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THE SECOND DUAL OF STRONGLY ZERO-PRODUCT PRESERVING MAPS

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Abstract. The notion of strongly Lie zero-product preserving maps on normed algebras as a generalization of Lie zero-product preserving maps are defined. We give a necessary and sufficient condition from which a linear map between normed algebras to be strongly Lie zero-product preserving. Also some hereditary properties of strongly Lie zero-product preserving maps are presented. Finally the second dual of a strongly zero-product, strongly Jordan zero-product and strongly Lie zero-product preserving map on a certain class of normed algebras are investigated.
Keywords: Strongly zero-product preserving map, strongly Jordan zero-product preserving map, strongly Lie zero-product preserving map, Arens regular.
MSC(2010): Primary: 46B99; Secondary: 46H99, 47B37, 65H04.

1. **Introduction and preliminaries**

Let A and B be two associative algebras over the same field $\mathbb{C}$. A linear map $\theta : A \longrightarrow B$ is said to be zero-product preserving if, $\theta(a)\theta(c) = 0$, whenever $ac = 0$. It is Jordan zero-product preserving if, $\theta(a) \circ \theta(c) = 0$, whenever $a \circ c = 0$, where $\circ$ is the Jordan product $a \circ c = ac + ca$. Also θ is Lie zero-product preserving if, $[\theta(a), \theta(c)] = 0$, whenever $[a, c] = 0$, where $[a, c] = ac - ca$, $a, c \in A$. A natural possibility for θ to preserve zero-products (Jordan zero-products or Lie zero-products) is to be of the form $\theta = b\varphi$, where b is a central element of B and $\varphi : A \longrightarrow B$ is a homomorphism (Jordan homomorphism or Lie homomorphism) that is,

$$\varphi(ac) = \varphi(a)\varphi(c) \quad \big(\varphi(a \circ c) = \varphi(a) \circ \varphi(c) \text{ or } \varphi([a, c]) = [\varphi(a), \varphi(c)]\big), a, c \in A.$$

But this characterization is not the case in general (see [4, Remark 2.5] and Example 2.2 in this paper). An interesting question is for which algebras A

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and B this natural possibility is the only possibility. These kind of questions have been studied since the 1970s.

As a generalization of the above mentioned notions, the notions of strongly zero-product preserving maps and strongly Jordan zero-product preserving maps are investigated in [3–5] on normed algebras. In this direction we recall some terminologies.

Let A and B be two normed algebras. A linear map $\theta : A \longrightarrow B$ is said to be :

- Strongly zero-product preserving if, for any two sequences $\{a_n\}_n$ and $\{c_n\}_n$ in A, $\theta(a_n)\theta(c_n) \longrightarrow 0$, whenever $a_n c_n \longrightarrow 0$.
- Strongly Jordan zero-product preserving if, for any two sequences $\{a_n\}_n$ and $\{c_n\}_n$ in A, $\theta(a_n) \circ \theta(c_n) \longrightarrow 0$, whenever $a_n \circ c_n \longrightarrow 0$.

Also in the sequel we will say θ is :

- Strongly Lie zero-product preserving if, for any two sequences $\{a_n\}_n$ and $\{c_n\}_n$ in A, $[\theta(a_n), \theta(c_n)] \longrightarrow 0$, whenever $[a_n, c_n] \longrightarrow 0$.

For an associative normed algebra A, let A^{**} be the second dual of A. We introduce the Arens products $\triangle$ and $\odot$ on the second dual A^{**} as follows. For $a, c \in A$, $f \in A^*$ and $m, n \in A^{**}$, $\langle f \cdot a, c\rangle = \langle f, ac\rangle$, $\langle n \cdot f, a\rangle = \langle n, f \cdot a\rangle$ and $\langle m \triangle n, f\rangle = \langle m, n \cdot f\rangle$. Similarly $\langle c, a \cdot f\rangle = \langle ca, f\rangle$, $\langle a, f \cdot n\rangle = \langle a \cdot f, n\rangle$ and $\langle f, m \odot n\rangle = \langle f \cdot m, n\rangle$. One can simply verify that $(A^{**}, \triangle)$ and $(A^{**}, \odot)$ are associative normed algebras.

The normed algebra A is called Arens regular if, $m \triangle n = m \odot n$ for all $m, n \in A^{**}$.

