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APPLICATIONS OF CONVOLUTION AND SUBORDINATION TO CERTAIN p-VALENT FUNCTIONS

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ABSTRACT. In this paper we consider some new classes of multivalent functions by using Aouf-Silverman-Srivastava operator and we derive some interesting results using convolution and subordination technique. These new classes are the extensions of some classes introduced before.
Keywords: Multivalent function, convolution, subordination, linear operator.
MSC(2010): Primary: 30C45; Secondary: 30C50.

1. Introduction

Let $\mathcal{A}(p)$, $p = 1, 2, 3, \ldots$, be the class of $p-$valent analytic functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \tag{1.1}$$

defined in the open unit disc $U = \{z : |z| < 1\}$.

We say $f \in \mathcal{A}(p)$ is subordinate to $g \in \mathcal{A}(p)$, written $f \prec g$, if and only if there exists a Schwarz function w, $w(0) = 0$ and $|w(z)| < 1$ in U such that $f(z) = g(w(z))$. The classes $\mathcal{S}^*$ and $\mathcal{K}$ of starlike and convex functions consist of all functions in $\mathcal{A}(1)$ such $\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$, $\frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}$, $z \in U$, respectively. Ma and Minda type, [15], starlike and convex functions are given by

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A}(1) : \frac{zf'(z)}{f(z)} \prec \varphi(z),\ z \in U \right\}, \tag{1.2}$$

$$\mathcal{K}(\varphi) = \left\{ f \in \mathcal{A}(1) : \left[1 + \frac{zf''(z)}{f'(z)}\right] \prec \varphi(z),\ z \in U \right\}, \tag{1.3}$$

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where φ is analytic in U with $\varphi(0) = 1$. For $\varphi(z) = \frac{1+z}{1-z}$ (1.2) and (1.3) reduce to $\mathcal{S}^*$ and $\mathcal{K}$ respectively. For different choice φ we can obtain some other well known classes investigated earlier.

For $p-$valent analytic functions f given by (1.1) and $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$, the Hadamard product (convolution) is defined as;

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p}.$$

The study of these operators play an important role in the geometric function theory. Many differential and integral operators can be written in terms of convolution of certain analytic functions. Libera [14] introduced an integral operator and showed that the $\mathcal{S}^*$ and $\mathcal{K}$ classes are closed under this operator. Bernardi [3] gave a generalized operator and studied its properties. Ruscheweyh [19], Noor and Noor [17, 18], Noor [16] and many others, for example, [4, 13], defined new operators and studied various classes of analytic and univalent functions generalizing a number of previously known classes and at times discovering new classes of analytic functions. Sokół introduced and studied certain classes of p-valent functions using Aouf-Silverman-Srivastava operator and derived some interesting results in [20], in the present paper we extend this work, by using convolution and subordination technique.

Next we discuss an important operator as;

Let $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1,\ B_1, \ldots, \beta_s,\ B_s$ $(q,\ s \in N)$ be positive real parameters such that

$$1 + \sum_{i=1}^{s} B_i - \sum_{i=1}^{q} A_i \geq 0.$$

The Wright generalized hypergeometric function

$${}_q\Psi_s\left[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s); z\right] = {}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right],$$

see [21], is defined by

$${}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] = \sum_{k=0}^{\infty} \frac{\Pi_{i=1}^{q} \Gamma(\alpha_i + kA_i)}{\Pi_{i=1}^{s} \Gamma(\beta_i + kB_i)} \frac{z^k}{k!}.$$

If $A_i = 1\,(i = 1, 2, \ldots, q)$ and $B_i = 1\,(i = 1, 2, \ldots, s)$, we have

$$\Omega_q \Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] = {}_q F_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

where ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is the generalized hypergeometric function (for details see [6, 8]) and

$$\Omega = \frac{\Pi_{i=1}^{q} \Gamma\left(\alpha_i\right)}{\Pi_{i=1}^{s} \Gamma\left(\beta_i\right)}. \tag{1.4}$$

