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NSE CHARACTERIZATION OF SOME LINEAR GROUPS

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Abstract. For a finite group G, let $\mathrm{nse}(G) = \{m_k \mid k \in \pi_e(G)\}$, where m_k is the number of elements of order k in G and $\pi_e(G)$ is the set of element orders of G. In this paper, we prove that $G \cong L_m(2)$ if and only if $p \mid |G|$ and $\mathrm{nse}(G) = \mathrm{nse}(L_m(2))$, where $m \in \{n, n+1\}$ and $2^n - 1 = p$ is a prime number.
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MSC(2010): Primary: 20D06; Secondary: 20D15.

1. Introduction

For a finite group G and a positive integer t, let $M_t(G)$ be the set of all elements of G satisfying the equation $x^t = 1$, that is $M_t(G) = \{g \in G \mid g^t = 1\}$. The groups G_1 and G_2 are called of the same order type if and only if $|M_t(G_1)| = |M_t(G_2)|$, $t = 1, 2, \ldots$. In 1987, J.G. Thompson posed a question as follows:

Thompson's Problem. Suppose that G_1 and G_2 are of the same order type. If G_1 is solvable, is it true that G_2 is necessarily solvable?

For a natural number n, let $\pi(n)$ be the set of prime divisors of n. We denote by $\pi(G)$ the set of prime divisors of $|G|$ and by $\pi_e(G)$ the set of element orders of G. Let $\mathrm{nse}(G) = \{m_k \mid k \in \pi_e(G)\}$, where m_k is the number of elements of order k in G. It is well known that if G_1 and G_2 are of the same order type, then $|G_1| = |G_2|$ and $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$. So it is natural to investigate Thompson's problem by $|G|$ and $\mathrm{nse}(G)$. The following example, due to Thompson, shows that there are finite groups which are not characterizable by $\mathrm{nse}(G)$ and $|G|$. For the groups $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ (which are maximal subgroups of the Mathieu group of degree 23), we have $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$ and $|G_1| = |G_2|$ but $G_1 \ncong G_2$.

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The influence of $\mathrm{nse}(G)$ on the structure of finite groups was studied by some authors (see [9,10,15]). We say that the group G is characterizable by nse in the class $\mathfrak{A}$ of groups, if every group $H \in \mathfrak{A}$ with $\mathrm{nse}(G) = \mathrm{nse}(H)$ is isomorphic to G. Recently, Shao and Jiang [14] showed that the group $L_2(p)$, where p is prime, is characterizable by nse in the class of finite groups whom orders are divisible by p. They also showed [13] that the group $L_2(2^a)$, where either $2^a - 1$ or $2^a + 1$ is a prime, is characterizable by its order and nse in the class of finite groups. Aslo in [1] and [2], the characterization of some alternating groups, projective Symplectic groups and projective special orthogonal groups have been studied. It is known that $L_3(2) \cong L_2(7)$ and $L_4(2) \cong A_8$ (see [4]). Authors in [9,10] showed that $L_2(7)$ and A_8 are characterizable by nse. In this paper, we focus on the group $L_m(2)$, where $2^n - 1 = p \geq 31$ is a prime number and $m \in \{n, n+1\}$. In fact, we are going to prove the following theorem:

Theorem 1.1 (Main Theorem)**.** *Let G be a finite group, $2^n - 1 = p \geq 31$ be a prime number and $m \in \{n, n+1\}$. Then $G \cong L_m(2)$ if and only if $p \mid |G|$ and $\mathrm{nse}(G) = \mathrm{nse}(L_m(2))$.*

To prove this theorem, we use the classification of finite simple groups with disconnected prime graph. The prime graph $GK(G)$ of G is the graph with the vertex set $\pi(G)$, where two distinct primes r and s are joined by an edge if G contains an element of order rs. Let $t(G)$ denote the number of connected components of G and let $\pi_1(G), \pi_2(G), \ldots, \pi_{t(G)}(G)$ be the sets of vertices of the connected components of $GK(G)$. We will use the notation π_i instead of $\pi_i(G)$, when it causes no ambiguity. If $2 \in \pi(G)$, then we always assume that $2 \in \pi_1(G)$. Also, $|G|$ can be expressed as a product of $OC_1,\ OC_2,\ \cdots, OC_{t(G)}$, where OC_i is a positive integer with $\pi(OC_i) = \pi_i$. The OC_i's are called the order components of G. In particular, an odd number OC_i is called an odd order component of G. The sets of order components of finite simple groups with disconnected prime graph can be obtained using [11] and [17]. For a natural number n and a prime number a, we use $|n|_a = a^e$, when $a^e \| n$, i.e., $a^e \mid n$ but $a^{e+1} \nmid n$. All further unexplained notations are standard and can be found in [4], for instance.

2. Preliminaries

In this section, we present some useful lemmas which will be used in the proof of the main theorem.

Lemma 2.1 ([6])**.** *Let G be a finite group and let t be a positive integer dividing $|G|$. If $M_t(G) = \{g \in G \mid g^t = 1\}$, then $t \mid |M_t(G)|$.*

Lemma 2.2 ([3])**.** *Let G be a Frobenius group of even order with kernel K and complement H. Then $t(G) = 2$, the prime graph components of G are $\pi(H)$ and $\pi(K)$, and the following assertions hold:*

(1) *K is nilpotent;*
(2) $|K| \equiv 1 \pmod{|H|}$.

