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HÖLDER CONTINUITY OF SOLUTION MAPS TO A PARAMETRIC WEAK VECTOR EQUILIBRIUM PROBLEM

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Abstract. In this paper, by using a new concept of strong convexity, we obtain sufficient conditions for Hölder continuity of the solution mapping for a parametric weak vector equilibrium problem in the case where the solution mapping is a general set-valued one. Without strong monotonicity assumptions, the Hölder continuity for solution maps to parametric weak vector optimization problems is discussed.
Keywords: Vector equilibrium problem, solution mapping, Hölder continuity, linear scalarization.
MSC(2010): Primary: 49K40; Secondary: 90C31, 47H09.

1. **Introduction**

Let X and Y be real topological vector spaces, and K a nonempty subset of X. Let C be a nonempty, closed, and convex cone in Y with nonempty interior, i.e., $int\, C \neq \emptyset$. Let $f : X \times X \to Y$ be a vector-valued bifunction. The weak vector equilibrium problem is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -int\, C, \quad \forall y \in K.$$

This model was extension and generalization from the equilibrium problem in [10]. It is well known that the vector equilibrium problem provides a unified model of several classes of problems, including, vector variational inequality problems, vector complementarity problems, vector optimization problems, and vector saddle point problems, so on.

The existence of solutions for weak vector equilibrium problems is the most popular topics that has been studied intensively and extensively; for example, see [7, 11, 19]. By the way, when constraint set and/or objective function are perturbed by parameters, the stability analysis is also interesting topics for optimization theory. Stability analysis considers the effect of slight change on

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a parameter to the solution of problem, for examples including, lower(upper) semicontinuity, continuity, and lower(upper) Lipschitz or lower(upper) Hölder continuity. Hölder stability is one of the most studied topic in the theory of stability for vector equilibrium problems and related problems (see [1–4, 6, 8, 9, 12, 14–18, 20–30]).

At first, the Hölder continuity of solution maps to parametric vector equilibrium problems depends upon the strong monotonicity assumptions, for example we refer the reader to [1–3, 8, 9, 23]. Such conditions lead to uniqueness of the solution and inapplicable for some special cases. In many situations, the solution sets may not be a singleton. Is there a way to avoid the monotonicity assumptions? In order to answer this problem, Li et al. [16, 18] introduced Hölder related assumptions and established the Hölder continuity of solutions to generalized vector (quasi)equilibrium problems. Although, these conditions do not allow a uniqueness solution it requires have information about a solution a solution set. Li and Li [17] introduced the concepts of Hölder strongly monotone and Hölder continuous with respect to the interior point of an ordering cone. By using a linear scalarization technique, they obtained the Hölder continuity of the set-valued solution sets for parametric weak vector equilibrium problems in metric spaces. Peng et al. [25] could omit the strong monotonicity assumption. Based on the linear scalarization technique, they obtained the Hölder continuity of set-valued approximate solution maps. That means the Hölder property holds for nearby original solution sets. For more related results, see [4,26]. Recently, Anh et al., [5] gave the sufficient condition for Hölder continuity of unique solution mappings to parametric equilibrium problems, by using the strong convexity assumption. Without strong monotonicity, their results can be applicable for parametric variational inequality and optimization problems.

Based on the above literature, the aim of this paper is to establish the Hölder continuity of solution maps to a parametric weak vector equilibrium problem without the strong monotonicity assumptions and requirement for information of solution sets. Inspired by [5, 17], we introduce the new concept of strong convexity with respect to an interior point in ordering cones and establish the sufficient conditions for Hölder continuity for set-valued exact solution maps to a parametric weak vector equilibrium problem, by using a linear scalarization method. Application to a parametric weak vector optimization problem is also discussed when strong monotonicity assumptions could omit.

The structure of the paper is as follows. Section 2 presents a parametric weak generalized vector equilibrium problem and materials used in the rest of this paper. We establish, in Section 3, a sufficient condition for the Hölder continuity of the solution mapping to a parametric weak vector equilibrium problem. We give some examples to illustrate that our main results are different from the corresponding ones in the literature. Section 4 is reserved for an application of the main result to a parametric weak vector optimization problem.

2. **Preliminaries**

In the sequel, $\|\cdot\|$ and $d(\cdot,\cdot)$ denote the norm and metric in any normed space and metric space, respectively. $B_X(\mathbf{0},\delta)$ denotes the closed ball with centre $\mathbf{0}\in X$ and radius $\delta>0$, int C stands for the interior of C. Throughout this paper, if not otherwise specified, X,Λ,M will denote three metric spaces and Y a linear normed space. Let Y^* be the topological dual space of Y. For any $\xi\in Y^*$, we introduce $\|\xi\|_*:=\sup\{|\langle\xi,x\rangle|:\|x\|=1\}$, where $\langle\xi,x\rangle$ denotes the value of ξ at x. Let $C\subset Y$ be a pointed, closed and convex cone with $int\,C\neq\emptyset$. Let

$$C^*:=\{\xi\in Y^*:\langle\xi,y\rangle\geq 0,\forall y\in C\},$$

be the dual cone of C. Since $int\,C\neq\emptyset$, the dual cone C^* of C has a weak* compact base. Let $e\in int\,C$. Then,

$$B_e^*:=\{\xi\in C^*:\langle\xi,e\rangle=1\},$$

is a weak* compact base of C^*.

