ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 6, pp. 1723–1737

Title:

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Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 6, pp. 1723–1737
Online ISSN: 1735-8515

HIGH-ACCURACY ALTERNATING SEGMENT EXPLICIT-IMPLICIT METHOD FOR THE FOURTH-ORDER HEAT EQUATION

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(Communicated by Davod Khojasteh Salkuyeh)

Abstract. Based on a group of new Saul'yev type asymmetric difference schemes constructed by author, a high-order, unconditionally stable and parallel alternating segment explicit-implicit method for the numerical solution of the fourth-order heat equation is derived in this paper. The truncation error is fourth-order in space, which is much more accurate than the known alternating segment explicit-implicit methods. Numerical simulations are performed to show the effectiveness of the present method that are in preference to the prior methods.

Keywords: Fourth-order heat equation, alternating segment explicit-implicit method, high accuracy, parallel computation, unconditional stability.
MSC(2010): Primary: 65M06; Secondary: 65M12, 65Y05.

1. Introduction

With the introduction of the parallel computer, the high-performance computers play an important role in scientific and engineering computations. Many numerical researchers have focused on constructing the parallel algorithms to various problems.

In the last few years, the alternating schemes were widely studied. Evans and Abdullah [2,3] first developed the alternating group explicit (AGE) scheme for solving parabolic equations. Then, the alternating segment explicit-implicit (ASE-I) scheme [18] and the alternating segment Crank-Nicolson (ASC-N) scheme [1] were proposed. The alternating scheme uses the explicit scheme and the implicit scheme alternately in the time and space direction, which has the intrinsic parallelism and unconditional stability. The results of numerical experiments show effectiveness of the methods. Afterwards, the alternating

Article electronically published on 30 November, 2017.
Received: 22 May 2015, Accepted: 9 October 2016.
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schemes have been extended to two-dimensional diffusion systems [16], dispersive equation [13, 14, 19–22], nonlinear three-order KdV equation [8] and fourth-order diffusion equation [4]. Meanwhile, the introduction of the alternating schemes leads to the rapid development of the domain decomposition parallel method [9, 11, 12, 17, 24]. However, the majority of the literature have focused their attentions on the parallelism. The major problem in the above algorithms is that the truncation error is only near the second order in space. The construction of the highly accurate parallel difference scheme has been considered by only a limited number of investigators. In [14, 19, 20], the fourth-order accurate AGE and ASC-N schemes have been constructed for the dispersive equation by a group of new high-order accurate asymmetric difference schemes.

In view of the limited information available of highly accurate parallel difference method, this paper undertakes a study of the construction of high-order accurate algorithm for the fourth-order heat equation.

$$Lu = \frac{\partial u}{\partial t} + \alpha \frac{\partial^4 u}{\partial x^4} = 0, \quad x \in [0, L], t \in [0, T], \tag{1.1}$$

The fourth-order heat equation is well-known as one of the applied equations, which can be found in image processing or thin film modeling. The numerical solving methods were widely studied [4, 6, 7, 10, 15].

Although the unconditionally stable general schemes with intrinsic parallelism for fourth-order heat equation have been derived in [4], the truncation error is only near the second order in space. In this work, a group of new Saul'yev asymmetric difference schemes is constructed. Basing on these schemes, we will derive a high-accuracy alternating segment explicit-implicit method. The new parallel ASE-I method is not only unconditionally stable but also has fourth-order accuracy in space. Its numerical simulations show better accuracy than the known ASE-I, AGE1 and AGE2 schemes in [4]. We hope the result of this paper makes an essential contribution in this direction.

Here is the outline of the paper. In Section 2, eight basic schemes for constructing the high-accuracy ASE-I method are given . In Section 3, the new ASE-I method is developed and the error analysis and the stability are discussed. Finally, in Section 4, the numerical experiments are performed.

2. The new asymmetric schemes

2.1. The asymmetric schemes. Divide the domain of definition $[0, L] \times [0, T]$ by parallel lines $x = x_j = jh (j = 0, 1, 2, \ldots, J)$, $t = t^n = n\tau (n = 0, 1, 2, \ldots, N)$, where $h = L/J$ is space mesh length, $\tau = T/N$ is time mesh length. J and N are positive integers. We use U_j^n to represent the approximate solution of $u(x_j, t^n)$, where $u(x, t)$ represents the exact solution of (1.1). We first introduce

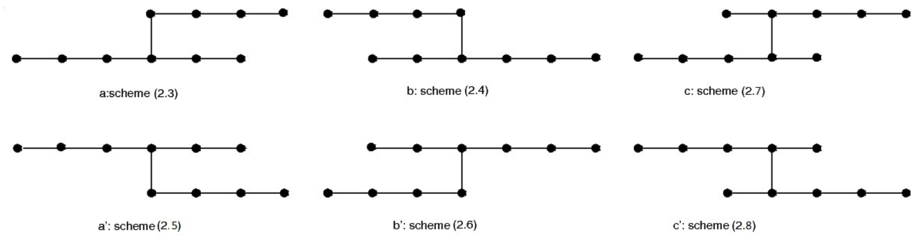

FIGURE 1. The new asymmetric schemes (2.3) – (2.8)