The group G is named a 2-Frobenius group, when there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.3 ([17, Theorem 2])**.** *Let G be a 2-Frobenius group of even order, which has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then*

(i) *$t(G) = 2$ and, $\pi_1 = \pi(H) \cup \pi(G/K)$ and $\pi_2 = \pi(K/H)$.*
(ii) *G/K and K/H are cyclic, $(|G/K|, |K/H|) = 1$ and $|G/K|$ divides $|\mathrm{Aut}(K/H)|$.*
(iii) *H is a nilpotent group and G is a solvable group.*

Lemma 2.4 ([17])**.** *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(i) *G is a Frobenius or 2-Frobenius group;*
(ii) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a non abelian simple group, H is a nilpotent group and $|G/H| \mid |\mathrm{Aut}(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H.*

For a natural number k and a prime p, if $p^m \| k$, then we say that p^m is a p-part of k and denote it by k_p. The following result of Zsigmondy is used to prove the main theorem.

Lemma 2.5 ([18, Zsigmondy's Theorem])**.** *Let n and a be integers greater than 1. Then there exists a prime divisor p of $a^n - 1$ such that p does not divide $a^i - 1$ for all i, $1 \leq i \leq n-1$, except in the following cases:*

(i) *$n = 2$ and $a = 2^k - 1$, where $k \geq 2$.*
(ii) *$n = 6$ and $a = 2$.*

The prime p in Lemma 2.5 is called a Zsigmondy prime of $a^n - 1$.

Lemma 2.6 ([5])**.** *Let p and q be prime and $m, n > 1$.*

(i) *With the exceptions of the relations $(239)^2 - 2(13)^4 = 1$ and $(3)^5 - 2(11)^2 = 1$ every solution of the equation $p^m - 2q^n = 1$ has exponents $m = n = 2$.*
(ii) *The only solution of the equation $p^m - q^n = 1$ is $3^2 - 2^3 = 1$.*

Remark 2.7. Let H be a finite group. Clearly, for $n \in \pi_e(H)$, $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in H and $\phi(n)$ the Euler totient function of n. By Lemma 2.1 and the discussion above we have:

$$\begin{cases} \phi(n) \mid m_n \\ n \mid \Sigma_{d|n} m_d. \end{cases} \tag{2.1}$$

If $n > 2$, then $\phi(n)$ is even and hence, m_n is even. If $2 \in \pi(H)$, then (2.1) shows that $2 \mid 1 + m_2$ and hence, m_2 is odd. This implies that $a \in \text{nse}(H)$ is odd if and only if $2 \in \pi(H)$ and $m_2(H) = a$.

3. Main results

Suppose that $cl_G(x)$ denotes the conjugacy class in G containing x. Throughout this section, let $n \geq 5$, $2^n - 1 = p$ be a prime, $m \in \{n, n+1\}$, G a finite group such that $p \mid |G|$ and $\text{nse}(G) = \text{nse}(L_m(2))$.

Lemma 3.1. *For every $1 \neq x \in L_m(2)$, either $p \mid |cl_{L_m(2)}(x)|$ or x has order p and $|cl_{L_m(2)}(x)| = \frac{|L_m(2)|}{p}$.*

Proof. Let $p \nmid |cl_{L_m(2)}(x)|$. Then [8] implies that there exists a divisor r of n such that $|cl_{L_m(2)}(x)| = \frac{|L_m(2)|}{|L_{n/r}(2^r)|}$ and $r \neq 1$. Since $2^n - 1$ is prime, n is prime and hence, $r = n$. This forces $|cl_{L_m(2)}(x)| = \frac{|L_m(2)|}{p}$. Thus $|C_{L_m(2)}(x)| = p$ and hence, x has order p, as claimed. □

Let $r \in \pi(G)$. We denote by $S_r(G)$, $\text{Syl}_r(G)$ and $n_r(G)$, a Sylow r-subgroup of G, the set of Sylow r-subgroups of G and $|\text{Syl}_r(G)|$, respectively. The following lemma is well-known and it can for example be extracted from [7]:

Lemma 3.2. $n_p(L_m(2)) = \frac{|L_m(2)|}{np}$.

Corollary 3.3. *For $u \in \pi_e(L_m(2))$, either $p \mid m_u(L_m(2))$ or $u = p$ and $m_p(L_m(2)) = \frac{(p-1)|L_m(2)|}{np}$.*

Proof. Since $|S_p(L_m(2))| = p$, we deduce that $S_p(L_m(2))$ is cyclic. Thus it is obvious that $m_p(L_m(2)) = \phi(p).n_p(L_m(2))$ and Lemma 3.2 shows that $m_p(L_m(2)) = \frac{(p-1)|L_m(2)|}{np}$.

On the other hand, $m_u(L_m(2)) = \sum_{\text{for some } y \in L_m(2) \text{ with } O(y)=u} |cl_{L_m(2)}(y)|$, so Lemma 3.1 completes the proof. □

Corollary 3.4. *For every $u \in \pi_e(G)$, $p \nmid m_u(G)$ if and only if $m_u(G) = m_p(L_m(2))$. In particular, $m_p(G) = m_p(L_m(2))$.*

Proof. Since $m_u(G) \in \text{nse}(L_m(2))$, Corollary 3.3 completes the proof. Also, by (2.1), $p \mid 1 + m_p(G)$, so $p \nmid m_p(G)$. Thus $m_p(G) = m_p(L_m(2))$, as claimed. □

Lemma 3.5 ([1]). *Let t be the number of cyclic subgroups of order n in G, namely $H_1, \ldots, H_t$ and let for $1 \leq i \leq t$, β_i be the number of cyclic subgroups of $C_G(H_i)$ of order r, where $\gcd(r, n) = 1$. If $\beta = \min\{\beta_i : 1 \leq i \leq t\}$, then $m_n \phi(r) \beta \leq m_{nr}(G)$.*

Lemma 3.6. *Let $n \neq 7$ and s be a Zsigmondy prime of $2^{n-1} - 1$.*

(i) *If $t = 2s$, then $t \in \pi_e(L_{n+1}(2))$ and $m_t(L_{n+1}(2)) = \frac{\phi(t)|L_{n+1}(2)|}{2(n-1)(2^{n-1}-1)}$.*

(ii) *If* $t = s$, *then* $t \in \pi_e(L_n(2))$ *and* $m_t(L_n(2)) = \frac{\phi(t)|L_n(2)|}{(n-1)(2^{n-1}-1)}$.

