Title:
The graph of equivalence classes and Isoclinism of groups

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THE GRAPH OF EQUIVALENCE CLASSES AND ISOCLINISM OF GROUPS

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(Communicated by Ali Reza Ashrafi)

Abstract. Let $G$ be a non-abelian group and let $\Gamma(G)$ be the non-commuting graph of $G$. In this paper we define an equivalence relation $\sim$ on the set of $V(\Gamma(G)) = G \setminus Z(G)$ by taking $x \sim y$ if and only if $N(x) = N(y)$, where $N(x) = \{u \in G \mid x$ and $u$ are adjacent in $\Gamma(G)\}$ is the open neighborhood of $x$ in $\Gamma(G)$. We introduce a new graph determined by equivalence classes of non-central elements of $G$, denoted $\Gamma_E(G)$, as the graph whose vertices are $\{[x] \mid x \in G \setminus Z(G)\}$ and join two distinct vertices $[x]$ and $[y]$, whenever $[x, y] \neq 1$. We prove that group $G$ is AC-group if and only if $\Gamma_E(G)$ is complete graph. Among other results, we show that the graphs of equivalence classes of non-commuting graph associated with two isoclinic groups are isomorphic.

Keywords: Non-commuting graph, graph of equivalence classes,  Isoclinism.


1. Introduction

Let $G$ be a group and $Z(G)$ be the center of $G$. The non-commuting graph $\Gamma(G)$ associated with $G$ is the graph whose vertex set is $G \setminus Z(G)$ and two distinct elements $x$ and $y$ are adjacent, denoted $x \sim y$, if and only if $[x, y] \neq 1$. According to [2] the non-commuting graph of a finite group $G$ was first considered by Paul Erdős in connection with the following problem. Let $G$ be a group whose non-commuting graph $\Gamma(G)$ has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of $\Gamma(G)$? B.H. Neumann [12] answered positively to this question. In [2] and [11], some graph theoretical properties of $\Gamma(G)$ and the relations between some properties of $\Gamma(G)$ and the structure of group $G$ were studied. Of course, there are some other ways to construct a graph associated with a given group.
The graph of equivalence classes

We may refer to the works of Bertram et al. [6] and Moghadamfar et al. [11] or recent papers on non-commuting graph, Engel graph and non-cyclic graph in [1, 2] and [4], respectively.

Two vertices \(a\) and \(b\) of a simple graph \(\Gamma\) are said to be equivalent, if their open neighborhoods are the same, i.e., \(a \sim b\) if and only if \(N(a) = N(b)\), where \(N(a) = \{e \in V(\Gamma) \mid a\) and \(c\) are adjacent in \(\Gamma\}\). One can see that \(\sim\) is an equivalence relation and we denote the class of \(a\) by \([a]\). The graph of equivalence classes of \(\Gamma\), denoted \(\Gamma_E\), is the graph associated with \(\Gamma\) whose vertex set is \([a]: a \in V(\Gamma)\) and two equivalence classes \([a]\) and \([b]\) are adjacent in \(\Gamma_E\) if \(a\) and \(b\) are adjacent in \(\Gamma\). In Section 2, we will introduce the graph of equivalence classes of the non-commuting graph \(\Gamma(G)\). We will state some of basic graph theoretical properties of \(\Gamma_E(G)\), for instance determining diameter, girth, dominating set, planarity of the graph and we give some relation between the graph properties of \(\Gamma(G)\) and \(\Gamma_E(G)\). In Section 3 of the paper, we state a connection between the graph of equivalence classes of the non-commuting graph and isoclinism of groups. We prove that the graphs of equivalence classes of two isoclinic groups are isomorphic. Moreover, we show that for any group \(G\) with \(\Gamma_E(G) < \infty\), there is a finite group \(K\) such that \(\Gamma_E(G) \cong \Gamma_E(K)\).

2. Definitions and basic results

Let \(\Gamma(G)\) be the non-commuting graph of a non-abelian group \(G\). For \(x, y \in G \setminus Z(G)\), we say that \(x \sim y\) if and only if \(G \setminus C_G(x) = N(x) = G \setminus C_G(y)\) if and only if \(C_G(x) = C_G(y)\), where \(N(x) = \{u \in G \mid x\) and \(u\) are adjacent in \(\Gamma(G)\}\). It is easy to see that \(\sim\) is an equivalence relation and we denote the class of \(x\) by \([x]\).

**Definition 2.1.** The graph of equivalence classes of \(\Gamma(G)\), denoted \(\Gamma_E(G)\), is the graph associated with \(G\) with vertex set \([x]: x \in G \setminus Z(G)\) such that two distinct vertices \([x]\) and \([y]\) are joined by an edge, denoted \([x] - [y]\), if and only if \([x, y] \neq 1\).