the following explicit and implicit schemes

$$
\begin{aligned}
U_j^{n+1} &= rU_{j-3}^n - 12rU_{j-2}^n + 39rU_{j-1}^n \\
&\quad + (1-56r)U_j^n + 39rU_{j+1}^n - 12rU_{j+2}^n + rU_{j+3}^n,
\end{aligned} \tag{2.1}
$$

$$
\begin{aligned}
-rU_{j-3}^{n+1} &+ 12rU_{j-2}^{n+1} - 39rU_{j-1}^{n+1} + (1+56r)U_j^{n+1} \\
&- 39rU_{j+1}^{n+1} + 12rU_{j+2}^{n+1} - rU_{j+3}^{n+1} = U_j^n.
\end{aligned} \tag{2.2}
$$

Then, we give the following six asymmetric schemes (see Figure 1).

$$
\begin{aligned}
&-rU_{j+3}^{n+1} + 6rU_{j+2}^{n+1} - 6rU_{j+1}^{n+1} + (1+r)U_j^{n+1} = \\
&-6rU_{j+2}^n + 33rU_{j+1}^n + (1-55r)U_j^n + 39rU_{j-1}^n - 12rU_{j-2}^n + rU_{j-3}^n,
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
&(1+r)U_j^{n+1} - 6rU_{j-1}^{n+1} + 6rU_{j-2}^{n+1} - rU_{j-3}^{n+1} = \\
&-6rU_{j-2}^n + 33rU_{j-1}^n + (1-55r)U_j^n + 39rU_{j+1}^n - 12rU_{j+2}^n + rU_{j+3}^n,
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
&6rU_{j+2}^{n+1} - 33rU_{j+1}^{n+1} + (1+55r)U_j^{n+1} - 39rU_{j-1}^{n+1} + 12rU_{j-2}^{n+1} - rU_{j-3}^{n+1} = \\
&rU_{j+3}^n - 6rU_{j+2}^n + 6rU_{j+1}^n + (1-r)U_j^n,
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
&-rU_{j+3}^{n+1} + 12rU_{j+2}^{n+1} - 39rU_{j+1}^{n+1} + (1+55r)U_j^{n+1} - 33rU_{j-1}^{n+1} + 6rU_{j-2}^{n+1} = \\
&(1-r)U_j^n + 6rU_{j-1}^n - 6rU_{j-2}^n + rU_{j-3}^n,
\end{aligned} \tag{2.6}
$$

$$
\begin{aligned}
&-rU_{j+3}^{n+1} + 12rU_{j+2}^{n+1} - 33rU_{j+1}^{n+1} + (1+28r)U_j^{n+1} - 6rU_{j-1}^{n+1} = \\
&6rU_{j+1}^n + (1-28r)U_j^n + 33rU_{j-1}^n - 12rU_{j-2}^n + rU_{j-3}^n,
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
&-6rU_{j+1}^{n+1} + (1+28r)U_j^{n+1} - 33rU_{j-1}^{n+1} + 12rU_{j-2}^{n+1} - rU_{j-3}^{n+1} = \\
&rU_{j+3}^n - 12rU_{j+2}^n + 33rU_{j+1}^n + (1-28r)U_j^n + 6rU_{j-1}^n,
\end{aligned} \tag{2.8}
$$

where $r = \alpha\tau/6h^4$.

Let $L_h^{(2.1)}, L_h^{(2.2)}, L_h^{(2.3)}, L_h^{(2.4)}, L_h^{(2.5)}, L_h^{(2.6)}, L_h^{(2.7)}$, and $L_h^{(2.8)}$ be the discretizied operators for L based on schemes (2.1)-(2.8). From the Taylor series expansion at (x_j, t^n), we obtain the following truncation error expressions for formulaes (2.1)-(2.8):

$$L_h^{(2.1)} u_j^n - [Lu]_j^n = \frac{\tau}{2}[\frac{\partial^2 u}{\partial t^2}]_j^n + O(\tau + h^4), \tag{2.9}$$

$$L_h^{(2.2)} u_j^n - [Lu]_j^n = \frac{\tau}{2}[\frac{\partial^2 u}{\partial t^2}]_j^n + 6rh^4[\frac{\partial^5 u}{\partial t \partial x^4}]_j^n + O(\tau + h^4), \tag{2.10}$$

$$\begin{aligned} L_h^{(2.3)} u_j^n - [Lu]_j^n =& 3rh[\frac{\partial^2 u}{\partial t \partial x}]_j^n + \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t \partial x^2}]_j^n \\ & + \frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t \partial x^3}]_j^n + \frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2 \partial x}]_j^n + O(\tau + h^4), \end{aligned} \tag{2.11}$$

$$\begin{aligned} L_h^{(2.4)} u_j^n - [Lu]_j^n =& -3rh[\frac{\partial^2 u}{\partial t \partial x}]_j^n + \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t \partial x^2}]_j^n \\ & - \frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t \partial x^3}]_j^n - \frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2 \partial x}]_j^n + O(\tau + h^4), \end{aligned} \tag{2.12}$$