Proof. It is known that $2s \in \pi_e(L_{n+1}(2))$ and $s \in \pi_e(L_n(2))$. Let x_1 be an element of $L_{n-1}(2)$ of order s and $S_1 \in \mathrm{Syl}_s(L_{n-1}(2))$. Then [7, P. 187, Satz 7.3] implies that all subgroups of $L_{n-1}(2)$ of order s are conjugate with $\langle x_1 \rangle$, $C_{L_{n-1}(2)}(\langle x_1 \rangle) = C_{L_{n-1}(2)}(S_1)$ is a cyclic group of order $2^{n-1} - 1$, and $|N_{L_{n-1}(2)}(S_1)| = |N_{L_{n-1}(2)}(\langle x_1 \rangle)| = (n-1)(2^{n-1}-1)$. Since $S = \{\mathrm{diag}(I_{m-n+1}, y) : y \in S_1\} \in \mathrm{Syl}_s(L_m(2))$, we can get that $\mathrm{diag}(I_{m-n+1}, x_1)$ is an element of $L_m(2)$ of order s and S is cyclic. On the other hand, we can see by Schur's lemma that

$$\begin{aligned} C_{L_m(2)}(S) &= C_{L_m(2)}(\langle \mathrm{diag}(I_{m-n+1}, x_1) \rangle) \\ &= \{\mathrm{diag}(y_1, y_2) : y_1 \in L_{m-n+1}(2),\ y_2 \in C_{L_{n-1}(2)}(S_1)\} \end{aligned}$$

and

$$\begin{aligned} N_{L_m(2)}(S) &= N_{L_m(2)}(\langle \mathrm{diag}(I_{m-n+1}, x_1) \rangle) \\ &= \{\mathrm{diag}(y_1, y_2) : y_1 \in L_{m-n+1}(2),\ y_2 \in N_{L_{n-1}(2)}(S_1)\}. \end{aligned}$$

This forces for every $g \in L_m(2)$, $S^g = S$ or $S^g \cap S = 1$ and hence,

$$m_s(L_m(2)) = \phi(s) n_s(L_m(2)) = \frac{\phi(s)|L_m(2)|}{|L_{m-n+1}(2)|(n-1)(2^{n-1}-1)}.$$

If $m = n+1$, then since $m_2(L_2(2)) = m_2(C_{L_m(2)}(\langle \mathrm{diag}(I_{m-n+1}, x_1) \rangle)) = 3$, Lemma 3.5 implies that $m_{2s}(L_{n+1}(2))) = \frac{\phi(s)|L_{n+1}(2)|}{2(n-1)(2^{n-1}-1)}$, as claimed in (i). If $m = n$, then $|L_{m-n+1}(2)| = 1$ and hence, (ii) follows. □

Lemma 3.7. *Let* $m \in \{7, 8\}$. *If* $127 \mid |G|$ *and* $\mathrm{nse}(G) = \mathrm{nse}(L_m(2))$, *then* $127^2 \nmid |G|$.

Proof. Let $P \in \mathrm{Syl}_{127}(G)$. If $m = 7$, then applying a simple GAP program [16] shows that for every $k \in \mathrm{nse}(L_7(2)) = \mathrm{nse}(G)$, $127^2 \nmid (1 + m_{127} + m_k)$ and hence $127^2 \notin \pi_e(G)$. Therefore, every non-trivial element of P has order 127 and hence, Lemma 2.1 forces $|P|$ to divide $1 + m_{127}(G) = 1 + 23222833643520$, considering Corollary 3.3. This implies that $|P| = 127$ and hence, $127^2 \nmid |G|$, as desired. If $m = 8$, then applying a simple GAP program [16] shows that $k := 2^{27}.3^2.5.17.31.127.331 \in \mathrm{nse}(L_8(2)) = \mathrm{nse}(G)$. Thus there exists $l \in \pi_e(G)$ such that $m_l(G) = k$. Since $126 \nmid m_l(G)$, we get $\gcd(127, l) = 1$. We claim that P acts fixed point freely on the set $\{x \in G : O(x) = l\}$. If not, there exists a natural number u such that $127^u l \in \pi_e(G)$ and hence, by Lemma 3.5, $m_{127^u l}(G) \geq \phi(127) m_l(G) > |L_8(2)|$, which is a contradiction. This forces $|P| \mid m_l(G)$ and hence $|P| = 127$, as desired. □

Lemma 3.8. *If* $p \mid |G|$ *and* $\mathrm{nse}(G) = \mathrm{nse}(L_m(2))$ *where* $m \in \{n, n+1\}$, *then* $p^2 \nmid |G|$.