2. **Strongly Lie zero-product preserving maps**

In this section we give a necessary and sufficient condition from which a linear map between normed algebras to be strongly Lie zero-product preserving. Also we investigate some hereditary properties of strongly Lie zero-product preserving maps.

Definition 2.1. Let A and B be two normed algebras. We shall say that a linear map $\theta : A \longrightarrow B$ is strongly Lie zero-product preserving, if for any two sequences $\{a_n\}_n$ and $\{c_n\}_n$ in A, $[\theta(a_n), \theta(c_n)] \longrightarrow 0$, whenever $[a_n, c_n] \longrightarrow 0$.

Example 2.2. (1) Let A and B be normed algebras. Then every continuous Lie homomorphism from A into B is a strongly Lie zero-product preserving map.

(2) Let $\mathbb{W}$ be a finite dimensional normed vector space with the basis $\beta = \{e_1, e_2, e_3\}$. Also let $f \in \mathbb{W}^*$ be a linear functional on $\mathbb{W}$ such that $f(e_1) = 1$ and $f(e_2) = f(e_3) = 0$. For $a, c \in \mathbb{W}$ define $a \cdot c = f(a)c$. Obviously $(\mathbb{W}, \cdot)$ is an associative normed algebra. We

denote it by $\mathbb{W}_f$. Let $\theta : \mathbb{W}_f \longrightarrow \mathbb{W}_f$ be a linear map such that $\theta(a) = g(a)e_2$ where $g \in \mathbb{W}^*$ is a linear functional such that $g(e_3) = 1$ and $g(e_1) = g(e_2) = 0$. A direct verification shows that θ is strongly Lie zero-product preserving. But θ is neither a Lie homomorphism nor a Lie homomorphism multiplied by a central element of $\mathbb{W}_f$.

(3) Let A and B be two normed algebras and let B be commutative. Then every linear map from A into B is a strongly Lie zero-product preserving map.

Clearly every strongly Lie zero-product preserving map is a Lie zero-product preserving map. But the converse is not the case in general. We give the following example to show this fact.

Example 2.3. Similar to Example 2.2 let $\mathbb{W}$ be an infinite dimensional normed vector space with the basis $\beta = \{e_1, e_2, e_3, \dots\}$ such that $\|e_n\| = 1$ for all $n \geq 1$. Also let $f \in \mathbb{W}^*$ be a bounded linear functional such that $f(e_1) = 1$ and $f(e_n) = 0$ for all $n \geq 2$. Assume that $\theta : \mathbb{W}_f \longrightarrow \mathbb{W}_f$ is a linear map such that $\theta(e_1) = e_1$ and $\theta(e_n) = 2^n e_2$ for all $n \geq 2$. A direct verification shows that θ is Lie zero-product preserving. We shall show that θ is not a strongly Lie zero-product preserving map. Let $a_n = \frac{e_1}{n}$ and $c_n = e_{n+1}$. Clearly $\lim_{n\to\infty}[a_n, c_n] = 0$. But $\lim_{n\to\infty} \|[\theta(a_n), \theta(c_n)]\| = \lim_{n\to\infty} \|\frac{2^{(n+1)}}{n} e_2\| = \lim_{n\to\infty} \frac{2^{(n+1)}}{n} = \infty$.

Theorem 2.4. *Let $\mathbb{W}$ be a non-zero normed vector space and let $f \in \mathbb{W}^*$ be a non-zero element such that $\|f\| \leq 1$. Then a linear map $\theta : \mathbb{W}_f \longrightarrow \mathbb{W}_f$ is strongly Lie zero-product preserving if and only if one of the following statements holds.*

(1) $f \circ \theta = 0$.
(2) $\theta(\ker f) \subseteq \ker f$ *and there exist a continuous linear map* $\varphi : \mathbb{W}_f \longrightarrow \ker f$ *and an element* $e \in f^{-1}(\{1\})$ *such that for all* $a \in \mathbb{W}_f$, $\theta(a) = f(a)\theta(e) + \theta \circ \varphi(a)$ *and* $\theta \circ \varphi|_{\ker f}$ *is continuous.*
(3) *There exist a continuous linear map* $\varphi : \mathbb{W}_f \longrightarrow \ker f$ *and an element* $e \in f^{-1}(\{1\})$ *such that* $\theta(a) = f(a)\theta(e) + f \circ \theta \circ \varphi(a)e,\ a \in \mathbb{W}_f$ *and one of the following statements holds.*
   (a) $f \circ \theta \circ \varphi|_{\ker f}$ *is continuous.*
   (b) $[\theta(e), e] = 0$.