$$\begin{aligned} L_h^{(2.5)} u_j^n - [Lu]_j^n =& -3rh[\frac{\partial^2 u}{\partial t \partial x}]_j^n - \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t \partial x^2}]_j^n \\ & - \frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t \partial x^3}]_j^n - \frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2 \partial x}]_j^n + O(\tau + h^4), \end{aligned} \tag{2.13}$$

$$\begin{aligned} L_h^{(2.6)} u_j^n - [Lu]_j^n =& 3rh[\frac{\partial^2 u}{\partial t \partial x}]_j^n - \frac{9}{2}rh^2[\frac{\partial^3 u}{\partial t \partial x^2}]_j^n \\ & + \frac{5}{2}rh^3[\frac{\partial^4 u}{\partial t \partial x^3}]_j^n + \frac{3}{2}rh\tau[\frac{\partial^3 u}{\partial t^2 \partial x}]_j^n + O(\tau + h^4), \end{aligned} \tag{2.14}$$

$$\begin{aligned} L_h^{(2.7)} u_j^n - [Lu]_j^n =& -6rh[\frac{\partial^2 u}{\partial t \partial x}]_j^n + 7rh^3[\frac{\partial^4 u}{\partial t \partial x^3}]_j^n \\ & - 3rh\tau[\frac{\partial^3 u}{\partial t^2 \partial x}]_j^n + O(\tau + h^4), \end{aligned} \tag{2.15}$$

$$\begin{aligned} L_h^{(2.8)} u_j^n - [Lu]_j^n =& 6rh[\frac{\partial^2 u}{\partial t \partial x}]_j^n - 7rh^3[\frac{\partial^4 u}{\partial t \partial x^3}]_j^n \\ & + 3rh\tau[\frac{\partial^3 u}{\partial t^2 \partial x}]_j^n + O(\tau + h^4). \end{aligned} \tag{2.16}$$

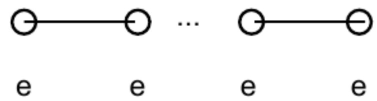

FIGURE 2. The diagram of the explicit segment

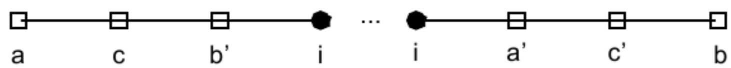

FIGURE 3. The diagram of the implicit segment

2.2. **The initial and boundary conditions.** We consider the following initial and boundary conditions [4,7]

$$u(x,0) = u_0(x), \quad x \in [0,l],$$
$$u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0 \quad t \in [0,T],$$

where $u_0(x)$ is a given function, α is a constant.

The discrete initial value condition is

$$u_j^0 = u_0(x_j), \quad j = 0,1,2,\ldots,J.$$

According to two-order central difference quotient operator in space, the discrete boundary value conditions are

$$\begin{aligned} &u_0^n = u_J^n = 0, \\ &u_{-1}^n + u_1^n = u_{-2}^n + u_2^n = 0, \\ &u_{J-1}^n + u_{J+1}^n = u_{J-2}^n + u_{J+2}^n = 0, \quad n = 0,1,2,\ldots,N. \end{aligned}$$

We can easily find that the truncation error for points on the boundary is $O(h^2)$.

3. The high-accuracy alternating segment explicit-implicit method

3.1. **The high-accuracy ASE-I method.** The new parallel ASE-I method is constructed as follow. Assuming $J-1 = k(2l+6)$, we consider the model of the segment at the $(n+1)$st and the $(n+2)$nd time levels, where n is an even number, and $l \geq 1$ is a positive integer. We first introduce the explicit segment and the implicit segment. The nodes in the explicit segment should be computed by (2.1) (see Figure 2), and the nodes in the implicit segment should be computed by (2.3)-(2.8) and (2.2) (see Figure 3).

We divide the nodes of the $(n+1)$ st time level into k implicit segments and $k+1$ explicit segments, and divide the nodes of the $(n+2)$ nd time level into k explicit segments and $k+1$ implicit segments:

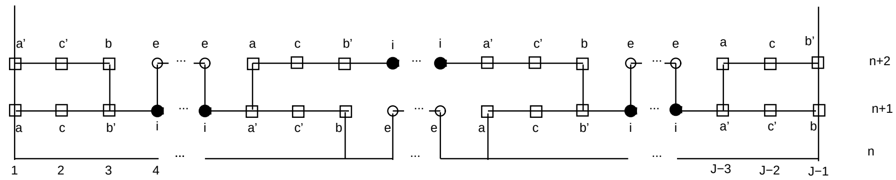

FIGURE 4. The diagram of the new ASE-I method

implicit -explicit -· · · -implicit   at odd time levels,
explicit -implicit -· · · -explicit   at even time levels.

At odd time level, each explicit segment contains l nodes, and each implicit segment contains $l+6$ nodes. At even time level, the first implicit segment contains 3 nodes, the difference schemes of the nodes are arranged according to the rule of "(a')-(c')-(b) ", the $(k+1)$st implicit segment contains 3 nodes, the difference schemes of the nodes are arranged according to the rule of "(a)-(c)-(b')". The difference schemes of the nodes in the explicit segments and the other implicit segments are according to the Figure 2 and Figure 3. Generally speaking, the asymmetric schemes are used alternately in pairs at two adjacent points of the $(n+1)$st and the $(n+2)$nd time level. The rule is displayed in Figure 4, we use "□" to denote the asymmetric schemes (2.3)-(2.8), " ○ " to denote the explicit scheme (2.1), " ● " to denote the implicit scheme (2.2).