Proof. If $n = 7$, then Lemma 3.7 completes the proof. Now let $n \neq 7$ and let s be a Zsigmondy prime of $2^{n-1} - 1$. If $m = n + 1$, then by Lemma 3.6(i), $t = 2s \in \pi_e(L_m(2))$ and $m_t(L_m(2)) = \frac{\phi(t)|L_{n+1}(2)|}{2.(n-1).(2^{n-1}-1)} \in \mathrm{nse}(G)$. Thus there exists $l \in \pi_e(G)$ such that $m_l(G) = m_t(L_m(2))$. Since $p - 1 \nmid m_l(G)$, $\gcd(p, l) = 1$. We claim that $P \in \mathrm{Syl}_p(G)$ acts fixed point freely on the set $\{x \in G : O(x) = l\}$. If not, then for some natural number u, $p^u l \in \pi_e(G)$ and by Lemma 3.5, $m_{p^u l}(G) \geq \phi(p)m_l(G) \geq |L_m(2)|$ and hence, $m_{p^u l}(G) \notin \mathrm{nse}(L_m(2))$, which is a contradiction. Thus the fixed point free action of P on $\{x \in G \mid O(x) = l\}$ forces $|P|$ to divide $m_l(G)$ and hence, $|P| \leq p$, as desired. If $m = n$, then it is enough to replace $t = 2s$ with $t = s$ and use Lemma 3.6(ii) in the above argument. □

Corollary 3.9. *If $p \mid |G|$, then for every $r \in \pi_e(G) - \{p\}$, $rp \notin \pi_e(G)$.*

Proof. Suppose on the contrary that $rp \in \pi_e(G)$. Since $p^2 \nmid |G|$, we deduce that $S_p(G)$ is cyclic and hence, $m_{rp}(G) = m_p(G)\phi(r)k$, for some natural number k. Thus $m_{rp}(G) = \frac{(p-1)|L_m(2)|\phi(r)k}{np}$ and hence, one of the following holds:

- $p \mid m_{rp}(G)$. Then $p \mid \phi(r)k$ and hence, $m_{rp}(G) \geq \frac{(p-1)|L_m(2)|}{n} = \frac{(2^n-2)|L_m(2)|}{n} > |L_m(2)|$, which is a contradiction.
- $p \nmid m_{rp}(G)$. Then Corollary 3.4 shows that $m_{rp}(G) = m_p(G)$ and hence, $r = 2$. But $m_2(G) = m_2(L_m(2))$. Thus Corollary 3.4 forces $p \mid m_2(G)$. On the other hand, (2.1) forces $2p \mid (1 + m_p + m_2 + m_{2p})$ and $p \mid (1+m_p)$. It follows that $p \mid m_{2p} = m_p$, which is a contradiction.

Hence $rp \notin \pi_e(G)$, as desired. □

Corollary 3.10. (i) $n_p(G) = \frac{|L_m(2)|}{np} \mid |G|$.
(ii) $|G| \mid \frac{(p-1)|L_m(2)|}{n}$.

Proof. Since $p \| |G|$, $S_p(G)$ is cyclic, so $m_p(G) = \phi(p)n_p(G)$. Thus Corollary 3.4 completes the proof of (i). Let $r \in \pi(G) - \{p\}$. Then by Corollary 3.9, the Sylow r-subgroup of G acts fixed point freely on the set of elements of order p in G and hence, $|G|_r \mid m_p(G)$. Also, $|G|_p = p$. This forces $|G| \mid \frac{(p-1)|L_m(2)|}{n}$. □

Corollary 3.11. *If $r \in \pi(L_u(2)) - \{2\}$, then $|L_u(2)|_r \leq 2^{3u/2}$. In particular, if $r \in \pi(G) - \{2\}$, then $|G|_r < 2^{2m}$.*

Proof. By Corollary 3.10, we can assume that r is a Zsigmondy prime of $2^t - 1$, where $2 \leq t \leq m$. Let $(2^t - 1)_r = r^s$. It is known that $(\prod_{i=1}^{m}(2^i - 1))_r \leq ((2^t - 1)_r)^{[\frac{m}{t}]}([\frac{m}{t}]!)_r < ((2^t - 1)_r)^{[\frac{m}{t}]} r^{\frac{m}{t(r-1)}}$ (it can for example be extracted from [12, Lemma 1]). Since $r \geq 3$, Corollary 3.10(ii) shows that $|G|_r \leq (2^{n-1} - 1)_r |L_m(2)|_r < 2^{2m}$, as claimed. □

Lemma 3.12. *G is neither a Frobenius group nor a 2-Frobenius group.*

Proof. Suppose on the contrary that, G is a Frobenius group with kernel K and complement H. We have $\pi(H) = \{p\}$ or $\pi(K) = \{p\}$. If $\pi(K) = \{p\}$, then since $K \trianglelefteq G$ and $p \| |G|$, we deduce that $S_p(G) = K$ is a normal and cyclic subgroup of G. Thus $m_p(G) = p - 1$, which is a contradiction with Corollaries 3.3 and 3.4. Now, let $\pi(H) = \{p\}$. By Corollary 3.10, we have $\frac{|L_m(2)|}{np} \mid |G|$ and $|G| \mid \frac{(p-1)|L_m(2)|}{n}$, so there exists a prime divisor r of $2^{n-2} - 1$ such that $|G|_r = |L_m(2)|_r$. Also, Lemma 2.1 shows that $\{\pi(K), \pi(H)\} = \{\pi_1(G), \pi_2(G)\}$. Thus $r \in \pi(K)$. Since K is nilpotent, $S_r(G)$ is a normal subgroup of G, so $S_p(G)$ acts fixed point freely on $S_r(G)$ and hence, $p \mid |S_r(G)| - 1$. This shows that either $m = 6$ and $31 \mid 49 - 1$ or $p \leq |S_r(G)| \leq 2^{n-2} - 1 < p$, which are impossible. If G is a 2-Frobenius group, then it follows from Lemma 2.3 that there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a cyclic group of order p and $|G/K| \mid (p-1)$. Also, K/H acts fixed point freely on H and hence, the previous argument rules out this case. □

The above results show that p is an isolated point in the prime graph of G and so $t(G) \geq 2$. Since G is not a Frobenius or 2-Frobenius group, Lemma 2.4 shows that there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group and p is an odd order component of K/H. In Theorem 3.13, fix $S := K/H$ and $L := L_m(2)$, where $m \in \{n, n+1\}$. In what follows we need the sets of order components of finite simple groups with disconnected prime graph, which are given in [11] and [17].