In [2] Aouf considered the linear operator

$${}_q\theta_s^p \left[\left(\alpha_i, A_i\right)_{1,q}; \left(\beta_i \ B_i\right)_{1,s}\right] : \mathcal{A}\left(p\right) \to \mathcal{A}\left(p\right)$$

defined by

$${}_q\theta_s^p \left[\left(\alpha_i, A_i\right)_{1,q}; \left(\beta_i \ B_i\right)_{1,s}\right] f(z) =_q \phi_s^p \left[\left(\alpha_i, A_i\right)_{1,q}; \left(\beta_i \ B_i\right)_{1,s}; z\right] * f(z).$$

For $f(z)$ of the form (1.1), we have

$${}_q\theta_s^p \left[\left(\alpha_i, A_i\right)_{1,q}; \left(\beta_i \ B_i\right)_{1,s}\right] f(z) = z^p + \sum_{k=1}^{\infty} \Omega \sigma_{k,p}(\alpha_1) a_{k+p} z^{k+p}, \tag{1.5}$$

where Ω is given by (1.4) and

$$\sigma_{k,p}\left(\alpha_1\right) = \frac{\Gamma\left(\alpha_1 + A_1\left(k-p\right)\right) \cdots \Gamma\left(\alpha_p + A_p\left(k-p\right)\right)}{\Gamma\left(\beta_1 + \beta_1\left(k-p\right)\right) \cdots \Gamma\left(\beta_q + \beta_q\left(k-p\right)\right)\left(k-p\right)!}. \tag{1.6}$$

For some interesting special cases we refer to [1]. Kanas introduced the concept of $k-$uniform convexity and $k-$starlikeness of a function, using conic domain

$$\Omega_{k,\gamma} = \gamma \Omega_k + (1-\gamma)$$

where

$$\Omega_k = \left\{u + vi : u > k\sqrt{(u-1)^2 + v^2}\right\}, \ \ (\text{see } [10]).$$

The boundary $\partial\Omega_k$ of the above set becomes the imaginary axis when $k = 0$, a hyperbola when $0 < k < 1$, a parabola when $k = 1$, and an ellipse when $k > 1$. All of these curves have the vertex at the point $\frac{k}{k+1}$. Therefore the domain $\Omega_{k,\gamma}$ is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$ and right half plane when $k = 0$; ever symmetric with respect to the real axis. The functions which play the role of extremal functions for these conic regions are given as

(1.7)

$$q_{k,\gamma}(z) = \begin{cases} \frac{1+(1-2\gamma)z}{1-z}, \ k=0; \qquad 1 + \frac{2\gamma}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1+\sqrt{z}}\right)^2, & k = 1; \\ 1, & 0 < k < 1; \\ 1 + \frac{\gamma}{k^2-1} \sin\left(\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}} \mathrm{d}x\right) + \frac{\gamma}{1-k^2} & \text{k} > 1, \end{cases}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, and $t \in (0,1)$ is chosen such that $k = \cosh(\frac{\pi R'(t)}{R(t)})$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and

$R'(t) = \sqrt{1-t^2}$ is the complementary integral of $R(t)$. Moreover, $q_{k,\gamma}(U) = \Omega_{k,\gamma}$ and $q_{k,\gamma}(U)$ is convex univalent in U (see [11, 12]. Using the operator ${}_q\theta_s^p\left[(\alpha_i, A_i)_{1,q}; (\beta_i\ B_i)_{1,s}\right]$, we shall introduce two new classes $P(A_i, \alpha_i, B_j, \beta_j, \lambda, p, k, \gamma)$ and $Q\,(A_i, \alpha_i, B_j\,,\ \beta_j, \lambda, p, \rho)$ of p-valent functions.