The high-accuracy ASE-I scheme can be expressed as

$$\begin{cases} (I+rG_1)U^{n+1}=(I-rG_2)U^n, \\ (I+rG_2)U^{n+2}=(I-rG_1)U^{n+1}, \end{cases} \quad n=0,2,4,6,\ldots, \tag{3.1}$$

where $U^n=(u_1^n,u_2^n,\ldots,u_{J-1}^n)^T$, and the matrices G_1 and G_2 are given by

$$G_1=\begin{pmatrix} Q_{(l+6)\times(l+6)} & & & & & \\ & Q_{l\times l} & & & & \\ & & Q_{(l+6)\times(l+6)} & & & \\ & & & \ddots & & \\ & & & & Q_{l\times l} & \\ & & & & & Q_{(l+6)\times(l+6)} \end{pmatrix},$$

$$G_2=\begin{pmatrix} Q^{(l)}_{(3)\times(3)} & & & & & P_{3\times3} \\ & Q_{l\times l} & & & & \\ & & \ddots & & & \\ & & & Q_{(l+6)\times(l+6)} & & \\ & & & & Q_{l\times l} & \\ P^T_{3\times3} & & & & & Q^r_{(3)\times(3)} \end{pmatrix},$$

$$Q_{(l+6)\times(l+6)} = \begin{pmatrix} 1 & -6 & 6 & -1 & & & & & \\ -6 & 28 & -33 & 12 & -1 & & & & \\ 6 & -33 & 55 & -39 & 12 & -1 & & & \\ -1 & 12 & -39 & 56 & -39 & 12 & -1 & & \\ & & \ddots & & \ddots & & \ddots & & \\ & & -1 & 12 & -39 & 56 & -39 & 12 & -1 \\ & & & -1 & 12 & -39 & 55 & -33 & 6 \\ & & & & -1 & 12 & -33 & 28 & -6 \\ & & & & & -1 & 6 & -6 & 1 \end{pmatrix},$$

$$Q_{l\times l} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \; P_{3\times 3} = \begin{pmatrix} -1 & 12 & -39 \\ 0 & -1 & 12 \\ 0 & 0 & -1 \end{pmatrix},$$

$$Q^l_{3\times 3} = \begin{pmatrix} -55 & 33 & -6 \\ 33 & -28 & 6 \\ -6 & 6 & -1 \end{pmatrix}, \; Q^r_{3\times 3} = \begin{pmatrix} -1 & 6 & -6 \\ 6 & -28 & 33 \\ -6 & 33 & -55 \end{pmatrix}.$$

3.2. **The analysis of the truncation error.** Let us give out the error analysis for the high-accuracy ASE-I method. In order to analyze the truncation errors of the ASE-I method, we change the scheme (3.1) into the equivalent segment schemes of three-level, i.e., adding scheme (a') at the point (x_j, t^{n+1}) and scheme (a) at the point (x_j, t^{n+2}) for two adjacent points of the $(n+1)$st time level and the $(n+2)$nd time level, we obtain three-level scheme (3.2). Similarly, adding (b') and (b), (a) and (a'), (b) and (b'), (c') and (c), (c) and (c') at the point (x_j, t^{n+1}) of the $(n+1)$st time level and the point(x_j, t^{n+2}) of the $(n+2)$nd time level respectively, we obtain three-level schemes (3.3)-(3.7), separately:

$$\begin{aligned} &-rU_{j+3}^{n+2} + 6rU_{j+2}^{n+2} - 6rU_{j+1}^{n+2} + (1+r)U_j^{n+2} = \\ &-12rU_{j+2}^{n+1} + 66rU_{j+1}^{n+1} - 110rU_j^{n+1} + 78rU_{j-1}^{n+1} - 24rU_{j-2}^{n+1} + 2rU_{j-3}^{n+1} + \\ &rU_{j+3}^{n} - 6rU_{j+2}^{n} + 6rU_{j+1}^{n} + (1-r)U_j^{n}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &(1+r)U_j^{n+2} - 6rU_{j-1}^{n+2} + 6rU_{j-2}^{n+2} - rU_{j-3}^{n+2} = \\ &-12rU_{j-2}^{n+1} + 66rU_{j-1}^{n+1} - 110rU_j^{n+1} + 78rU_{j+1}^{n+1} - 24rU_{j+2}^{n+1} + 2rU_{j+3}^{n+1} + \\ &(1-r)U_j^{n} + 6rU_{j-1}^{n} - 6rU_{j-2}^{n} + rU_{j-3}^{n}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &6rU_{j+2}^{n+2} - 33rU_{j+1}^{n+2} + (1+55r)U_j^{n+2} - 39rU_{j-1}^{n+2} + 12rU_{j-2}^{n+2} - rU_{j-3}^{n+2} = \\ &2rU_{j+3}^{n+1} - 12rU_{j+2}^{n+1} + 12rU_{j+1}^{n+1} - 2rU_j^{n+1} - \end{aligned}$$