Proof. Let θ be a strongly Lie zero-product preserving map such that $f \circ \theta \neq 0$. Also let $e \in \mathbb{W}_f$ be an element such that $f(e) = 1$. So, for each $a \in \mathbb{W}_f$, $a = f(a)e + \varphi(a)$, where $\varphi : \mathbb{W}_f \longrightarrow \ker f$ and $\varphi(a) = a - f(a)e$. Hence for each $a, c \in \mathbb{W}_f$,

$$\theta(a) = f(a)\theta(e) + \theta \circ \varphi(a),\ \theta(c) = f(c)\theta(e) + \theta \circ \varphi(c).$$

If $\theta(\ker f) \subseteq \ker f$ then,

$$\begin{aligned}[\theta(a), \theta(c)] &= [f(a)\theta(e) + \theta \circ \varphi(a), f(c)\theta(e) + \theta \circ \varphi(c)] \\ &= f(a)[\theta(e), \theta \circ \varphi(c)] + f(c)[\theta \circ \varphi(a), \theta(e)] \\ &= [\theta(e), \theta \circ \varphi(ac)] + [\theta \circ \varphi(ca), \theta(e)] \\ &= [\theta(e), \theta \circ \varphi(ac)] - [\theta(e), \theta \circ \varphi(ca)] \\ &= [\theta(e), \theta \circ \varphi(ac - ca)] = [\theta(e), \theta \circ \varphi([a, c])] \\ &= f \circ \theta(e)\theta \circ \varphi([a, c]), \quad a, c \in \mathbb{W}_f.\end{aligned}$$

Obviously $f \circ \theta(e) \neq 0$. Indeed, the assumption $f \circ \theta(e) = 0$ implies,

$$\begin{aligned}f \circ \theta(a) &= f(a) f \circ \theta(e) + f \circ \theta \circ \varphi(a) \\ &= 0.\end{aligned}$$

That is a contradiction. Let $\{a_n\}_n \subseteq \ker f$ be a sequence such that $a_n \longrightarrow 0$. So, $[e, a_n] \longrightarrow 0$. As θ is strongly Lie zero-product preserving, we can conclude that,

$$\begin{aligned}\|\theta \circ \varphi(a_n)\| &= \|\theta \circ \varphi([e, a_n])\| \\ &= \frac{\|[\theta(e), \theta(a_n)]\|}{|f \circ \theta(e)|} \longrightarrow 0.\end{aligned}$$

This shows that $\theta \circ \varphi$ is continuous on $\ker f$.

If $\theta(\ker f) \not\subseteq \ker f$, then there exists $a_0 \in \ker f$ such that $f(\theta(a_0)) = 1$. Set $e = \theta(a_0)$. For each $a, c \in \ker f$ the equality $[a, c] = 0$ implies, $[\theta(a), \theta(c)] = 0$. So,

$$f(\theta(a))\theta(c) = f(\theta(c))\theta(a). \tag{2.1}$$

Upon substituting $a = a_0$ in (2.1), we obtain

$$\theta(c) = f(\theta(c))\theta(a_0) = f \circ \theta(c)e, \quad c \in \ker f. \tag{2.2}$$

So, for each $a \in \mathbb{W}_f$, $a = f(a)e + \varphi(a)$, where $\varphi : \mathbb{W}_f \longrightarrow \ker f$ and $\varphi(a) = a - f(a)e$. Hence for each $a, c \in \mathbb{W}_f$,

$$\theta(a) = f(a)\theta(e) + \theta \circ \varphi(a), \ \theta(c) = f(c)\theta(e) + \theta \circ \varphi(c).$$

Since $\varphi(a), \varphi(c) \in \ker f$, applying (2.2) yields,

$$\theta(a) = f(a)\theta(e) + f \circ \theta \circ \varphi(a)e, \ \theta(c) = f(c)\theta(e) + f \circ \theta \circ \varphi(c)e, \quad a, c \in \mathbb{W}_f.$$