It is easy to check \([x] \mapsto C_G(x)\) establishes a one-to-one correspondence between \(V(\Gamma_E(G))\) and the set of all proper centralizers of group \(G\). Hence \(|V(\Gamma_E(G))| = \#Cent(G) - 1\), where \(Cent(G)\) denote the set of centralizers of single elements of \(G\) and \(#Cent(G)\) is the size of \(Cent(G)\).

Recall that a clique of a graph is a set of mutually adjacent vertices, and that the maximum size of a clique of a graph \(\Gamma\), the clique number of \(\Gamma\), is denoted \(\omega(\Gamma)\). Moreover, a clique of a graph \(\Gamma\) is called a maximum clique if its size is \(\omega(\Gamma)\).

**Lemma 2.2.** Assume that \(A = \{[x]: C_G(x)\ is\ an\ abelian\ group\}\). Then \(A\) is a clique in \(\Gamma_E(G)\).
Proof. By the structure of $\mathcal{A}$, it will be enough to prove the induced subgraph on $\mathcal{A}$ is a complete graph. Hence, suppose that $[x]$ and $[y]$ are two distinct elements of $\mathcal{A}$. We claim that $[x]$ and $[y]$ are adjacent in $\Gamma_E(G)$, or equivalently, $[x, y] \neq 1$. If not, then for every $a \in C_G(y)$, we get $[a, x] = 1$, since $x \in C_G(y)$ which is an abelian group. Thus, $a \in C_G(x)$, and so $C_G(y) \subseteq C_G(x)$. Similarly $C_G(x) \subseteq C_G(y)$, and hence $C_G(x) = C_G(y)$. But then, by definition we have $x \sim y$, which forces $[x] = [y]$, a contradiction. \□

Let $\mathcal{B} = \{[x] : C_G(x) \text{ is minimal among all centralizers of } G\}$ i.e., if $[x] \in \mathcal{B}$ and $C_G(y) \subseteq C_G(x)$, then $C_G(y) = C_G(x)$. Assume that $\mathcal{A}$ is as in Lemma 2.2, $[x] \in \mathcal{A}$ and $C_G(y) \subseteq C_G(x)$. For every $a \in C_G(x)$, $[a, y] = 1$, since $a, y \in C_G(x)$ which is abelian, and so $a \in C_G(y)$. Therefore $[x] \in \mathcal{B}$. It follows that $\mathcal{A} \subseteq \mathcal{B}$. In the following we will give some other facts on the sets $\mathcal{A}$ and $\mathcal{B}$.

For a graph $\Gamma$ and a subset $S$ of vertices, denote by $N_G[S]$ the set of vertices in $\Gamma$ which are in $S$ or adjacent to a vertex in $S$. If $N_G[S] = V(\Gamma)$, then $S$ is called a dominating set for $\Gamma$. The dominating number $\gamma(\Gamma)$ of $\Gamma$ is the minimum size of a dominating set of the vertices of $\Gamma$.

Lemma 2.3. Assume that $G$ is a non-abelian group, and $\mathcal{A}$ and $\mathcal{B}$ defined as above.

(i) If $[y] \in \mathcal{B}$, then $[y]$ is adjacent to all elements of $\mathcal{A} \setminus \{[y]\}$ in $\Gamma_E(G)$.

(ii) If $[z] \in V(\Gamma_E(G)) \setminus \mathcal{B}$, then $[z]$ is not adjacent to all elements of $\mathcal{B}$ in $\Gamma_E(G)$.

(iii) If $\mathcal{A} = \mathcal{B}$, then every vertex of $\Gamma_E(G)$ is adjacent to at least a vertex of $\mathcal{A}$. Moreover, $\mathcal{A}$ is a maximal clique of $\Gamma_E(G)$ and if $\Gamma_E(G)$ is finite, then $\omega(\Gamma_E(G)) = |\mathcal{A}|$.

Proof. (i) For $[y] \in \mathcal{B}$, if $[y] \in \mathcal{A}$ and $[x, y] = 1$ for some $[x] \in \mathcal{A}$, then $[x] = [y]$ which is a contradiction. Then $[y]$ is adjacent to all elements of $\mathcal{A}$. Now assume that $[y] \in \mathcal{B} \setminus \mathcal{A}$ and $[x, y] = 1$ for some $[x] \in \mathcal{A}$, then $C_G(x) \subseteq C_G(y)$ and so $C_G(x) = C_G(y)$, since $[y] \in \mathcal{B}$. Thus $[x] = [y]$, a contradiction and so $[y]$ is adjacent to all elements of $\mathcal{A}$.