$$(3.4)\qquad 6rU^n_{j+2}+33rU^n_{j+1}+(1-55r)U^n_j+39rU^n_{j-1}-12rU^n_{j-2}+rU^n_{j-3},$$
$$-rU^{n+2}_{j+3}+12rU^{n+2}_{j+2}-39rU^{n+2}_{j+1}+(1+55r)U^{n+2}_j-33rU^{n+2}_{j-1}+6rU^{n+2}_{j-2}=$$
$$-2rU^{n+1}_j+12rU^{n+1}_{j-1}-12rU^{n+1}_{j-2}+2rU^{n+1}_{j-3}+$$

$$(3.5)\qquad 6rU^n_{j-2}+33rU^n_{j-1}+(1-55r)U^n_j+39rU^n_{j+1}-12rU^n_{j+2}+rU^n_{j+3},$$
$$-rU^{n+2}_{j+3}+12rU^{n+2}_{j+2}-33rU^{n+2}_{j+1}+(1+28r)U^{n+2}_j-6rU^{n+2}_{j-1}=$$
$$12rU^{n+1}_{j+1}-56rU^{n+1}_j+66rU^{n+1}_{j-1}-24rU^{n+1}_{j-2}+2rU^{n+1}_{j-3}+$$

$$(3.6)\qquad rU^n_{j+3}-12rU^n_{j+2}+33rU^n_{j+1}+(1-28r)U^n_j+6rU^n_{j-1},$$
$$-6rU^{n+2}_{j+1}+(1+28r)U^{n+2}_j-33rU^{n+2}_{j-1}+12rU^{n+2}_{j-2}-rU^{n+2}_{j-3}=$$
$$2rU^{n+1}_{j+3}-24rU^{n+1}_{j+2}+66rU^{n+1}_{j+1}+-56rU^{n+1}_j+12rU^{n+1}_{j-1}+$$

$$(3.7)\qquad 6rU^n_{j+1}+(1-28r)U^n_j+33rU^n_{j-1}-12rU^n_{j-2}+rU^n_{j-3}.$$

From the Taylor series expansion at (x_j, t^{n+1}), we obtain the following truncation error expressions for formulae (3.2)-(3.7)

$$(3.8)\qquad T_{3.2}=3r\tau h[\frac{\partial^3 u}{\partial x\partial t^2}]^{n+1}_j+\frac{9}{2}r\tau h^2[\frac{\partial^4 u}{\partial x^2\partial t^2}]^{n+1}_j+o(\tau+h^4),$$

$$(3.9)\qquad T_{3.3}=-3r\tau h[\frac{\partial^3 u}{\partial x\partial t^2}]^{n+1}_j-\frac{9}{2}r\tau h^2[\frac{\partial^4 u}{\partial x^2\partial t^2}]^{n+1}_j+o(\tau+h^4),$$

$$(3.10)\qquad T_{3.4}=-3r\tau h[\frac{\partial^3 u}{\partial x\partial t^2}]^{n+1}_j+\frac{9}{2}r\tau h^2[\frac{\partial^4 u}{\partial x^2\partial t^2}]^{n+1}_j+o(\tau+h^4),$$

$$(3.11)\qquad T_{3.5}=3r\tau h[\frac{\partial^3 u}{\partial x\partial t^2}]^{n+1}_j-\frac{9}{2}r\tau h^2[\frac{\partial^4 u}{\partial x^2\partial t^2}]^{n+1}_j+o(\tau+h^4),$$

$$(3.12)\qquad T_{3.6}=-6r\tau h[\frac{\partial^3 u}{\partial x\partial t^2}]^{n+1}_j+o(\tau+h^4),$$

$$(3.13)\qquad T_{3.7}=6r\tau h[\frac{\partial^3 u}{\partial x\partial t^2}]^{n+1}_j+o(\tau+h^4).$$

On the same time level, the schemes (3.2) and (3.3), (3.4) and (3.5), (3.6) and (3.7) are used symmetrically in the space direction, respectively, the signs terms with the parameter h in (3.8) and (3.9) are opposite, and the signs of the terms with the parameter h in the (3.10) and (3.11), (3.12) and (3.13), are also opposite. Thus the effect of the terms with h in the errors can be canceled. Very similar discussion for the explicit scheme and the implicit scheme can be carried out. Therefore, the truncation error of the new ASE-I method is approximately $o(h^4)$ in space.

3.3. **The analysis of the unconditional stability.** To prove the stability, we have to introduce the following Kellogg Lemma [5].