Definition 1.1. A function $f \in \mathcal{A}\,(p)$ is said to be in the class $P(A_i, \alpha_i, B_j, \beta_j, \lambda, p, k, \gamma)$ if it satisfies the condition

$$\frac{1}{p-\lambda}\left[\frac{[{}_q\theta_s^p f(z)]'}{z^{p-1}} - \lambda\right] \prec q_{k,\gamma}(z),\ \ (z \in U), \tag{1.8}$$

where

(1.9)

$${}_q\theta_s^p f(z) := [{}_q\theta_s^p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}]f(z) = z^p + \sum_{k=1}^{\infty}\Omega\sigma_{k,p}(\alpha_1)a_{k+p}z^{k+p}.$$

Definition 1.2. A function $f \in \mathcal{A}\,(p)$ is said to be in the class $Q\,(A_i, \alpha_i, B_j, \beta_j, \lambda, p, \rho)$ if it satisfies the condition

$$\frac{1}{p-\lambda}\left[\frac{[{}_q\theta_s^p f(z)]'}{z^{p-1}} - \lambda\right] \prec \frac{1+[(1-\rho)A+\rho B]\,z}{1+Bz} \tag{1.10}$$

or, equivalently if

$$\left|\frac{\frac{({}_q\theta_s^p f(z))'}{z^{p-1}} - 1}{B\left(\frac{({}_q\theta_s^p f(z))'}{z^{p-1}}\right) - (pB + (1-\rho)\,(A-B))\,(p-\lambda)}\right| < 1, \tag{1.11}$$

where $-1 \le A < B \le 1,\ \ 0 \le \rho \le 1$.

For $\alpha_i = 1$, $\beta_i = 1$, $A_1 = a$, $B_1 = c$ and $\rho = 0$, the class $Q\,(A_i, \alpha_i, B_j, \beta_j, \lambda, p, \rho)$ was studied in [20].

2. Main results

Theorem 2.1. *Let $0 \le \lambda < p$, $\gamma \in \mathbb{C}\backslash\{0\}$ and $0 < k \le 1$. A function $f \in \mathcal{A}(p)$ is in the class $P(A_i, \alpha_i, B_j, \beta_j, \lambda, p, k, \gamma)$ if and only if*

$$\frac{\left(f * \widehat{\phi}_p(A_1, B_1, \lambda)\right)}{z^p} + \frac{z}{p}\left[\frac{\left(f * \widehat{\phi}_p(A_1, B_1, \lambda)\right)(z)}{z^p}\right]' \prec q_{k,\gamma}(z) \quad (z \in U), \tag{2.1}$$

where

$$\widehat{\phi}_p(A_1, B_1, \lambda) = z^p + \frac{p}{p-\lambda}\sum_{k=1}^{\infty}\Omega\sigma_{k,p}(\alpha_1)z^{k+p}. \tag{2.2}$$

Proof. Let $f \in \mathcal{A}(p)$. Then it is enough to show that

$$\frac{1}{p-\lambda}\left[\frac{[{}_q\theta_s^p f(z)]'}{z^{p-1}}-\lambda\right]=\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)}{z^p}+\frac{z}{p}\left[\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}\right]'.$$

Therefore,

$$\begin{aligned}
&\frac{1}{p-\lambda}\left[\frac{[{}_q\theta_s^p[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s}]f(z)]'}{z^{p-1}}-\lambda\right]\\
=\;&\frac{1}{p-\lambda}\left[\frac{(z^p+\sum\limits_{k=1}^{\infty}\Omega\sigma_{k,p}(\alpha_1)a_{k+p}z^{k+p})'}{z^{p-1}}-\lambda\right]\\
=\;&\frac{f(z)}{z^p}*\left\{1+\frac{1}{p-\lambda}\sum_{k=1}^{\infty}(k+p)\Omega\sigma_{k,p}(\alpha_1)z^k\right\}\\
=\;&\frac{f(z)}{z^p}*\left\{\frac{\widehat{\phi}_p(A_1,B_1,\lambda)}{z^p}+\frac{z}{p}\left(\frac{\widehat{\phi}_p(A_1,B_1,\lambda)}{z^p}\right)'\right\}\\
=\;&\frac{[f*\widehat{\phi}_p(A_1,B_1,\lambda)](z)}{z^p}+\frac{z}{p}\left[\frac{[f*\widehat{\phi}_p(A_1,B_1,\lambda)](z)}{z^p}\right]'.
\end{aligned}$$

This completes the proof. □

Now, We need the following important Lemma for our next investigation.