Theorem 3.13. *S is isomorphic to L.*

Proof. By the classification of finite simple groups, we proceed the proof in the following steps.

Step 1. S can not be an alternating group $\mathbb{A}_r$, $r \geq 5$.

Proof. If $S \cong \mathbb{A}_r$, then since $2^n - 1 = p \in \pi(S)$, $r \geq 2^n - 1$. Thus there exists a prime number $u \in \pi(\mathbb{A}_r)$ such that $2^{n-1} - 1 = \frac{(p-1)}{2} < u < p$. But $|G|$ divides $\frac{(p-1)|L|}{n}$. Therefore $u \in \pi(\frac{(p-1)|L|}{n})$, which is a contradiction. □

Step 2. S is not a sporadic simple group.

Proof. Suppose that S is a sporadic simple group. Since one of the odd order components of S is p, which is a Mersenne prime, we deduce, by considering the odd order components of sporadic simple groups, that $p = 7$ or $p = 31$. This forces $n = 3$ or $n = 5$. By our assumption $n \geq 5$. Also considering the order of sporadic simple groups with 31 as one of their odd order components shows that $|S| \nmid \frac{(p-1)|L|}{n}$, and so $|G| \nmid \frac{(p-1)|L|}{n}$, which is a contradiction. □

Step 3. $S \cong L$. By Steps 1 and 2, and the classification of finite simple groups, S is a simple group of Lie type with disconnected prime graph. We continue the proof in the following cases:

Case 1. Let $t(S) = 2$. Then $OC_2(S) = 2^n - 1$.

1.1. If $S \cong C_{n'}(q)$, where $n' = 2^t \geq 2$, then $\frac{q^{n'}+1}{(2,q-1)} = 2^n - 1$. Thus p is a Zsigmondy prime of $q^{2n'} - 1$, and hence Fermat's little theorem shows that $2n' \mid p - 1 = 2(2^{n-1} - 1)$. This forces $n' = 1$, which is a contradiction. The same reasoning rules out the case when either $S \cong B_{n'}(q)$ or $S \cong {}^2D_{n'}(q)$, where $n' = 2^t \geq 4$.

1.2. If $S \cong C_r(3)$ or $B_r(3)$, then $\frac{3^r-1}{2} = 2^n - 1$. Thus $2^{n+1} - 3^r = 1$, which is a contradiction with Lemma 2.6. The same reasoning rules out the case when $S \cong D_r(3)$ or $S \cong D_{r+1}(3)$.

1.3. If $S \cong C_r(2)$, then $2^r - 1 = 2^n - 1$, and hence $r = n$. This implies that $2^{n^2} \mid |G|$ and so $|G| \nmid \frac{(p-1)|L|}{n}$, which is a contradiction. The same reasoning rules out the cases when $S \cong D_r(2)$ or $S \cong D_{r+1}(2)$.

1.4. If $S \cong D_r(5)$, where $r \geq 5$, then $(5^r - 1)/4 = (2^n - 1)$. Thus $5^r - 1 = 2^{n+2} - 4$ and hence, $5(5^{r-1}+1) = 2(2^{n+1}+1)$. But $5^{r-1}+1 \mid |S|$, so $2^{n+1}+1 \mid |G|$. Let r be a Zsigmondy prime of $2^{2(n+1)} - 1$, then $r \mid 2^{n+1} + 1$. Thus $r \mid |G|$, and hence $r \mid \frac{(p-1)|L|}{n}$, which is impossible.

1.5. If $S \cong {}^2D_{n'}(3)$, where $9 \leq n' = 2^m + 1$ and n' is not prime, then $\frac{3^{n'-1}+1}{2} = 2^n - 1$, and hence $3^{n'-1} = 2^{n+1} - 3$, which is a contradiction.

1.6. If $S \cong {}^2D_{n'}(2)$, where $n' = 2^m + 1 \geq 5$, then $2^{n'-1} + 1 = 2^n - 1$, and hence $2^{n'-1} = 2(2^{n-1} - 1)$, which is a contradiction.

1.7. If $S \cong {}^2D_r(3)$, where $5 \leq r \neq 2^m + 1$, then $\frac{3^r+1}{4} = 2^n - 1$, and hence $3^r = 2^{n+2} - 5 = 4(2^n + 1) - 9$. Thus $9 \mid 2^n + 1$. So $9 = \gcd(2^3 + 1, 2^n + 1)$ and hence, $3 \mid n$. But n is prime, and hence $n = 3$, which is a contradiction.

1.8. If $S \cong G_2(q)$, where $2 < q \equiv \epsilon \pmod 3$ and $\epsilon = \pm 1$, then $q^2 - \epsilon q + 1 = 2^n - 1$. First, assume that q is an odd number. Then $q^2 - \epsilon q = 2(2^{n-1} - 1)$, and hence $q(q - \epsilon) = 2(2^{\frac{n-1}{2}} - 1)(2^{\frac{n-1}{2}} + 1)$. Thus either $q \mid (2^{\frac{n-1}{2}} - 1)$ or $q \mid (2^{\frac{n-1}{2}} + 1)$. If $q \mid (2^{\frac{n-1}{2}} - 1)$, then $2^{\frac{n-1}{2}} - 1 = kq$. Therefore, $q(q - \epsilon) = 2kq(kq + 2)$ and hence, $q - \epsilon = 2k^2q + 4k$. Thus $-\epsilon - 4k = q(2k^2 - 1)$, which is a contradiction, since the right hand side is positive and the left hand side is negative. If $q \mid (2^{\frac{n-1}{2}} + 1)$, then $2^{\frac{n-1}{2}} + 1 = kq$. Thus $q(q - \epsilon) = 2kq(kq - 2)$ and hence, $q - \epsilon = 2k^2q - 4k$. This implies that $4k - \epsilon = q(2k^2 - 1)$. Thus $q = \frac{4k-\epsilon}{2k^2-1} \in \mathbb{N}$. This forces $k = 1$ and so, $q = 5$. Thus $2^n = 32$ and hence, $n = 5$. This gives that $|S| \nmid \frac{(p-1)|L|}{n}$, which is a contradiction.