Lemma 3.1. *If $\rho > 0$, $C + C^T$ is nonnegative definite, then $(I + \rho C)^{-1}$ exists and there holds*

$$||(I + \rho C)^{-1}||_2 \leq 1.$$

Lemma 3.2. *Under the conditions of Lemma 3.1, the following inequality holds*

$$||(I - \rho C)(I + \rho C)^{-1}||_2 \leq 1.$$

Lemma 3.3. *For any real number r, and the symmetric non-negative matrices G_1 and G_2, matrices rG_1 and rG_2 are both symmetric and non-negative definite.*

Theorem 3.4. *For any real number r, the new ASE-I method (3.1) is unconditionally stable.*

Proof. By eliminating U^{n+1} from (3.1), we obtain $U^{n+2} = GU^n$, where G is the growth matrix

$$G = (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}(I - rG_2).$$

For any even number n, there holds

$$\begin{aligned} G^n \quad = \quad & (I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1} \\ & \cdot[(I - rG_2)(I + rG_2)^{-1}(I - rG_1)(I + rG_1)^{-1}]^{n-1}(I - rG_2). \end{aligned}$$

Since G_1 and G_2 are all symmetric, for any real number r, we can obtain the following inequality from the Kellogg Lemma

$$\begin{aligned} \|G^n\|_2 \quad \leq \quad & \|(I + rG_2)^{-1}\|_2 \cdot \|(I - rG_1)(I + rG_1)^{-1}\|_2^n \\ & \cdot\|(I - rG_2)(I + rG_2)^{-1}\|_2^{n-1} \cdot \|(I - rG_2)\|_2. \end{aligned}$$

Hence

$$\|G^n\|_2 \leq \|(I - rG_2)\|_2 \leq \sqrt{\|(I - rG_2)\|_\infty \cdot \|(I - rG_2)\|_1} \leq \sqrt{1 + 160r}$$

This shows that the new ASE-I method is unconditionally stable. □

4. Numerical experiments

In this section, we perform numerical experiments for (1.1). Let $u_0(x) = \sin x$, $\alpha = 1$, and $L = \pi$. The exact solution of this problem is

$$u(x, t) = e^{-t} \sin x.$$

We first illustrate the convergence rates in space for the schemes (3.1). Let $v_j^n = u(x_j, t^n)$ be the exact solution of the problem (1.1) and u_j^n be the approximate solution. We introduce the following $L_\infty-$norm error and L_2-norm error

$$E_{\infty,h} = \max_j |v_j^n - u_j^n|, \; E_{2,h} = \Big(\sum_j |v_j^n - u_j^n|^2 h\Big)^{\frac{1}{2}}.$$

Thus, we can calculate the rates of convergence by the following definitions

$$rate = \frac{\log(E_{\infty,h_1}/E_{\infty,h_2})}{\log(h_1/h_2)},\ rate = \frac{\log(E_{2,h_1}/E_{2,h_2})}{\log(h_1/h_2)}.$$

where h_1 and h_2 are the space mesh steps.

Let 'NASE-I' represents the new high-accuracy ASE-I method described above, and 'ASE-I' represents the ASE-I scheme, 'AGE4' represents the AGE 4-points scheme, and 'AGE8' represents the AGE 8-points scheme in [4], respectively. We choose $l = 4$ in the NASE-I method and $l = 6$ in the ASE-I method. For the NASE-I, ASE-I, AGE4, AGE8 methods, we give the $L_\infty-$norm errors, L_2-norm errors and the convergence rates at the different time t in Tables 1-3, respectively. We can see from these tables that the convergence rate of the new ASE-I method appears to be $\mathrm{O}(h^4)$ in space, which is coincident with our theoretical analysis, while the ASE-I, AGE4 and AGE8 methods appear to be $\mathrm{O}(h^2)$ in space in [4]. Because of the particularity of the experimental example, although the boundary formulation could reduce the accuracy, it does not affect the $\mathrm{O}(h^4)$ in space

Table 1. The convergence rate of the NASE-I at $t = 0.1, \tau = 10^{-8}$

h	$\pi/11$	$\pi/25$	$\pi/39$	$\pi/53$	$\pi/67$
L_∞	1.7173E-5	6.5534E-7	1.1165E-7	3.2737E-8	1.3096E-8
Rate	–	3.9781	3.9799	3.9998	3.9087
L_2	2.1744E-5	8.2297E-7	1.4005E-7	4.1046E-8	1.6411E-8
Rate	–	3.9881	3.9824	4.0013	3.9110

Table 2. The convergence rate of the NASE-I at $t = 0.01, \tau = 10^{-8}$

h	$\pi/11$	$\pi/25$	$\pi/39$	$\pi/53$	$\pi/67$
L_∞	1.8790E-6	7.1706E-8	1.2217E-8	3.5829E-9	1.4350E-9
Rate	–	3.9781	3.9798	3.9991	3.9036
L_2	2.3792E-6	9.0047E-8	1.5324E-8	4.4911E-9	1.7957E-9
Rate	–	3.9881	3.9824	4.0013	3.9108

Next, we compare the errors for the NASE-I with the ASE-I, AGE4 and the AGE8 methods at the different time t in Tables 4–5, respectively, where the absolute error $ae = |u_j^n - u(x_j,t^n)|$, the relative error $pe = \dfrac{|u_j^n - u(x_j,t^n)|}{|u(x_j,t^n)|} \times \%$, and 'Exact' represents the values of the exact solution $u(x_j,t^n)$. The results show that the NASE-I method is more accurate than the ASE-I, AGE4 and AGE8 methods in [4]. In addition, from Figures 5–8, we can see clearly that the