Lemma 2.2. *Let h be an analytic and convex univalent function in U. Let f be analytic in U with $h(0)=f(0)=1$. If $\gamma\neq 0$, $\mathfrak{Re}\gamma\geq 0$ and*

$$f(z)+\frac{zf'(z)}{\gamma}\prec h(z)\quad (z\in U),$$

then

$$f(z)\prec q(z)\prec h(z),$$

where

$$q(z)=\frac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1}h(t)dt.$$

Moreover the function $q(z)$ is convex univalent and it is the best dominant of the above in the sense that if there exist a function q_1 such that $f\prec q_1$, then also $q\prec q_1$.

Lemma 2.2 is due to Hallenbeck and Ruscheweyh [9].

Theorem 2.3. *Let* $f \in P(A_i, \alpha_i, B_j, \beta_j, \lambda, p, k, \gamma)$*. Then*

$$\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p} \prec g(z) = \frac{p}{z^p}\int_0^z t^{p-1}h(t)dt \prec h(z) \ (z \in U). \tag{2.3}$$

This result is best possible.
Moreover the function $g(z)$ *is convex univalent and it is the best dominant.*

Proof. If $f \in P(A_i, \alpha_i, B_j, \beta_j, \lambda, p, k, \gamma)$ then by using Theorem 2.1, we have

$$\frac{\left(f * \widehat{\phi}_p(A_1, B_1, \lambda)\right)}{z^p} + \frac{z}{p}\left[\frac{\left(f * \widehat{\phi}_p(A_1, B_{1,}\lambda)\right)(z)}{z^p}\right]' \prec q_{k,\gamma}(z), \ (z \in U),$$

also $q_{k,\gamma}(z)$ is convex univalent (see [11, 12]) so, by Lemma 2.2, we obtain

$$\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p} \prec g(z) = \frac{p}{z^p}\int_0^z t^{p-1}h(t)dt \prec h(z) \ (z \in U).$$

Hence, we have the required result. □

Theorem 2.4. *Let* $f \in \mathcal{A}(p)$*. Then*

$$\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda+1)](z)}{z^p} + \frac{z}{p}\left[\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda+1)](z)}{z^p}\right]' \prec q_{k,\gamma}(z) \tag{2.4}$$

if and only if for $z \in U$

$$\left\{\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p} + \frac{z}{p}\left[\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p}\right]'\right\}$$

$$+\frac{z}{\xi}\left\{\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p} + \frac{z}{p}\left[\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p}\right]'\right\}' \prec q_{k,\gamma}(z). \tag{2.5}$$

Proof. We want to show the equality of left-hand sides of (2.4) and (2.5). Notice that

$$\widehat{\phi}_p(A_1, B_1, \lambda+1) = \widehat{\phi}_p(A_1, B_1, \lambda) * \left\{z^p + \sum_{k=1}^{\infty}\frac{\xi+k}{\xi}z^{k+p}\right\}.$$

Consider

$$
\begin{aligned}
&\frac{[f*\widehat{\phi}_p(A_1,B_1,\lambda+1)](z)}{z^p}+\frac{z}{p}\left[\frac{[f*\widehat{\phi}_p(A_1,B_1,\lambda+1)](z)}{z^p}\right]' \\
= \quad &\frac{\left[f*\left(\widehat{\phi}_p(A_1,B_1,\lambda)*\left\{z^p+\sum\limits_{k=1}^{\infty}\frac{\xi+k}{\xi}z^{k+p}\right\}\right)\right](z)}{z^p} \\
&+\frac{z}{p}\left[\frac{\left[f*\left(\widehat{\phi}_p(A_1,B_1,\lambda)*\left\{z^p+\sum\limits_{k=1}^{\infty}\frac{\xi+k}{\xi}z^{k+p}\right\}\right)\right](z)}{z^p}\right]' \\
= \quad &\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}*\left\{1+\sum_{k=1}^{\infty}\frac{\xi+k}{\xi}z^k\right\} \\
&+\frac{1}{p}\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}*\left\{\sum_{k=1}^{\infty}\frac{k(\xi+k)}{\xi}z^k\right\} \\
= \quad &\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}*\left\{1+\sum_{k=1}^{\infty}\left(1+\frac{k}{p}+\frac{k}{\xi}+\frac{k^2}{\xi p}\right)z^k\right\} \\
= \quad &\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}*\left\{1+\sum_{k=1}^{\infty}z^k+\frac{1}{p}\sum_{k=1}^{\infty}kz^k\right\} \\
&+\frac{1}{\xi}\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}*\left\{\sum_{k=1}^{\infty}kz^k+\frac{1}{p}\sum_{k=1}^{\infty}k^2z^k\right\} \\
= \quad &\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}+\frac{z}{p}\left[\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}\right]' \\
&+\frac{z}{\xi}\left\{\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}+\frac{z}{p}\left[\frac{\left(f*\widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}\right]'\right\}'.
\end{aligned}
$$