Now, let $q = 2^t > 2$, then $2^t(2^t - \epsilon) = 2(2^{n-1} - 1)$. This forces $t = 1$ and hence $q = 2$, which is a contradiction.

1.9. If $S \cong F_4(q)$, where q is odd, then $q^4 - q^2 + 1 = 2^n - 1$ and hence, $q^2(q - 1)(q + 1) = 2(2^{n-1} - 1)$. This shows that $4 \mid 2(2^{n-1} - 1)$, which is a contradiction. The same reasoning rules out the case when $S \cong^3 D_4(q)$.

1.10. If $S \cong {}^2F_4(2)'$, then $|S| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$. Thus $2^n - 1 = 13$, which is impossible.

1.11. If $S \cong {}^2A_3(2)$, then $|S| = 2^6 \cdot 3^4 \cdot 5$. Thus $2^n - 1 = 5$, which is impossible.

1.12. Let r be an odd prime and S be isomorphic to the one of the following simple groups:

a. Let $S \cong L_r(q)$, where $(r, q) \neq (3, 2),\ (3, 4)$ and for a prime u, $q = u^\alpha$. First let $u \neq 2$. Since $\frac{q^r-1}{(r,q-1)(q-1)} = 2^n - 1$, $2^n < q^r$. So Corollary 3.11 forces $2^{n(r-1)/2} < |S|_u = q^{\frac{r(r-1)}{2}} \leq |G|_u \leq 2^{2m}$. Thus either $r = 3$ or $r = 5$ and $m = n + 1$. This shows that $\frac{q^5-1}{(r,q-1)(q-1)} = 2^n - 1$ or $\frac{q^3-1}{(r,q-1)(q-1)} = 2^n - 1$. If $r = 5$, then $q(q^4 - 1)/(q - 1) = 5(2^n - 1) - 1 = 2(5.2^{n-1} - 5 + 2)$ or $q(q^4 - 1)/(q - 1) = 2(2^{n-1} - 1)$. Thus $u \nmid 2^{n-1} - 1 = (p - 1)/2$, and hence Corollary 3.11 guarantees that $|G|_u \leq |L_m(2)|_u \leq 2^{3m/2}$. Thus $2^{2n} < |S|_u = q^{\frac{r(r-1)}{2}} \leq |G|_u \leq 2^{3m/2}$, which is a contradiction. If $r = 3$, then $\frac{q^3-1}{(3,q-1)(q-1)} = 2^n - 1 = p$. Thus by Fermat's little theorem $3\alpha \mid p - 1 = 2(2^{n-1} - 1)$, and hence if w is a Zsigmondy prime of $2^{n-2} - 1$, then an easy computation shows that $w \notin \pi(S)$. Also, $\bar{G}/S \leq \mathrm{Out}(S)$, where $\bar{G} = G/H$. So $|\bar{G}/S| = 2(3, q - 1)\alpha$. This forces $w \in \pi(H)$ and $|H|_w = |L|_w$. But H is nilpotent, so $S_p(G)$ acts fixed point freely on $S_w(H)$ and hence, $p \mid |S_w(H)| - 1$. Thus either $m = 6$ and $w = 7$ and hence, $31 \mid 49 - 1$ or $2^n - 1 = p < 2^{n-2} - 1$, which are impossible. Now let $u = 2$. Then p is a Zsigmondy prime of $2^n - 1$ and $2^{r\alpha} - 1$. Thus $n = r\alpha$. But n is prime, so $\alpha = 1$ and $n = r$. If $m = n$, then $S \cong L_n(2)$, as claimed. Now let $m = n + 1$ and r be a Zsigmondy prime of $2^{n+1} - 1$. Then $r \nmid |\mathrm{Out}(S)||S| = 2|S|$ and hence, $r \mid |H|$. But $|H|_r = |L|_r = |2^{n+1} - 1|_r$ and hence, applying the previous argument leads us to get a contradiction. If $n = 5$, then replacing r with 7 in the above argument leads us to get a contradiction.

b. Let $S \cong L_{r+1}(q)$, where $(q - 1) \mid (r + 1)$ and for a prime u, $q = u^\alpha$. First let $u \neq 2$. Since $\frac{q^r-1}{q-1} = 2^n - 1$, $2^n < q^r$. So Corollary 3.11 forces $2^{n(r+1)/2} < |S|_u = q^{\frac{r(r+1)}{2}} \leq |G|_u \leq 2^{2m}$. Thus $m = n + 1$, $r = 3$ and $q \in \{3, 5\}$. So $\frac{3^3-1}{2} = 2^n - 1$ or $\frac{5^3-1}{4} = 2^n - 1$. This forces $q = n = 5$. But $5^3 \mid |S|$, while $5^3 \nmid (p-1)|L_6(2)|/5$, and hence $|S| \nmid |G|$, which is a contradiction. Now let $u = 2$. Then p is a Zsigmondy prime of $2^n - 1$ and $2^{r\alpha} - 1$. Thus $n = r\alpha$. But n is prime, so $\alpha = 1$ and $n = r$. This forces $S \cong L_{n+1}(2)$. If $m = n$, then, $|S|_2 > |G|_2$, which is a contradiction. If $m = n + 1$, then $S \cong L$, as claimed.

c. Let $S \cong {}^2A_{r-1}(q)$. Then applying the same reasoning as that in Subcase (a) we get a contradiction.

d. Let $S \cong {}^2A_r(q)$, where $(q + 1) \mid (r + 1)$ and $(r, q) \neq (3, 3), (5, 2)$. Then applying the same reasoning as that of in Subcase (b) we get a contradiction.