(ii) Assume that $[z] \in V(\Gamma_E(G)) \setminus \mathcal{B}$. Then there is $[w] \in \mathcal{B}$ such that $C_G(w) \subseteq C_G(z)$ and so $[z, w] = 1$. This means that $[z]$ is not adjacent to $[w]$, as required.

(iii) Suppose that $\mathcal{A} = \mathcal{B}$. If there is $[z] \in V(\Gamma_E(G))$ such that $[z]$ is not adjacent to elements in $\mathcal{A}$, then $[z, x] = 1$ for all $[x] \in \mathcal{A} = \mathcal{B}$ and so $\bigcup_{[x] \in \mathcal{A}} C_G(x) \subseteq C_G(z)$. On the other hand, there is a non-central element $w \in G$ such that $[z]$ and $[w]$ are adjacent. Thus $w \notin C_G(z)$ and so $[w]$ is adjacent to all elements of $\mathcal{A} = \mathcal{B}$, which will contradict part (ii). Therefore, $\mathcal{A}$ is a dominating set for $\Gamma_E(G)$. Moreover, assume that $\mathcal{A} \cup \{[y]\}$ is a clique of $\Gamma_E(G)$, where $[y] \notin \mathcal{A}$. Then there is $[x] \in \mathcal{B} = \mathcal{A}$ such that $C_G(x) \subseteq C_G(y)$ and so $[x, y] = 1$, a contradiction. Therefore, $\mathcal{A}$ is a maximal clique of $\Gamma_E(G)$. Now suppose that $\Gamma_E(G)$ is finite and $Y$ is a maximum clique of $\Gamma_E(G)$ such
Lemma 2.4. A non-abelian group $G$ is an AC-group if and only if $\Gamma_E(G)$ is a complete graph. In particular, if $G$ is an AC-group with $n = \#\text{Cent}(G) < \infty$, then $\Gamma_E(G) \cong K_{n-1}$, where $K_{n-1}$ is a complete graph with $n-1$ vertices.

Proof. Suppose that $\Gamma_E(G)$ is a complete graph and $a, b \in C_G(x) \setminus Z(G)$. Then $[a] = [x] = [b]$ and so $[a, b] = 1$. Therefore, $C_G(x)$ is an abelian group.

Now, assume that $G$ is an AC-group. Then $V(\Gamma_E(G)) = A$ and the result follows from Lemma 2.2. Furthermore, assume that $\#\text{Cent}(G) = n < \infty$. If $C_G(x_1), C_G(x_2), \ldots, C_G(x_{n-1})$ are all proper centralizers of $G$, then $[x_i, x_j] \neq 1$ for all $1 \leq i \neq j \leq n$ and so $V(\Gamma_E(G)) = \{[x_1], [x_2], \ldots, [x_{n-1}]\}$ is the maximum clique of $\Gamma_E(G)$. Thus $\Gamma_E(G) \cong K_{n-1}$. □

Corollary 2.5. Let $G$ and $H$ be two non-abelian groups. If $\Gamma_E(G) \cong \Gamma_E(H)$, then $G$ is an AC-group if and only if $H$ is an AC-group.

Corollary 2.6.

(i) Let $G = D_{2n} = \langle a, b|a^n = b^2 = (ab)^2 = 1 \rangle$ be the dihedral group of order $2n$. Then

$$\Gamma_E(G) \cong \begin{cases} K_{\frac{n+1}{2}} & \text{n is odd} \\ K_{\frac{n}{2}+1} & \text{n is even} \end{cases}$$

(ii) If $G = Q_{4n}$ is the generalized quaternion group of order $4n$, then $\Gamma_E(G) \cong K_{n+1}$.

Proof. It is easy to check that $\text{Cent}(G) = \{G, C_G(a), C_G(a^i b), 0 \leq i \leq n-1\}$, when $n$ is odd and $\text{Cent}(G) = \{G, C_G(a), C_G(a^i b), 0 \leq i \leq n/2 - 1\}$ when $n$ is even. Furthermore $\text{Cent}(G) = \{G, C_G(a), C_G(a^i b), 0 \leq i \leq n-1\}$ when $G = Q_{4n}$. Now the results follows from Theorem 2.4. □

From Theorem 2.4 we note that in some cases the graph of the equivalence classes of groups are not complete. The smallest counterexample is the symmetric group $S_4$. It is easy to check that $C_G((1\ 2\ 3\ 4))$ and $C_G((1\ 3)(2\ 4))$ are non-abelian and distinct, and the vertices $[[1\ 2\ 3\ 4]\] and $[[1\ 3\](2\ 4)]$ are not adjacent in $\Gamma_E(S_4)$.