TABLE 3. The convergence rate of the NASE-I at $t = 0.001, \tau = 10^{-8}$

h	$\pi/11$	$\pi/25$	$\pi/39$	$\pi/53$	$\pi/67$
L_∞	1.8690E-7	7.2354E-9	1.2330E-9	3.6146E-10	1.4456E-10
Rate	–	3.9781	3.9793	4.0004	3.9098
L_2	2.1744E-5	8.2297E-7	1.4005E-7	4.1046E-8	1.6411E-8
Rate	–	3.9881	3.9825	4.0011	3.9108

TABLE 4. The comparison for the results at $J = 24, \tau = 10^{-5}, t = 1$.

scheme	error	j=4	j=7	j=10	j=12	j=15	j=21
NASE-I	$ae(10^{-8})$	2.0654	7.3191	9.7617	9.9138	9.7616	2.0654
	$pe(10^{-7})$	1.1654	2.5821	2.7901	2.8635	2.7900	1.1654
ASE-I	$ae(10^{-4})$	4.6684	7.4609	9.2091	9.6638	9.2091	4.6648
	$pe(10^{-3})$	2.6321	2.6321	2.6321	2.6321	2.6321	2.6321
AGE4	$ae(10^{-4})$	4.6643	7.4601	9.2081	9.6629	9.2081	4.6643
	$pe(10^{-3})$	2.6318	2.6318	2.6318	2.6318	2.6318	2.6318
AGE8	$ae(10^{-4})$	4.6649	7.4609	9.2092	9.6640	9.2092	4.6649
	$pe(10^{-3})$	2.6231	2.6231	2.6231	2.6231	2.6231	2.6231
Exact	(10^{-1})	1.7723	2.8346	3.4987	3.6715	3.4987	1.7723

TABLE 5. The comparison for the results at $J = 80, \tau = 10^{-7}, t = 0.01$.

scheme	error	j=10	j=25	j=30	j=40	j=55	j=70
NASE-I	$ae(10^{-9})$	3.6985	4.9936	5.0853	5.3752	5.0002	3.8462
	$pe(10^{-9})$	9.8775	6.1161	5.5939	5.4303	5.9699	9.3879
ASE-I	$ae(10^{-6})$	1.0864	2.0467	2.2789	2.4814	2.0996	1.0270
	$pe(10^{-6})$	2.5068	2.5068	2.5068	2.5068	2.5068	2.5068
AGE4	$ae(10^{-6})$	1.0864	2.0467	2.2789	2.4814	2.0996	1.0270
	$pe(10^{-6})$	2.5068	2.5068	2.5068	2.5068	2.5068	2.5068
AGE8	$ae(10^{-6})$	1.0863	2.0468	2.2789	2.4814	2.1000	1.0270
	$pe(10^{-6})$	2.5069	2.5069	2.5069	2.5069	2.5069	2.5069
Exact	(10^{-1})	3.7444	8.1647	9.0908	9.8986	8.3757	4.0969

NASE-I solutions are much more accurate than the ASE-I, AGE4 and AGE8 solutions.

Third, we verified the stability of the NASE-I method. From Tables 6 and 7, we can easily find that the high-accuracy NASE-I method is unconditionally stable.