1.13. If $S \cong E_6(q)$, where $q = u^\alpha$, then $\frac{(q^6+q^3+1)}{(3,q-1)} = 2^n - 1$. First let $u \neq 2$. Thus $q^9 > 2^n$, and hence Corollary 3.11 shows that $2^{4n} < q^{36} = |S|_u \leq |G|_u < 2^{2m}$, which is a contradiction. Now let $u = 2$. Then p is a Zsigmondy prime of

$2^n - 1$ and $2^{9f} - 1$. Thus $n = 9f$, which is a contradiction, because n is prime. The same reasoning rules out the case when $S \cong {}^2E_6(q)$, where $q > 2$.

Case 2. Let $t(S) = 3$. Then $2^n - 1 \in \{OC_2(S),\ OC_3(S)\}$.

2.1. If $S \cong L_2(q)$, where $4 \mid q+1$, then $\frac{q-1}{2} = 2^n - 1$ or $q = 2^n - 1$.
If $q = 2^n - 1$, then $q = p$ and

$$|S| = |L_2(p)| = \frac{1}{(2, p-1)} p(p^2 - 1) = 2^n(2^{n-1} - 1)(2^n - 1).$$

On the other hand, $S \leq G/H \leq \mathrm{Aut}(S)$ and $\mathrm{Out}(S) \cong \mathbb{Z}_2$. Therefore $2^{n-2} - 1 \mid |H|$. Let r be a Zsigmondy prime of $2^{n-2} - 1$. Since H is nilpotent, $S_r(H) \trianglelefteq G$. Thus Corollary 3.9 shows that $S_p(G)$ acts fixed point freely on $S_r(H)$. Therefore, $|S_p(G)| \mid |S_r(H)| - 1$, and hence Corollary 3.10 shows that either $p = 2^n - 1 < 2^{n-2} - 1$ or $31 \mid 49 - 1$, which is a contradiction.

If $\frac{q-1}{2} = 2^n - 1$, then $q = 2^{n+1} - 1$, and hence Lemma 2.6 shows that q is prime. But $3 = 2^2 - 1 \mid 2^{n+1} - 1 = q$, and hence $3 = q = 2^{n+1} - 1$, which is impossible.

2.2. If $S \cong L_2(q)$, where $4 \mid q - 1$, then $q = 2^n - 1$ or $\frac{q+1}{2} = 2^n - 1$. If $q = 2^n - 1$, then $q - 1 = 2(2^{n-1} - 1)$. But $4 \mid q - 1$, which is a contradiction. If $\frac{q+1}{2} = 2^n - 1$, then $q = 2^{n+1} - 3$. Thus $|S| = q(q^2 - 1)/(2, q-1) = 4(2^{n+1} - 3)(2^n - 1)(2^{n-1} - 1)$. Therefore $2^{n-2} - 1 \mid |H|$, and hence repeating the same argument as that of in Case 2.1 leads us to get a contradiction.

2.3. If $S \cong L_2(q)$, where $q > 2$ and q is even, then $|S| = q(q-1)(q+1)$. If $q - 1 = 2^n - 1$, then $q = 2^n$. Thus $|S| = 2^n(2^n - 1)(2^n + 1) \mid |G|$, and hence $(2^n+1) \mid \frac{(p-1)|L|}{n}$, which is a contradiction by considering the Zsigmondy prime of $2^{2n} - 1$. If $q + 1 = 2^n - 1$, then $q = 2(2^{n-1} - 1)$. But q is a power of 2 and $q > 2$, so $2 \mid (2^{n-1} - 1)$, which is a contradiction.

2.4. If $S \cong {}^2A_5(2)$ or $S \cong A_2(2)$, then $|S| = 2^{15} \cdot 3^6 \cdot 7 \cdot 11$ or $|S| = 8 \cdot 3 \cdot 7$. Clearly, $2^n - 1 \neq 11$. If $2^n - 1 = 7$, then $n = 3$, which is a contradiction.

2.5. If $S \cong {}^2D_r(3)$, where $r = 2^t + 1 \geq 5$, then $\frac{3^r+1}{4} = 2^n - 1$ or $\frac{3^{r-1}+1}{2} = 2^n - 1$. If $\frac{3^r+1}{4} = 2^n - 1$, then the same reasoning as that of in Subcase 1.7 shows that $r = 3 < 5$, which is a contradiction. If $\frac{3^{r-1}+1}{2} = 2^n - 1$, then $2^{n+1} = 3^{r-1} - 3$, which is impossible.

2.6. If $S \cong {}^2D_{r+1}(2)$, where $r = 2^{n'} - 1$ and $n' \geq 2$, then $2^r + 1 = 2^n - 1$ or $2^{r+1} + 1 = 2^n - 1$. Hence $2 \mid 2^{n-1} - 1$, which is a contradiction.

2.7. If $S \cong G_2(q)$, where $q \equiv 0 \pmod 3$, then $q^2 - q + 1 = 2^n - 1$ or $q^2 + q + 1 = 2^n - 1$, and hence $q(q \pm 1) = 2(2^{n-1} - 1)$. Thus the same reasoning as that of in Subcase 1.8 leads us to get a contradiction.