Proposition 2.7. Assume that $G$ is a non-abelian group. Then $\text{diam}(\Gamma_E(G)) < 2$ and $\text{girth}(\Gamma_E(G)) = 3$. In particular $\Gamma_E(G)$ is connected.

Proof. Let $[x]$ and $[y]$ be two distinct vertices of $\Gamma_E(G)$. If $[x] \neq [y]$ then $d([x], [y]) = 1$. Thus we may assume that $[x] = [y]$. Since $x, y$ are non-central, there exist $[x'], [y'] \in V(\Gamma_E(G))$ such that $[[x], [x']]$ and $[[y], [y']]$
are edges. If $[y] - [x']$ or $[x] - [y']$ then $d([x],[y]) = 2$. Otherwise the vertex $[x'y']$ is adjacent to both $[x]$ and $[y]$ and again $d([x],[y]) = 2$. Therefore, $diam(\Gamma_E(G)) \leq 2$. Moreover, for every edge $\{[x],[y]\}$ of $\Gamma_E(G)$, $\{[x],[y],[xy]\}$ is a triangle. Hence the girth of $\Gamma_E(G)$ is 3.

A subset $X$ of the vertices of a graph $\Gamma$ is called an independent set if the induced subgraph on $X$ has no edges. The maximum size of an independent set in a graph $\Gamma$ is called the independence number of $\Gamma$ and denoted by $\alpha(\Gamma)$.

**Proposition 2.8.** Let $G$ be a non-abelian group.

(i) If $\mathcal{C}$ is a dominating set for $\Gamma(G)$, then $\overline{\mathcal{C}} = \{[x] : x \in \mathcal{C}\}$ is a dominating set for $\Gamma_E(G)$.

(ii) If $\mathcal{D}$ is an independent set of $\Gamma(G)$, then $\overline{\mathcal{D}} = \{[x] : x \in \mathcal{D}\}$ is an independent set of $\Gamma_E(G)$.

**Proof.**

(i) Let $[x]$ be a vertex of $\Gamma_E(G)$ that is not in $\overline{\mathcal{C}}$ so $x \notin \mathcal{C}$. Therefore there is $y \in \mathcal{C}$ such that $[x,y] \neq 1$ and this means that there is $[y] \in \overline{\mathcal{C}}$ such that $[x] - [y]$.

(ii) If $[x]$ and $[y]$ are two elements of $\overline{\mathcal{D}}$, then $x,y \in \mathcal{D}$ and this means that $[x,y] = 1$. Therefore $[x]$ and $[y]$ are not adjacent. □

By Proposition 2.8 one can see that $\gamma(\Gamma_E(G)) \leq \gamma(\Gamma(G))$ and $\alpha(\Gamma_E(G)) \leq \alpha(\Gamma(G))$. In the following, we will prove that the graphs $\Gamma(G)$ and $\Gamma_E(G)$ have the same clique number and vertex chromatic number.

Let $k > 0$ be an integer. A $k$-vertex coloring of a graph $\Gamma$ is an assignment of $k$ colors to the vertices of $\Gamma$ such that no two adjacent vertices have the same color. The vertex chromatic number $\chi(\Gamma)$ of a graph $\Gamma$, is the minimum $k$ for which $\Gamma$ has a $k$-vertex coloring.

**Proposition 2.9.** Let $G$ be a finite non-abelian group. Then $\omega(\Gamma_E(G)) \leq \omega(\Gamma(G))$, $\chi(\Gamma_E(G)) \leq \chi(\Gamma(G))$.

**Proof.** Since $\Gamma_E(G)$ is isomorphic to a subgraph of $\Gamma(G)$, then $\omega(\Gamma_E(G)) \leq \omega(\Gamma(G))$. Assume that $\omega(\Gamma(G)) = n$ and $\{x_1,x_2,\ldots,x_n\}$ is a maximum clique of $\Gamma(G)$. Then for every $1 \leq i \neq j \leq n$, $[x_i,x_j] \neq 1$ and so, $[x_i]$ and $[x_j]$ are adjacent for all $1 \leq i \neq j \leq n$. This means that $\{[x_1],[x_2],\ldots,[x_n]\}$ is a clique in $\Gamma_E(G)$ and so $\omega(\Gamma(G)) \leq \omega(\Gamma_E(G))$.