So,

$$\begin{aligned}[\theta(a),\theta(c)] &= [f(a)\theta(e)+f\circ\theta\circ\varphi(a)e, f(c)\theta(e)+f\circ\theta\circ\varphi(c)e] \\ &= f(a)[\theta(e), f\circ\theta\circ\varphi(c)e] - f(c)[\theta(e), f\circ\theta\circ\varphi(a)e] \\ &= [\theta(e), f\circ\theta\circ\varphi(ac)e] - [\theta(e), f\circ\theta\circ\varphi(ca)e] \\ &= [\theta(e), f\circ\theta\circ\varphi([a,c])e] \\ &= f\circ\theta\circ\varphi([a,c])[\theta(e),e], \quad a,c\in\mathbb{W}_f. \end{aligned} \tag{2.3}$$

Let $[\theta(e),e]\neq 0$ and let $\{a_n\}_n\subseteq \ker f$ be a sequence such that $a_n\longrightarrow 0$. So, $[e,a_n]\longrightarrow 0$. As θ is strongly Lie zero-product preserving, we can conclude that,

$$\begin{aligned}|f\circ\theta\circ\varphi(a_n)| &= |f\circ\theta\circ\varphi([e,a_n])| \\ &= \frac{\|[\theta(e),\theta(a_n)]\|}{\|[\theta(e),e]\|}\longrightarrow 0.\end{aligned}$$

This shows that $f\circ\theta\circ\varphi$ is continuous on $\ker f$.

For the converse if $f\circ\theta=0$ then clearly θ is a strongly Lie zero-product preserving map. In the cases (2) and (3), one can easily verify that,

$$[\theta(a),\theta(c)] = f\circ\theta(e)\theta\circ\varphi([a,c]), \quad a,c\in\mathbb{W}_f$$

and

$$[\theta(a),\theta(c)] = f\circ\theta\circ\varphi([a,c])[\theta(e),e], \quad a,c\in\mathbb{W}_f,$$

respectively. Let $[a_n,c_n]\longrightarrow 0$. So the continuity of $f\circ\theta$ and $f\circ\theta\circ\varphi$ on $\ker f$ implies, $[\theta(a_n),\theta(c_n)]\longrightarrow 0$. This shows that θ is strongly Lie zero-product preserving. □

Corollary 2.5. *Let $\mathbb{W}$ be a non-zero normed vector space and let $f\in\mathbb{W}^*$ be non-zero. Also let $\theta:\mathbb{W}_f\longrightarrow\mathbb{W}_f$ be a continuous linear map such that $\theta(\ker f)\subseteq\ker f$. Then θ is strongly Lie zero-product preserving.*

Proof. By Theorem 2.4 it is obvious. □

In the following we present a characterization of strongly Lie zero-product preserving maps on normed algebras.

Theorem 2.6. *Let A and B be two normed algebras. Then a linear map $\theta:A\longrightarrow B$ is strongly Lie zero-product preserving if and only if there exists an $M>0$ such that for all $a,c\in A$,*

$$\|[\theta(a),\theta(c)]\|\leq M\|[a,c]\|.$$

Proof. For the sake of contradiction similar to [3, Theorem 3.1] and [5, Theorem 4.1], suppose there is no such M. Then for each $n\in\mathbb{N}$ there exist $a_n,c_n\in A$ such that,

$$\|[\theta(a_n),\theta(c_n)]\| > n\|[a_n,c_n]\|.$$

So

$$\|[\frac{a_n}{\|[\theta(a_n),\theta(c_n)]\|},c_n]\| < \frac{1}{n}.$$

Set $a'_n = \frac{a_n}{\|[\theta(a_n),\theta(c_n)]\|}$ and $c'_n = c_n$. Clearly $[a'_n, c'_n] \longrightarrow 0$. It follows that

$$[\theta(a'_n),\theta(c'_n)] \longrightarrow 0,$$

that is a contradiction. Indeed

$$\|[\theta(a'_n),\theta(c'_n)]\| = \frac{\|[\theta(a_n),\theta(c_n)]\|}{\|[\theta(a_n),\theta(c_n)]\|} \longrightarrow 1.$$

The converse is obvious. □

Corollary 2.7. *Let A and B be two normed algebras. Then every continuous Lie homomorphism from A into B is strongly Lie zero-product preserving.*

We present some hereditary properties of strongly Lie zero-product preserving maps.