Let $N(\lambda_0)\subset\Lambda$ and $N(\mu_0)\subset M$ be neighborhoods of considered points λ_0 and μ_0, respectively. Let $K:\Lambda\rightrightarrows X$ be a set-valued mapping and $f:X\times X\times M\to Y$ be a vector-valued mapping. For each $\lambda\in N(\lambda_0)$ and $\mu\in N(\mu_0)$, we consider the following parametric weak vector equilibrium problem: Find $\bar{x}\in K(\lambda)$ such that

$$f(\bar{x},y,\mu)\notin -int\,C,\ \forall y\in K(\lambda). \tag{2.1}$$

For each $\lambda\in N(\lambda_0)$ and $\mu\in N(\mu_0)$, the weak solution set of (2.1) is defined by

$$S_W(\lambda,\mu):=\{x\in K(\lambda):f(x,y,\mu)\notin -int\,C,\ \forall y\in K(\lambda)\}.$$

For each $\xi\in C^*\setminus\{0\},\lambda\in N(\lambda_0)$ and $\mu\in N(\mu_0)$, the ξ-solution set of (2.1) is defined by

$$S(\xi,\lambda,\mu):=\{x\in K(\lambda):\langle\xi,f(x,y,\mu)\rangle\geq 0,\ \forall y\in K(\lambda)\}.$$

In this paper, we focus on stability properties of this class of problems, so we shall always assume that $S(\xi,\lambda,\mu)$ is nonempty in a neighborhood of the considered point (λ_0,μ_0). For proving our main result, we will refer to the following useful result and existence result for the ξ-solution.

We first introduce the new concept of strong convexity with respect to (w.r.t.) an interior point of an ordering cone C.

Definition 2.1. Let $\varphi:X\to Y$, $B\subseteq X$ and $h,\beta>0$. A vector-valued mapping φ is said to be

(i) *C-convex on B*, with B being convex if and only if for any $x, y \in B$ and $t \in (0, 1)$,

$$t\varphi(x) + (1-t)\varphi(y) \in \varphi(tx + (1-t)y) + C.$$

(ii) *$h.\beta$-strongly C-convex on B w.r.t. $e \in int\, C$*, with B being convex if and only if for any $x, y \in X$ and $t \in (0, 1)$,

$$t\varphi(x) + (1-t)\varphi(y) \in \varphi(tx + (1-t)y) + ht(1-t)d^{\beta}(x, y)e + C.$$

Remark 2.2. (1) From the previous definition, it is clear that (ii) $\Rightarrow$ (i).

(2) If for each $\mu \in N(\mu_0)$ and $x \in K(N(\lambda_0))$, $f(x, \cdot, \mu)$ satisfies (i) on $K(N(\lambda_0))$ w.r.t. $e \in int\, C$, then $f(x, K(N(\lambda_0)), \mu) + C$ is a convex set.

(3) In the case where $Y = \mathbb{R}$, $C = [0, +\infty)$ and $e = 1 \in int\, C = (0, +\infty)$, Definition 2.1 (ii) is reduced to $h.\beta$-strongly C-convex in [5, Definition 2.2].

Lemma 2.3 ([17, Theorem 2.1], [13, Theorem 3.1])**.** *If for each $\mu \in N(\mu_0)$ and $x \in K(N(\lambda_0))$, $f(x, K(N(\lambda_0)), \mu) + C$ is a convex set, then*

$$S_W(\lambda, \mu) = \bigcup_{\xi \in C^* \setminus \{0\}} S(\xi, \lambda, \mu) = \bigcup_{\xi \in B_e^*} S(\xi, \lambda, \mu).$$

Lemma 2.4 ([17, Theorem 2.2])**.** *Suppose that the following conditions hold:*

(i) *For each $\mu \in N(\mu_0)$ and each $x \in K(N(\lambda_0))$, $f(x, x, \mu) \in C$;*

(ii) *For each $\mu \in N(\mu_0)$ and each $y \in K(N(\lambda_0))$, $f(\cdot, y, \mu)$ is continuous on $K(N(\lambda_0))$;*

(iii) *For each $x \in K(N(\lambda_0))$ and each $\mu \in N(\mu_0)$, $f(x, \cdot, \mu)$ is C-convex on $K(N(\lambda_0))$;*

(iv) *For each $\lambda \in N(\lambda_0)$, $K(\lambda)$ is a nonempty, compact and convex set.*

Then

(a) *For each $\xi \in B_e^* \subseteq C^*$ and $(\lambda, \mu) \in N(\lambda_0) \times N(\mu_0)$, $S(\xi, \lambda, \mu) \neq \emptyset$.*

(b) *For each $(\lambda, \mu) \in N(\lambda_0) \times N(\mu_0)$, $S_W(\lambda, \mu)$ is a nonempty compact set.*