Now, let $\chi(\Gamma_E(G)) = t$ and $\chi(\Gamma(G)) = k$. By [2, Lemma 4.1] $k$ is the minimum number of abelian subgroups of $G$ whose union is $G$, then $G$ is covered by abelian subgroups $A_1,\ldots,A_k$. Assume that $V(\Gamma_E(G)) = \{[x_1],\ldots,[x_n]\}$, $T = \{x_1,\ldots,x_n\}$ and $T_i = \{[x] : x \in T \cap A_i\}$ for $1 \leq i \leq k$. Then the vertices of $\Gamma_E(G)$ in $T_i$ are independent and so $t \leq k$. Now suppose that $B_1,\ldots,B_t$ are independent subsets of $V(\Gamma_E(G))$ such that $\bigcup_{j=1}^t B_j = V(\Gamma_E(G))$. Then $A_j = \langle \bigcup_{x \in B_j} [x], Z(G) \rangle$ is an abelian subgroup of $G$, for $1 \leq j \leq t$ and $G$ is covered by these $t$ abelian subgroups. It follows that $k \leq t$. □
 Proposition 2.10. If $\Gamma(G) \cong \Gamma(H)$, then $\Gamma_E(G) \cong \Gamma_E(H)$.

Proof. Let $\phi : V(\Gamma(G)) \rightarrow V(\Gamma(H))$ be a bijective map such that for every two distinct elements $x, y \in V(\Gamma(G))$, we have $[x, y] = 1$ if and only if $[\phi(x), \phi(y)] = 1$. Define $\psi : V(\Gamma_E(G)) \rightarrow V(\Gamma_E(H))$ such that $\psi([x]) = [\phi(x)]$. One can check that $\psi$ is a bijective map and $[x] - [y]$ in $\Gamma_E(G)$ if and only if $\psi([x]) = [\phi(x)] - [\phi(y)] = \psi([y])$ in $\Gamma_E(H)$. $\square$

 Proposition 2.11. Let $G$ or $H$ be non-abelian AC-groups. Then $\omega(\Gamma_E(G \times H)) = \omega(\Gamma_E(G)) \omega(\Gamma_E(H))$.

Proof. Assume that $G$ is an AC-group and $V(\Gamma_E(G)) = \{(x_1), \ldots, (x_n)\}$ and $\{(y_1), \ldots, (y_m)\}$ are maximum clique of $\Gamma_E(G)$ and $\Gamma_E(H)$, respectively. We show that $\Omega = \{(x_i, y_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a maximum clique of $\Gamma_E(G \times H)$. Suppose, for a contradiction, that $\Omega \neq \Omega \cup \{(u, v)\}$ is a clique of $\Gamma_E(G \times H)$. Then $[x_i, y_j] - [u, v]$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Case 1. $[u] = [x_t]$ or $[v] = [y_s]$ for some $1 \leq t \leq n$ and $1 \leq s \leq m$. It is clear that $\{(y_1), \ldots, (y_m), [u]\}$ or $\{(x_1), \ldots, (x_n), [u]\}$ is a clique of $\Gamma_E(H)$ or $\Gamma_E(G)$, respectively, which is a contradiction.

Case 2. $[u] \neq [x_i]$ for all $1 \leq i \leq n$. Since $G$ is an AC-group, $[u]$ is adjacent to $[x_i]$ for every $1 \leq i \leq n$ and so $\{(x_1), \ldots, (x_n), [u]\}$ is a clique of $\Gamma_E(G)$, a contradiction. $\square$

 Proposition 2.12. Let $H$ be a finite subgroup of $G$. Then $\Gamma_E(G) \cong \Gamma_E(H)$ if and only if $G = HZ(G)$.

Proof. Define $\phi : V(\Gamma_E(G)) \rightarrow V(\Gamma_E(H))$ by $\phi([h]) = [h]$, which is a bijective map. Since $[h_1z_1, h_2z_2] = 1$ if and only if $[h_1, h_2] = 1$, then $[h_1z_1]$ and $[h_2z_2]$ are adjacent in $\Gamma_E(G)$ if and only if $[h_1]$ and $[h_2]$ are adjacent in $\Gamma_E(H)$ and so $\phi$ is a graph isomorphism.

Conversely, assume that $V(\Gamma_E(H)) = \{[h_1], \ldots, [h_n]\}$, so that we may have $V(\Gamma_E(G)) = \{[h_1], \ldots, [h_n]\}$ and $Z(H) \subseteq Z(G)$. Since the map $\phi : HZ(H) \rightarrow Z(G)$ by $\phi(hZ(H)) = hZ(G)$ is an isomorphism, for every $g \in G$ there exists $h \in H$ such that $gZ(G) = \phi(hZ(H)) = hZ(G)$. Thus $h^{-1}g \in Z(G)$ and so $g = hZ(G)$ and the proof is complete. $\square$

Corollary 2.13. $\Gamma_E(G \times A) \cong \Gamma_E(G)$ if and only if $A$ is an abelian group.