Proposition 2.8. *Let A, B, C, D be normed algebras and let $\varphi : A \longrightarrow B$ and $\psi : C \longrightarrow D$ be two strongly Lie zero-product preserving maps. Then $\varphi \oplus \psi : A \oplus C \longrightarrow B \oplus D$ is strongly Lie zero-product preserving .*

Proof. By Theorem 2.6, there exist $M, N > 0$ such that,

$$\|[\varphi(a),\varphi(a')]\| \leq M\|[a,a']\|, a, a' \in A,$$

and

$$\|[\psi(c),\psi(c')]\| \leq N\|[c,c']\|, c, c' \in C.$$

So,

$$\begin{aligned}
\|[\varphi\oplus\psi(a,c),\varphi\oplus\psi(a',c')]\| &= \|[(\varphi(a),\psi(c)),(\varphi(a'),\psi(c'))]\| \\
&= \|(\varphi(a)\varphi(a') - \varphi(a')\varphi(a), \psi(c)\psi(c') - \psi(c')\psi(c))\| \\
&= \|([\varphi(a),\varphi(a')],[\psi(c),\psi(c')])\| \\
&= \|[\varphi(a),\varphi(a')]\| + \|[\psi(c),\psi(c')]\| \\
&\leq M\|[a,a']\| + N\|[c,c']\| \\
&\leq (M+N)(\|[a,a']\| + \|[c,c']\|) \\
&= (M+N)\|([a,a'],[c,c'])\| \\
&= (M+N)\|[(a,c),(a',c')]\|.
\end{aligned}$$

Applying Theorem 2.6 implies that $\varphi \oplus \psi$ is strongly Lie zero-product preserving. □

Proposition 2.9. *Let A, B, C be normed algebras and let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be two strongly Lie zero-product preserving maps. Then $\psi \circ \varphi : A \longrightarrow C$ is strongly Lie zero-product preserving.*

Proof. As φ and ψ are strongly Lie zero-product preserving maps so, there exist $M, N > 0$ such that

$$\|[\varphi(a), \varphi(a')]\| \leq M\|[a, a']\|, a, a' \in A,$$

and

$$\|[\psi(b), \psi(b')]\| \leq N\|[b, b']\|, b, b' \in B.$$

So,

$$\begin{aligned}\|[\psi \circ \varphi(a), \psi \circ \varphi(a')]\| &= \|[\psi(\varphi(a)), \psi(\varphi(a'))]\| \\ &\leq N\|[\varphi(a), \varphi(a')]\| \\ &\leq MN\|[a, a']\|, a, a' \in A.\end{aligned}$$

This shows that $\psi \circ \varphi$ is strongly Lie zero-product preserving. □

3. **Main results**

In this section we investigate the second dual of a strongly zero-product (strongly Jordan zero-product and strongly Lie zero-product) preserving map, defined on a certain class of normed algebras.
Let A and B be normed algebras. For a bounded linear map $\theta : A \longrightarrow B$, it is obvious that if $\theta^{**} : A^{**} \longrightarrow B^{**}$ is strongly (strongly Jordan and strongly Lie) zero-product preserving, then so is $\theta : A \longrightarrow B$.
An interesting question is for which algebras A and B the converse is true. In the sequel, by $a * c$ we mean one of the following products,

$$ac, \quad a \circ c \quad \text{or} \quad [a, c].$$

And also by a strongly $*-$zero-product preserving map we mean a linear map between normed algebras, that is strongly zero-product preserving, strongly Jordan zero-product preserving or strongly Lie zero-product preserving. A $*-$homomorphism is a homomorphism, Jordan homomorphism or Lie homomorphism.

Theorem 3.1. *Let A and B be two Arens regular normed algebras and let $\theta : A \longrightarrow B$ be a bounded linear map. Also let there exists $M \geq 0$ such that for each $a, c \in A$ and each $g \in \overline{B_1^{(0)}}$ (the closed unit ball of B^*) the inequality,*

$$|\langle\theta(a) * \theta(c), g\rangle| \leq M|\langle a * c, \theta^*(g)\rangle| \tag{3.1}$$

holds. Then $\theta^{**} : A^{**} \longrightarrow B^{**}$ is strongly $*-$zero-product preserving.