$\square$

Using the steps as outlines in Theorem 2.1 and Theorem 2.3 the following results can be easily obtained.

Theorem 2.5. *A function $f \in \mathcal{A}(p)$ is in the class $Q(A_i, \alpha_i, B_j, \beta_j, \lambda, p, \rho)$ if and only if*

(2.6)
$$\frac{\left(f * \widehat{\phi}_p(A_1, B_1, \lambda)\right)}{z^p} + \frac{z}{p}\left[\frac{\left(f * \widehat{\phi}_p(A_1, B_1, \lambda)\right)(z)}{z^p}\right]' \prec \frac{1 + [(1-\rho)A + \rho B]\, z}{1 + Bz}.$$

Theorem 2.6. *Let $f \in \mathcal{A}(p)$ and h be a convex univalent function. If*

$$\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p} + \frac{z}{p}\left[\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p}\right]' \prec h(z),$$

then

$$\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p} \prec g(z) = \frac{p}{z^p}\int_0^z t^{p-1} h(t)dt \prec h(z) \ (z \in U). \tag{2.7}$$

Moreover the function $g(z)$ given by (2.7) is convex univalent and it is the best dominant.

Corollary 2.7. *Let $f \in \mathcal{A}(p)$ is in the class $Q(A_i, \alpha_i, B_j, \beta_j, \lambda, p, \rho)$ and $\widehat{\phi}_p$ is given by (2.2), then*

$$\frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p} \prec g_m(z) \prec \frac{1 + [(1-\rho)A + \rho B]\, z}{1 + Bz}(z \in U),$$

where

$$\begin{aligned} g_m(z) &= 1 + \frac{p}{p+1}\left[\{(1-\rho)A + \rho B\} - B\right] z \\ &\quad + p\left[\{(1-\rho)A + \rho B\} - B\right] \sum_{k=1}^{\infty} \frac{(-B)}{p+k} z^k (z \in U). \end{aligned}$$

Proof. From Theorem 2.6, substituting

$$h(t) = \frac{1 + [(1-\rho)A + \rho B]\, t}{1 + Bt}$$

in (*2.7*) we obtain

$$\begin{aligned} & \frac{[f * \widehat{\phi}_p(A_1, B_1, \lambda)](z)}{z^p} \prec g_m(z) \\ = & \ \frac{p}{z^p}\int_0^z t^{p-1} \frac{1 + [(1-\rho)A + \rho B]\, t}{1 + Bt} \mathrm{d}t \prec \frac{1 + [(1-\rho)A + \rho B]\, z}{1 + Bz}. \end{aligned}$$

For $B \neq 0$ the function $g_m(z)$ becomes

$$g_m(z) = 1 + p\left[\{(1-\rho)A + \rho B\} - B\right] \sum_{k=1}^{\infty} \frac{(-B)^{k-1}}{p+k} z^k.$$

If we consider the function f_p such that $\left(f_p * \widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)=z^p g_m(z)$, then we have

$$f_p(z) = z^p + (p-\lambda)\left[\{(1-\rho)A+\rho B\} - B\right] \sum_{k=1}^{\infty} \frac{(-B)^{k-1}}{(p+k)\Omega\sigma_{k,p}(\alpha_1)} z^{k+p}.$$