 Proposition 2.14. Let $N$ be a normal subgroup of $G$. Then $\Gamma_E(G) \cong \Gamma_E(G/N)$, if $N \cap G' = 1$.

Proof. Assume that $N \cap G' = 1$. Then the map $\phi : V(\Gamma_E(G)) \rightarrow V(\Gamma_E(G/N))$ by $\phi([x]) = [xN]$ is a bijection. Now assume that $[x]$ is not adjacent to $[y]$ in $\Gamma_E(G)$. Since $[x, y] = 1$, then $[xN, yN] = 1$ and so $[xN] \neq [yN]$. On the other hand, if $[xN]$ and $[yN]$ are not adjacent in $\Gamma_E(G/N)$, then $[xN, yN] = 1_{G/N}$. Thus $[x, y] \in N \cap G' = 1$ and so $[x]$ and $[y]$ are not adjacent in $\Gamma_E(G)$. $\square$
Proposition 2.15. Assume that $\Gamma_E(G_1) \cong \Gamma_E(H_1)$ and $\Gamma_E(G_2) \cong \Gamma_E(H_2)$. Then $\Gamma_E(G_1 \times G_2) \cong \Gamma_E(H_1 \times H_2)$.

Proof. Let $\varphi_i : V(\Gamma_E(G_i)) \to V(\Gamma_E(H_i))$ be a graph isomorphism for $i = 1, 2$. Then it is easy to see that $\psi : V(\Gamma_E(G_1 \times G_2)) \to V(\Gamma_E(H_1 \times H_2))$ such that $\psi([[x, y]]) = [(\varphi_1(x), \varphi_2(y))]$, where $C_{H_1}(\varphi_1(x)) := \varphi_1(C_{G_1}(x))$ and $C_{H_2}(\varphi_2(y)) := \varphi_2(C_{G_2}(y))$ is a graph isomorphism between $\Gamma_E(G_1 \times G_2)$ and $\Gamma_E(H_1 \times H_2)$. □

3. Isoclinism classes and the graph of equivalences classes

The notion of isoclinism of groups was introduced by Philip Hall [9] as the following

Definition 3.1. Let $G$ and $H$ be two groups; a pair $(\varphi, \psi)$ is called an isoclinism from $G$ to $H$ if:

1. $\varphi$ is an isomorphism from $G/Z(G)$ to $H/Z(H)$;
2. $\psi$ is an isomorphism from $G'$ to $H'$;
3. the following diagram is commutative:

$$
\begin{array}{ccc}
\frac{G}{Z(G)} \times \frac{G}{Z(G)} & \xrightarrow{\varphi \times \varphi} & \frac{H}{Z(H)} \times \frac{H}{Z(H)} \\
\downarrow a_G & & \downarrow a_H \\
G' & \xrightarrow{\psi} & H'
\end{array}
$$

where, $a_G(g_1Z(G), g_2Z(G)) = [g_1, g_2]$ and $a_H(h_1Z(H), h_2Z(H)) = [h_1, h_2]$.

If there is an isoclinism from $G$ to $H$, we say that $G$ and $H$ are isoclinic and denote it by $G \sim H$. One may easily check that the groups $Q_8$ and $D_8$ are isoclinic but are not isomorphic.

In the following, we prove that the graphs of equivalences classes of non-commuting graphs associated with two isoclinic groups are isomorphic.

Theorem 3.2. Assume that $G$ and $H$ are two isoclinic groups. Then $\Gamma_E(G) \cong \Gamma_E(H)$.

Proof. Let $(\varphi, \psi)$ be an isoclinism from $G$ to $H$, where $\varphi : G/Z(G) \to H/Z(H)$ and $\psi : G' \to H'$ are isomorphisms and if $\varphi(g_1Z(G)) = h_1Z(H)$, $\varphi(g_2Z(G)) = h_2Z(H)$, then $\psi([g_1, g_2]) = [h_1, h_2]$. Now assume that $C_G(x) \in \Cent(G)$ such that $\varphi(C_G(x)/Z(G)) = K/Z(H)$, we will show that $K = C_H(y)$, where $\varphi(xZ(G)) = yZ(H)$. For $k \in K$ there is $g \in C_G(x)$ such that $\varphi(gZ(G)) = kZ(H)$. Since $[x, g] = 1$ and by commutativity

$$[[y, k]] = a_H(\varphi \times \varphi(xZ(G), gZ(G))) = \psi(a_G(xZ(G), gZ(G))) = \psi([x, g]),$$

where $a_G(xZ(G), gZ(G)) = [g_1, g_2]$.
then $[y, k] = 1$ and so $k \in C_H(y)$. Conversely, for $h \in C_H(y)$ there is $g \in G$ such that $\varphi(gZ(G)) = hZ(H)$ and

$$\psi([x, g]) = \psi(a_G(xZ(G), gZ(G))) = a_H(\varphi \times \varphi(xZ(G), gZ(G))) = [y, k] = 1.$$ 