Proof. We only prove the case when $*$ is Lie product. Let $m, n \in A^{**}$ and let $(a_\alpha)_\alpha$ and $(c_\beta)_\beta$ be two nets in A such that

$$m = w^* - \lim_\alpha a_\alpha, \quad n = w^* - \lim_\beta c_\beta.$$

By the Arens regularity of A and B, it is clear that

$$[m, n] = w^* - \lim_\alpha w^* - \lim_\beta [a_\alpha, c_\beta] \tag{3.2}$$

and

$$[\theta^{**}(m), \theta^{**}(n)] = w^* - \lim_{\alpha} w^* - \lim_{\beta}[\theta(a_\alpha), \theta(c_\beta)]. \tag{3.3}$$

So by (3.1), (3.2) and (3.3), for all $g \in \overline{B_1^{(0)}}$ we have,

$$|\langle[\theta^{**}(m), \theta^{**}(n)], g\rangle| \leq M|\langle[m, n], \theta^*(g)\rangle|.$$

This shows that

$$\|[\theta^{**}(m), \theta^{**}(n)]\| \leq M\|\theta\|\|[m, n]\|.$$

Applying Theorem 2.6 shows that, θ^{**} is strongly Lie zero-product preserving. In the case when $*$ is Jordan product or original product, a similar argument can be applied. Note that in the case when $*$ is the original product, the conclusion without the hypotheses of Arens regularity of A and B is valid. □

Corollary 3.2. *Let A and B be Arens regular normed algebras and let $\theta : A \longrightarrow B$ be a bounded $*-$ homomorphism. Then θ^{**} is strongly $*-$zero-product preserving.*

Proof. As θ is $*-$homomorphism so for all $a, c \in A$, $\theta(a * c) = \theta(a) * \theta(c)$. It follows that

$$\begin{aligned}|\langle\theta(a) * \theta(c), g\rangle| &= |\langle\theta(a * c), g\rangle| \\ &= |\langle a * c, \theta^*(g)\rangle|, \quad g \in \overline{B_1^{(0)}}, a, c \in A.\end{aligned}$$

Applying Theorem 3.1 implies that θ^{**} is strongly $*-$zero-product preserving. Note that in the case when $*$ is the original product, the condition of Arens regularity of A and B is surplus. □

Theorem 3.3. *Let $\mathbb{W}$ be a non-zero normed vector space and let $f \in \mathbb{W}^*$ be a non-zero element such that $\|f\| \leq 1$. Also let $\theta : \mathbb{W}_f \longrightarrow \mathbb{W}_f$ be a bounded strongly $*-$zero-product preserving map. Then $\theta^{**} : (\mathbb{W}_f)^{**} \longrightarrow (\mathbb{W}_f)^{**}$ is strongly $*-$zero-product preserving.*

Proof. As by [6, proposition 2.1] $\mathbb{W}_f$ is Arens regular, it is enough to show that the inequality (3.1) holds. Let $*$ be original product or Jordan product and θ be strongly $*-$zero-product preserving. Then by [4, Theorem 3.6 and Theorem 3.7] $f \circ \theta = 0$ or $\theta(\ker f) \subseteq \ker f$.

In the case when $*$ is Lie product and θ is strongly Lie zero-product preserving then by Theorem 2.4 $f \circ \theta = 0$ or $\theta(\ker f) \subseteq \ker f$ or $\theta(a) = f(a)\theta(e) + f \circ \theta \circ \varphi(a)e$, for some continuous linear map $\varphi : \mathbb{W}_f \longrightarrow \ker f$ and for some $e \in f^{-1}(\{1\})$. We obtain the conclusion in all of the following cases.

- $f \circ \theta = 0$. In this case $\theta(a) * \theta(c) = 0$ for all $a, c \in \mathbb{W}_f$. So the inequality (3.1) holds for all $M \geq 0$. Hence in the case when $f \circ \theta = 0$, θ^{**} is strongly $*-$zero-product preserving.