2.8. If $S \cong {}^2G_2(q)$, where $q = 3^{2t+1} > 3$, then $q - \sqrt{3q} + 1 = 2^n - 1$ or $q + \sqrt{3q} + 1 = 2^n - 1$. Thus $3^{t+1}(3^t + \epsilon) = 2(2^{n-1} - 1)$, where $\epsilon = \pm 1$. Thus the same reasoning as that of in subcase 1.8 leads us to get a contradiction.

2.9. If $S \cong F_4(q)$, where q is even, then $q^4 + 1 = 2^n - 1$ or $q^4 - q^2 + 1 = 2^n - 1$. If $q^4 + 1 = 2^n - 1$, then $2 \mid (2^{n-1} - 1)$, which is a contradiction. If

$q^4 - q^2 + 1 = 2^n - 1$, then $q^2(q^2-1) = 2(2^{n-1}-1)$, which is a contradiction because q is a power of 2.

2.10. If $S \cong {}^2F_4(q)$, where $q = 2^{2t+1} > 2$, then $OC_2 = q^2 + \sqrt{2q^3} + q - \sqrt{2q} + 1$ and $OC_3 = q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1$. Thus $2^n - 1 = 2^{2(2t+1)} + \epsilon 2^{3t+2} + 2^{2t+1} + \epsilon 2^{t+1} + 1$, where $\epsilon = \pm 1$, and hence $2(2^{n-1}-1) = 2^{t+1}(2^{3t+1} + \epsilon 2^{2t+1} + \epsilon 2^t - 1)$. This forces $t = 0$, which is a contradiction.

2.11. If $S \cong E_7(2)$, then $2^n - 1 \in \{73, 127\}$. It is evident $2^n - 1 \neq 73$, and hence $2^n - 1 = 127$, so $n = 7$. Thus $|S| = |E_7(2)| \mid \frac{126 \cdot |L_7(2)|}{7}$ or $|S| \mid \frac{126 \cdot |L_8(2)|}{7}$, which is impossible.

2.12. If $S \cong E_7(3)$, then $2^n - 1 \in \{757, 1093\}$, which is impossible.

Case 3. Let $t(S) \in \{4, 5\}$. Then
$2^n - 1 \in \{OC_2(S),\ OC_3(S),\ OC_4(S),\ OC_5(S)\}$.

3.1. If $S \cong A_2(4)$, then $n = 3$ or $n = 2$, which is a contradiction. The same reasoning rules out the case when $S \cong {}^2E_6(2)$.

3.2. If $S \cong {}^2B_2(q)$, where $q = 2^{2t+1}$ and $t \geq 1$, then $2^n - 1$ is one of the following values: $q - 1 = 2^n - 1$. Thus $2^{2t+1} - 1 = 2^n - 1$, and hence $n = 2t+1$. But $(q - \sqrt{2q} + 1)(q + \sqrt{2q} + 1) = (q^2 + 1) \mid |S|$, so $(2^{2n} + 1) \mid |G|$, and hence Corollary 3.10 shows that $(2^{2n} + 1) \mid \frac{(p-1)|L|}{n}$, which is a contradiction. If $q \pm \sqrt{2q} + 1 = 2^n - 1$, then $2^{t+1}(2^t \pm 1) = 2(2^{n-1} - 1)$. This forces $t = 0$, which is a contradiction.

3.3. If $S \cong E_8(q)$, then $2^n - 1 = \frac{q^{10} + q^5 + 1}{q^2 - q + 1} = q^8 - q^7 + q^5 - q^4 + q^3 - q + 1$ or $2^n - 1 = \frac{q^{10} - q^5 + 1}{q^2 - q + 1} = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$ or $2^n - 1 = \frac{q^{10} + 1}{q^2 + 1} = q^8 - q^6 + q^4 - q^2 + 1$ or $2^n - 1 = q^8 - q^4 + 1$. Thus $q(q-1)(q+1)(q^5 - q^4 + q^3 + 1) = 2(2^{n-1} - 1)$ or $q(q-1)(q+1)(q^5 + q^4 + q^3 - 1) = 2(2^{n-1} - 1)$ or $q^2(q-1)(q+1)(q^4 + q^2 - 1) = 2(2^{n-1} - 1)$ or $q^4(q-1)(q+1)(q^2+1) = 2(2^{n-1} - 1)$. If q is odd, then $q - 1$ and $q + 1$ are even and so, 2 divides $(2^{n-1} - 1)$, which is a contradiction. If q is even, then $q = 2$ and $2^n < 2^{10}$. Thus $2^{120} = |E_8(2)|_2 > |G|_2$, which is a contradiction.

The above steps show that $S \cong L$, as claimed. □

Proof of the Main Theorem. By Lemma 2.4, we have $L \leq G/H \leq \mathrm{Aut}(L)$. Since $|\mathrm{Out}(L)| = 2$, we have $G/H \cong L$ or $G/H \cong \mathrm{Aut}(L)$. Thus Corollary 3.10 shows that $|H| \mid 2(2^{n-1} - 1)/n$. However, by Corollary 3.9, $S_p(G)$ acts fixed point freely on H, so $p = 2^n - 1 \mid |H| - 1$, while $|H| < 2^{n-1} - 1$. Thus $H = 1$, and hence either $G \cong L_m(2)$ or $G \cong \mathrm{Aut}(L_m(2))$. But $m_2(L_m(2)) < m_2(\mathrm{Aut}(L_m(2)))$ and since by Remark 2.7, $m_2(G)$ is the only odd element of $\mathrm{nse}(G)$ and $m_2(\mathrm{Aut}(L_m(2)))$ is an odd number too, we deduce that $G \ncong \mathrm{Aut}(L_m(2))$. Thus $G \cong L_m(2)$, as claimed. □

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