Therefore $g \in C_G(x)$ and so $h \in K$, as required. Thus $\#\text{Cent}(G) = \#\text{Cent}(H)$ and so $\theta : V(\Gamma_E(G)) \rightarrow V(\Gamma_E(H))$ by $\theta([x]) = [y]$ is a bijective map, where $\varphi(C_G(x)/Z(G)) = C_H(y)/Z(H)$. 

Furthermore if $[a]$ and $[b]$ are adjacent in $\Gamma_E(G)$, then $C_G(a) \neq C_G(b)$ and $[a, b] \neq 1$. Since $(\varphi, \psi)$ is an isoclinism, $[a']$ and $[b']$ are adjacent in $\Gamma_E(H)$, where $\varphi(C_G(a)/Z(G)) = C_H(a')/Z(H)$ and $\varphi(C_G(b)/Z(G)) = C_H(b')/Z(H)$. In the same way we can show that, $[a'] - [b']$ in $\Gamma_E(H)$ implies that $[a] - [b]$ in $\Gamma_E(G)$ and so $\Gamma_E(G) \cong \Gamma_E(H)$. $\square$

**Theorem 3.3.** Let $G$ be a group such that $\Gamma_E(G)$ is finite. Then there is a finite group $K$ such that $Z(K) \cong K'$, $K \sim G$ and $\Gamma_E(K) \cong \Gamma_E(G)$.

**Proof.** Thanks to [10, Proposition 2.5] there is a group $K$ isoclinic to $G$ such that $Z(K) \subseteq K'$. On the other hand, since $\Gamma_E(G)$ is finite, $\omega(\Gamma_E(G)) = \omega(\Gamma_E(G)) < \infty$ and so by the main theorem of [13], we have $[K : Z(K)] = [G : Z(G)] \leq e^{\omega(\Gamma_E(G))}$ for some constant $c$. Now, Schur’s Theorem follows that $K'$ and so $Z(K)$ is finite. Therefore $K$ is finite and the result follows from Theorem 3.2. $\square$

**Theorem 3.4.** $\Gamma_E(G) \cong K_3$ if and only if $G \sim D_8$.

**Proof.** It is clear that if $G$ is isoclinic to $D_8$, then $\Gamma_E(G) \cong K_3$. 

Now assume that $\Gamma_E(G) \cong K_3$. By Theorem 3.3, we might as well assume that $G$ is a finite group such that $Z(G) \subseteq G'$ and by Theorem 2.4, $G$ is an AC-group. Since group $G$ is the union of its proper centralizers, there are proper centralizers $C_G(x), C_G(y), C_G(z)$ of $G$ such that $G = C_G(x) \cup C_G(y) \cup C_G(z)$. On the other hand, $[xy, x] \neq 1$ and $[xy, y] \neq 1$. Thus, $C_G(xy) = C_G(z)$ and so $G = C_G(x) \cup C_G(y) \cup C_G(xy)$.

We claim that for every $u \in G \setminus Z(G)$, $u^2 \in Z(G)$. Suppose, for a contradiction, that $u^2$ is not central and for example $u \in C_G(x)$, then $C_G(x) = C_G(u)$ and so $[u^2, x] \neq 1$ and $[u^2, y] \neq 1$. Hence, $C_G(u^2) = C_G(xy)$, since $G$ has four distinct centralizers. Moreover, $C_G(xy) = C_G(uy)$, since $[uy, x] \neq 1$ and $[uy, y] \neq 1$. Then $C_G(u^2y) = C_G(uy)$ and so $uy = yu$, which is a contradiction. Therefore, $G/Z(G)$ is an elementary abelian 2-group. 