□

Corollary 2.8. *If* $\lambda = 0$ *or* $\mathfrak{Re}(\lambda) > 0$, *then*

$$Q\left(A_i,\alpha_i,B_j,\beta_j,\lambda+1,p,\rho\right) \subset Q\left(A_i,\alpha_i,B_j,\beta_j,\lambda,p,\rho\right). \tag{2.8}$$

Moreover if $f \in Q\left(A_i,\alpha_i,B_j,\beta_j,\lambda,p,\rho\right)$ *and* $\lambda \neq 0$, *then*

$$\frac{\left(f * \widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p} + \frac{z}{p}\left[\frac{\left(f * \widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}\right]' \prec g_\lambda(z)$$

$$\prec \frac{1+\left[(1-\rho)A+\rho B\right]z}{1+Bz} \quad (z \in U), \tag{2.9}$$

where

$$\begin{aligned} g_\lambda(z) &= 1 + \frac{\lambda}{\lambda+1}\left[\{(1-\rho)A+\rho B\} - B\right] z \\ &\quad + \lambda\left[\{(1-\rho)A+\rho B\} - B\right] \sum_{k=2}^{\infty} \frac{(-B)^{k-1}}{\lambda+k} z^k. \end{aligned}$$

Moreover, the function $g_\lambda(z)$ *is convex univalent and it is the best dominant of* (2.9).

Proof. The inclusion (2.8) is trivial for $\lambda = 0$. Now let $f \in Q\left(A_i,\alpha_i,B_j,\beta_j,\lambda+1,p,\rho\right)$ with $\lambda \neq 0$, so we have

$$\frac{\left(f * \widehat{\phi}_p(A_1,B_1,\lambda+1)\right)(z)}{z^p} + \frac{z}{p}\left[\frac{\left(f * \widehat{\phi}_p(A_1,B_1,\lambda+1)\right)(z)}{z^p}\right]' \prec \frac{1+\left[(1-\rho)A+\rho B\right]z}{1+Bz}. \tag{2.10}$$

Using the steps as outlined in the proof of Theorem 2.4, and some properties of convolution, we obtain

$$\begin{aligned} &\frac{\left(f * \widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p} + \frac{z}{p}\left[\frac{\left(f * \widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}\right]' \\ &+ \frac{z}{\xi}\left\{\frac{\left(f * \widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p} + \frac{z}{p}\left[\frac{\left(f * \widehat{\phi}_p(A_1,B_1,\lambda)\right)(z)}{z^p}\right]'\right\}' \prec \frac{1+\left[(1-\rho)A+\rho B\right]z}{1+Bz}. \end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
&\frac{\left(f * \widehat{\phi}_p(A_1, B_1, \lambda)\right)(z)}{z^p} + \frac{z}{p}\left[\frac{\left(f * \widehat{\phi}_p(A_1, B_1, \lambda)\right)(z)}{z^p}\right]' \\
\prec \quad & g_\lambda(z) = \frac{\lambda}{z^\lambda}\int_0^z t^{\lambda-1}\left[\frac{1+[(1-\rho)A+\rho B]\,t}{1+Bt}\right] dt \\
\prec \quad & \frac{1+[(1-\rho)A+\rho B]\,z}{1+Bz},
\end{aligned}$$

where

$$g_\lambda(z) = 1+\frac{\lambda}{\lambda+1}\left[\{(1-\rho)A+\rho B\}-B\right]z+\lambda\left[\{(1-\rho)A+\rho B\}-B\right]\sum_{k=2}^{\infty}\frac{(-B)^{k-1}}{\lambda+k}z^k.$$

So $f \in Q\left(A_i, \alpha_i, B_j, \beta_j, \lambda, p, \rho\right)$. Finally,

$$Q\left(A_i, \alpha_i, B_j, \beta_j, \lambda+1, p, \rho\right) \subset Q\left(A_i, \alpha_i, B_j, \beta_j, \lambda, p, \rho\right).$$

□

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