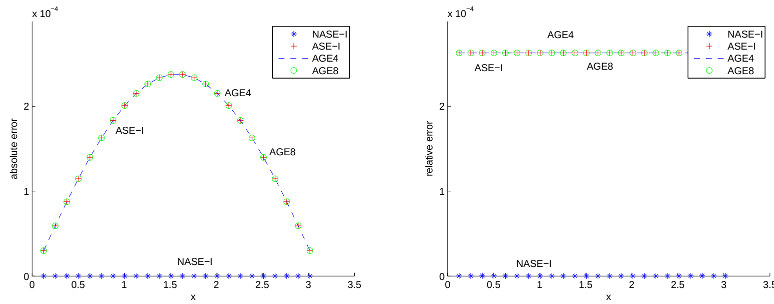

FIGURE 5. Comparison of the errors, $t = 0.1, h = \pi/25, \tau = 10^{-5}$

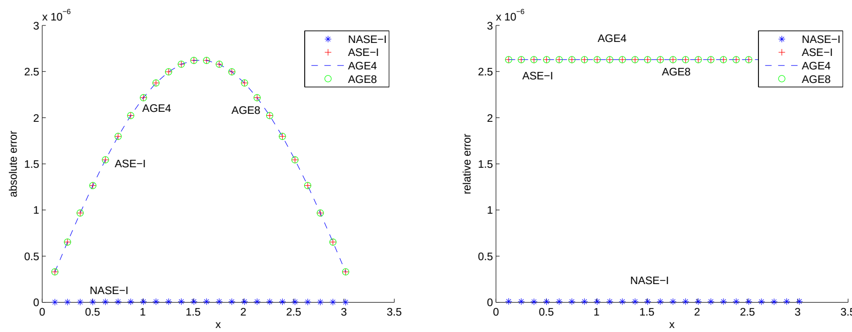

FIGURE 6. Comparison of the errors, $t = 0.001, h = \pi/25, \tau = 10^{-6}$

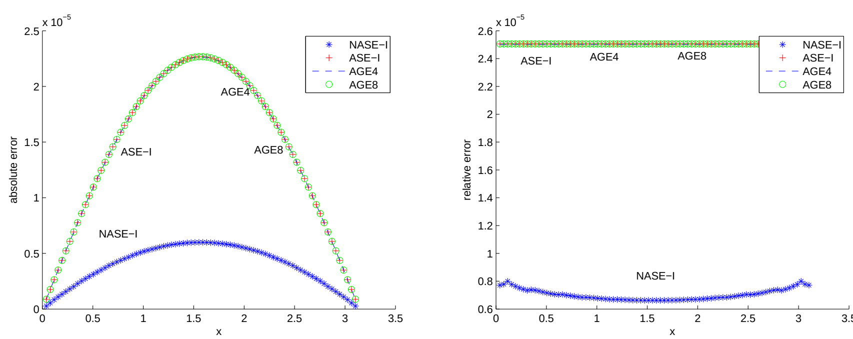

FIGURE 7. Comparison of the errors, $t = 0.1, h = \pi/81, \tau = 10^{-6}$

Table 6. The errors of numerical solution at $J = 10, \tau = 10^{-7}, t = 0.001, r = \tau/(6h^4)$.

r	error	j=2	j=4	j=6	j=8	j=10
$r_1 = r$	$ae(10^{-7})$	1.0356	1.7423	1.8959	1.4416	5.3963
	$re(10^{-7})$	1.9173	1.9173	1.9173	1.9173	1.9173
$r_2 = 10r$	$ae(10^{-7})$	1.0357	1.7424	1.8959	1.4473	5.3968
	$re(10^{-7})$	1.9176	1.9173	1.9173	1.9170	1.9175
$r_3 = 100r$	$ae(10^{-7})$	1.0509	1.7479	1.8977	1.4228	5.4423
	$re(10^{-7})$	1.9458	1.9235	1.9191	1.8847	1.9337

Table 7. The errors of numerical solution at $J = 94, \tau = 10^{-6}, t = 0.1, r = \tau/(6h^4)$.

r	error	j=15	j=30	j=45	j=60	j=75
$r_1 = r$	$ae(10^{-5})$	7.8778	1.3216	1.5449	1.4321	9.9860
	$re(10^{-5})$	1.8293	1.7446	1.7132	1.7283	1.7968
$r_2 = 5r$	$ae(10^{-6})$	1.9681	3.3016	3.8594	3.5777	2.4948
	$re(10^{-6})$	4.1700	4.3585	4.2800	4.3177	4.4889
$r_3 = 10r$	$ae(10^{-7})$	7.7257	1.2901	1.5131	1.4029	9.7897
	$re(10^{-7})$	1.7940	1.7094	1.6779	1.6930	1.7615

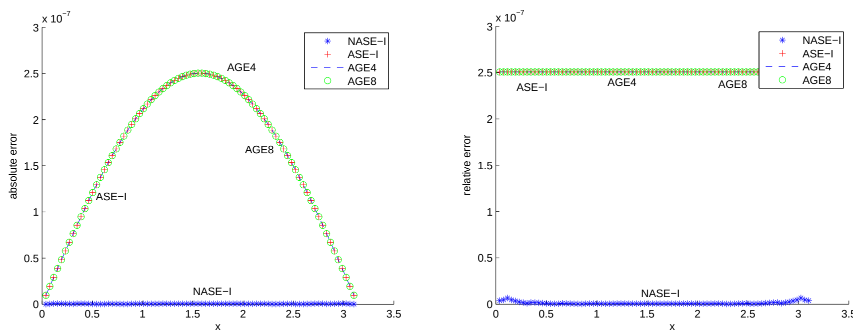

Figure 8. Comparison of the errors, $t = 0.001, h = \pi/81, \tau = 10^{-7}$

At last, let's give a brief discussion on parallelism of the NASE-I method. The major difficulty to solve partial differential equations on massive parallel computers is how to compute the values on the sub-domain boundaries. In this paper, we have constructed a group of asymmetric difference schemes (2.3)−(2.8), when we compute the interface values by the asymmetric difference schemes, the global domain of definition is divided into some small independent segments (see Figure 4), and can be computed in parallel, the parallelism is clarity.

5. Conclusion

We have constructed a new kind of alternating parallel difference scheme with unconditional stability and fourth-order accuracy for the fourth-order heat equation. The design of the method is new and simple as well. The theoretical analysis and the numerical experiments indicate that the high-accuracy ASE-I method constructed in this paper is more accurate than the known alternating schemes, and it can be extended to apply for nonlinear equations and two dimensional problems.

Acknowledgements

This work is jointly supported by the Tianjin Research Program of Application Foundation and Advanced Technology (Youth Research Program) (Grant No: 15JCQNJC01600), the Youth Foundation of Tianjin University of Technology and Education (Grant No: KJ15-13), the Foundation of Tianjin Municipal Education Commission (Grant No: JWK1603), the Research Abroad Fund of Tianjin University of Technology and Education (Grant No:J10011060321) and the National Natural Science Foundation of China (Grant Nos. 11272024 and 11526157).

The authors would like to thank the reviewers and editors of this paper for their valuable comments and suggestions.

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