On the other hand, one can see that for every $u \in G \setminus Z(G)$ if for example $u \in C_G(x)$, then $uZ(G) = xZ(G)$ and so $[G : Z(G)] = 4$, which implies that $G/Z(G) \cong C_2 \times C_2$ and so $Z(G) = G'$. Now by [14, Corollary 3.1], $G' \cong C_2$ and one can see that $G$ is isoclinic to $D_8$, as required. $\square$

**Theorem 3.5.** $\Gamma_E(G) \cong K_4$ if and only if $G$ is isoclinic to $S_3$ or an extra special group of order 27.
Proof. If \( G \sim S_3 \), then \( \Gamma_E(G) \cong \Gamma_E(S_3) \cong K_4 \) by Theorem 3.2. Moreover, assume that \( E \) is an extra special group of order 27 and \( G \) is isoclinic to \( E \). By using GAP, we can see that \( E \) has four abelian proper centralizers and so \( \Gamma_E(G) \cong \Gamma_E(E) \cong K_4 \). For the converse, let \( \Gamma_E(G) \cong K_4 \). From Theorem 3.3, we may assume, up to isoclinism, that \( G \) is a finite group such that \( Z(G) \cong G' \). If \( S \) is a Sylow \( p \)-subgroup of \( G \) (\( p \) is a prime). Then \( SZ(G)/Z(G) \in \text{Syl}_p(G/Z(G)) \) and Sylow subgroups of \( G/Z(G) \cong S_3 \) are cyclic groups of order 2 or 3. Thus, \( SZ(G)/Z(G) \) is cyclic which implies that \( SZ(G) \) is abelian and so is \( S \). Hence all Sylow subgroups of \( G \) are abelian and by [7, Corollary 4.5], \( G' \cap Z(G) = 1 \) and this implies that \( Z(G) = 1 \). Then we conclude that \( G \sim S_3 \), as required.

Now let \( G/Z(G) \cong C_3 \times C_3 \). By a similar argument as Theorem 3.4, we can see that \( G' = G' \) and \( G'' \cong C_3 \). It remains to prove that \( G \) is isoclinic to an extra special group of order 27. Let \( G/Z(G) = \langle xZ(G) \rangle \times \langle yZ(G) \rangle \) and \( E \) be an extra special group of order 27 i.e., \( E' = Z(E) \cong C_3 \) and \( E/Z(E) \cong C_3 \\times C_3 = \langle aZ(E) \rangle \times \langle bZ(E) \rangle \), where \( x, y \in G \) and \( a, b \in E \).

Define \( \varphi : G/Z(G) \to E/Z(E) \) by the rules \( \varphi(xZ(G)) = aZ(E) \) and \( \varphi(yZ(G)) = bZ(E) \), which is an isomorphism. Furthermore, one can check that \( G' = \langle x, y \rangle \) and \( G'' = \langle a, b \rangle \) and so the map \( \psi : G' \to E' \) by the rule \( \psi([x, y]) = [a, b] \) is an isomorphism. Now we prove that the following diagram is commutative:

\[
\begin{array}{ccc}
G/Z(G) \times G/Z(G) & \xrightarrow{\varphi \times \varphi} & E/Z(E) \times E/Z(E) \\
\downarrow_{a_G} & & \downarrow_{a_E} \\
G' & \xrightarrow{\psi} & E'
\end{array}
\]

For any \( 0 \leq t, s, u, v \leq 2 \), we have

\[
a_Ea_G \varphi(x^ty^vZ(G), x^suZ(G)) = a_E(a^tb^s\text{Z}(E), a^ub^v\text{Z}(E))
\]

\[
= [a^tb^s, a^ub^v] = [a, b]^{tv-su}
\]

\[
= \psi([x, y]^{tv-su}) = \psi([x^ty^s, x^uy^v])
\]

\[
= \psi a_G(x^ty^sZ(G), x^uy^vZ(G)).
\]

Therefore \( G \) and \( E \) are isoclinic by Definition 3.1 as required.

\( \square \)

**Corollary 3.6.** Let \( G \) be an AC-group. Then \( \Gamma_E(G) \) is planar if and only if \( G \) is isoclinic to \( S_3, D_8 \) or an extra special group of order 27.

**Proposition 3.7.** If \( \omega(\Gamma_E(G)) \leq 20 \), then \( G \) is solvable.
Proof. By Theorem 3.3, we may assume that \( G \) is a finite group with \( \omega(\Gamma_E(G)) = \omega(\Gamma(G)) \leq 20 \). Therefore, the result follows from [3, Theorem 1.4].

It must be noted that \( A_5 \) is an AC-group with \( \#\text{Cent}(A_5) = 22 \) and so \( \omega(\Gamma_E(A_5)) = 21 \), by Theorem 2.4. Therefore, the above upper bound is sharp.

Acknowledgements

The authors would like to thank anonymous referees for providing helpful and constructive comments and suggestions.

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