{s_1}(b_{s_1}), \ldots, j_{s_1}(b_{s_n}))) \\ &= j_s(o_A(b_{s_1}, \ldots, b_{s_n})). \end{aligned}$$

So $\ker j_s$ is compatible with operations. □

Now, we are ready to present the definite relationship between Nuc(A) and Con(A).

Theorem 5.5. *Let A be a sup-Σ-algebra. Then there is an isomorphism ψ : Nuc(A) $\to$ Con(A) of posets. Moreover, for each $j \in$ Nuc(A) with the S-indexed family of closure operators j_s, $s \in S$, $A_j \cong A/\psi(j)$ as sup-Σ-algebras.*

Proof. Define a mapping $\psi : \mathrm{Nuc}(A) \to \mathrm{Con}(A)$ by

$$\psi(j) := \ker j,$$

for each $j \in \mathrm{Nuc}(A)$.

By Lemma 5.4, $\ker j$ is a congruence on A. By Lemma 5.1(3), ψ is an order-embedding. Next we show that ψ is surjective. If $\rho \in \mathrm{Con}(A)$, then we consider the natural surjection $\pi : A \to A/\rho$. By Lemma 5.2, $\pi^*\pi$ is a nucleus on A, and by Lemma 5.3, $\psi(\pi^*\pi) = \ker(\pi^*\pi) = \ker\pi = \rho$, so ψ is surjective.

Next, for a nucleus j on A, let $f : A/\ker j \to A_j$ be an S-indexed family of mappings $f_s : A_s/\ker j_s \to A_{s_{j_s}}$, $s \in S$, which defined by

$$f_s([a_s]_{\ker j_s}) := j_s(a_s), \ \forall a_s \in A_s,$$

and g: $A_j \to A/\ker j$ an S-indexed family of mappings $g_s : A_{s_{j_s}} \to A_s/\ker j_s$, $s \in S$, which defined by

$$g_s(a_s) := [a_s]_{\ker j_s}, \ \forall a_s \in A_{s_{j_s}}.$$

Then f and g are well-defined. For $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \ldots s_n \to s$, $a_{s_i} \in A_{s_i}$, $i \in \{1, \ldots, n\}$, since

$$\begin{aligned} f_s(o_{A/\ker j}([a_{s_1}]_{\ker j_{s_1}}, \ldots, [a_{s_n}]_{\ker j_{s_n}})) &= f_s([o_A(a_{s_1}, \ldots, a_{s_n})]_{\ker j_s}) \\ &= j_s(o_A(a_{s_1}, \ldots, a_{s_n})) \\ &= j_s(o_A(j_{s_1}(a_{s_1}), \ldots, j_{s_n}(a_{s_n}))) \\ &= o_{A_j}(j_{s_1}(a_{s_1}), \ldots, j_{s_n}(a_{s_n})) \\ &= o_{A_j}(f_{s_1}([a_{s_1}]_{\ker j_{s_1}}), \ldots, f_{s_n}([a_{s_n}]_{\ker j_{s_n}})), \end{aligned}$$

and for $a_k \in A_s$,

$$f_s(\bigvee_{k\in I}[a_k]_{\mathrm{ker}j_s}) = f_s([\bigvee_{k\in I} a_k]_{\mathrm{ker}j_s}) = j_s(\bigvee_{k\in I} a_k)$$
$$= j_s(\bigvee_{k\in I} j_s(a_k)) = \bigvee_{k\in I} j_s(a_k) = \bigvee_{k\in I} f_s([a_k]_{\mathrm{ker}j_s}),$$

one can conclude that f is a homomorphism of sup-Σ-algebras.

Similarly, for $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \ldots s_n \to s$, $a_{s_i} \in A_{s_i}$, $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned} o_{A/\mathrm{ker}j}(g_{s_1}(a_{s_1}), \ldots, g_{s_n}(a_{s_n})) &= o_{A/\mathrm{ker}j}([a_{s_1}]_{\mathrm{ker}j_{s_1}}, \ldots, [a_{s_n}]_{\mathrm{ker}j_{s_n}}) \\ &= [o_A(a_{s_1}, \ldots, a_{s_n})]_{\mathrm{ker}j_s} \\ &= [j_s(o_A(a_{s_1}, \ldots, a_{s_n}))]_{\mathrm{ker}j_s} \\ &= [o_{A_j}(a_{s_1}, \ldots, a_{s_n})]_{\mathrm{ker}j_s} \\ &= g_s(o_{A_j}(a_{s_1}, \ldots, a_{s_n})), \end{aligned}$$

and for $a_k \in A_s$,

$$g_s(\bigvee_{k\in I} a_k) = g_s(j_s(\bigvee_{k\in I} a_k)) = [j_s(\bigvee_{k\in I} a_k)]_{\mathrm{ker}j_s}$$
$$= [\bigvee_{k\in I} a_k]_{\mathrm{ker}j_s} = \bigvee_{k\in I}[a_k]_{\mathrm{ker}j_s} = \bigvee_{k\in I} g_s(a_k),$$

which imply that g is a sup-Σ-algebra homomorphism.

Finally, for every $a_s \in A_{s_{j_s}}$, $s \in S$,

$$f_s g_s(a_s) = f_s([a_s]_{\mathrm{ker}j_s}) = j_s(a_s) = a_s,$$

and

$$g_s f_s([a_s]_{\mathrm{ker}j_s}) = g_s(f_s(a_s)) = [j_s(a_s)]_{\mathrm{ker}j_s} = [a_s]_{\mathrm{ker}j_s},$$

for any $[a_s]_{\mathrm{ker}j_s} \in A_s/\mathrm{ker}j_s$. We obtain that $A_j \cong A/\mathrm{ker}(j)$ as needed. □

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