- $\theta(\ker f) \subseteq \ker f$. Let $e \in f^{-1}(\{1\})$. Define $\varphi : \mathbb{W}_f \longrightarrow \ker f$ such that $\varphi(a) = a - f(a)e, a \in \mathbb{W}_f$. So we can conclude that
$$a = f(a)e + \varphi(a),$$
and
$$\theta(a) = f(a)\theta(e) + \theta \circ \varphi(a), a \in \mathbb{W}_f.$$
Hence
$$\begin{aligned}\langle \theta(a) * \theta(c), g\rangle &= \langle (f(a)\theta(e) + \theta \circ \varphi(a)) * (f(c)\theta(e) + \theta \circ \varphi(c)), g\rangle \\ &= f \circ \theta(e)\langle a * c, \theta^*(g)\rangle, \ g \in \overline{B_1^{(0)}}. \qquad (3.4)\end{aligned}$$
It follows that
$$\begin{aligned}|\langle \theta(a) * \theta(c), g\rangle| &= |f \circ \theta(e)\langle a * c, \theta^*(g)\rangle| \\ &\leq \|f\|\|e\|\|\theta\||\langle a * c, \theta^*(g)\rangle| \\ &\leq \|e\|\|\theta\||\langle a * c, \theta^*(g)\rangle|, a, c \in \mathbb{W}_f, \ g \in \overline{B_1^{(0)}}.\end{aligned}$$
So the inequality (3.1) holds for $M = \|e\|\|\theta\|$, that implies θ^{**} is strongly $*-$zero-product preserving.
Note that the accuracy of the equality (3.4) for example in the case when $*$ is Jordan product is as follows.
$$\begin{aligned}\theta(a) * \theta(c) &= \theta(a) \circ \theta(c) \\ &= (f(a)\theta(e) + \theta \circ \varphi(a)) \circ (f(c)\theta(e) + \theta \circ \varphi(c)) \\ &= f(f(a)\theta(e) + \theta \circ \varphi(a))\theta(c) + f(f(c)\theta(e) + \theta \circ \varphi(c))\theta(a) \\ &= f(a)f \circ \theta(e)\theta(c) + f(c)f \circ \theta(e)\theta(a) \\ &= f \circ \theta(e)\theta(f(a)c) + f \circ \theta(e)\theta(f(c)a) \\ &= f \circ \theta(e)(\theta(ac) + \theta(ca)) \\ &= f \circ \theta(e)\theta(a \circ c) \\ &= f \circ \theta(e)\theta(a * c), \quad a, c \in \mathbb{W}_f.\end{aligned}$$
- $*$ is Lie product and $\theta(a) = f(a)\theta(e) + f \circ \theta \circ \varphi(a)e$, for some continuous linear map $\varphi : \mathbb{W}_f \longrightarrow \ker f$ and for some $e \in f^{-1}(\{1\})$.
Let $a, c \in \mathbb{W}_f$. So by (2.3),
$$\begin{aligned}[\theta(a), \theta(c)] &= f \circ \theta \circ \varphi([a, c])[\theta(e), e] \\ &= \theta \circ \varphi([a, c]) \cdot [\theta(e), e], \quad a, c \in \mathbb{W}_f.\end{aligned}$$
It follows that
$$\begin{aligned}|\langle [\theta(a), \theta(c)], g\rangle| &= |\langle \theta \circ \varphi([a, c]) \cdot [\theta(e), e], g\rangle| \\ &= |\langle \theta \circ \varphi([a, c]), [\theta(e), e] \cdot g\rangle| \\ &= |\langle [a, c], (\theta \circ \varphi)^*([\theta(e), e] \cdot g)\rangle|, \ g \in \overline{B_1^{(0)}}.\end{aligned}$$

Similar to the proof of Theorem 3.1 one can simply verify that,

$$|\langle[\theta^{**}(m), \theta^{**}(n)], g\rangle| = |\langle[m, n], (\theta \circ \varphi)^*([\theta(e), e] \cdot g)\rangle|, m, n \in (\mathbb{W}_f)^{**}.$$

So we can conclude that

$$\|[\theta^{**}(m), \theta^{**}(n)]\| \leq \|(\theta \circ \varphi)^*\| \|[\theta(e), e]\| \|[m, n]\|.$$

This shows that θ^{**} is strongly Lie zero-product preserving.

□

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