Definition 2.5. A vector-valued mapping $g : X \times X \to Y$ is said to be C-monotone on $S \subseteq X$ if and only if for any $x, y \in S$,

$$g(x, y) + g(y, x) \in -C.$$

Definition 2.6 ([17])**.** For $h, \beta > 0$. A vector-valued mapping $g : X \times X \to Y$ is said to be *$h.\beta$-Hölder strongly monotone w.r.t.* $e \in int\, C$ on $S \subset X$ if and only if for any $x, y \in S$ with $x \neq y$,

$$g(x, y) + g(y, x) + hd^{\beta}(x, y)e \in -C.$$

Remark 2.7. It is clear that Hölder strongly monotone implies monotone but the converse may not be true. An easy example is that

$$g(x, y) = h(y) - h(x),\ \forall x, y \in X,$$

where $h : X \to Y$, we see that g is C-monotone but not Hölder strongly monotone. Indeed, for any $x, y \in X$ with $x \neq y$,

$$g(x, y) + g(y, x) = (h(y) - h(x)) + (h(x) - h(y)) = 0_Y \in -C.$$

In order to obtain Hölder properties for solution maps to a parametric weak vector equilibrium problem, the Hölder strong monotonicity assumption has been imposed in [17, Theorem 3.1]. This causes the theorem cannot be applied to the parametric weak vector optimization problem.

Now, we recall definitions of Hölder continuity for real-valued and vector-valued mappings, respectively.

Definition 2.8 ([2])**.** For $m, \beta > 0$ and $\theta \geq 0$. A function $f : X \times X \times M \to \mathbb{R}$ is said to be *$m.\beta$-Hölder continuous around μ_0, θ-uniformly in $S \subset X$* if and only if there is a neighborhood $N(\mu_0)$ such that for any $\mu_1, \mu_2 \in N(\mu_0)$ and $x, y \in S$ with $x \neq y$,

$$|f(x, y, \mu_1) - f(x, y, \mu_2)| \leq m d^{\beta}(\mu_1, \mu_2) d^{\theta}(x, y).$$

Now, we further recall the concept of Hölder continuity with respect to the element in interior of a fixed cone C for vector-valued mappings, which is generalized from real-valued mappings.

Definition 2.9. For $m, \beta > 0$ and $\theta \geq 0$. A vector-valued mapping $f : X \times X \times M \to Y$ is said to be *$m.\beta$-Hölder continuous around μ_0, θ-uniformly over a subset $S \subseteq X$* w.r.t. $e \in \mathit{int}\, C$ if and only if there is a neighborhood $N(\mu_0)$ of μ_0 such that for any $\mu_1, \mu_2 \in N(\mu_0)$ and $x, y \in S$ with $x \neq y$,

$$f(x, y, \mu_1) \in f(x, y, \mu_2) + \, m d^{\theta}(x, y) d^{\beta}(\mu_1, \mu_2)[-e, e],$$

where $[-e, e] := \{x : x \in e - C \text{ and } x \in -e + C\}$.

Remark 2.10. If $\beta = 1$, then f is said to be Lipschitz continuity. Note that, if $\theta = 0$, we say that f is *$m.\beta$-Hölder continuous around μ_0* which was first introduced in [17, Definition 2.6]. The dependence of x, y helps to sharpen the Hölder continuity result. In particular, if S is bounded, we can take $\theta = 0$ because $d(x, y) \leq L$ for some positive real number L for all $x, y \in S$.

Next, we recall the definition of Hölder continuity for general set-valued mappings.

Definition 2.11 (Classical notation)**.** For $l, \alpha > 0$, a set-valued mapping $G : (\Lambda, d_{\Lambda}) \rightrightarrows (X, d_X)$ is said to be *$l.\alpha$-Hölder continuous around* λ_0 if and only if there is a neighborhood $N(\lambda_0)$ of λ_0 such that for any $\lambda_1, \lambda_2 \in N(\lambda_0)$,

$$G(\lambda_1) \subseteq G(\lambda_2) + l B(0, d^{\alpha}(\lambda_1, \lambda_2)).$$

3. **Hölder continuity of solution maps**

To obtain the Hölder continuity of the solution mapping around (λ_0, μ_0), we assume that there exist neighborhood $N(\lambda_0) \times N(\mu_0)$ of (λ_0, μ_0) which satisfied the following assumptions:

(H_1) $K(\cdot)$ is $l.\beta$-Hölder continuous around μ_0 on $N(\lambda_0)$ and has midpoint convex valued, i.e., for any $\lambda \in N(\lambda_0)$, $\frac{x+y}{2} \in K(\lambda), \forall x, y \in K(\lambda)$.

(H_2) For each $\mu \in N(\mu_0), f(\cdot, \cdot, \mu)$ is C-monotone on $K(N(\lambda_0)) \times K(N(\lambda_0))$.

(H_3) For each $\mu \in N(\mu_0)$ and $x \in K(N(\lambda_0)), f(x, \cdot, \mu)$ is $n.\delta$-Hölder continuous as well as $h.\alpha$-strongly C-convex w.r.t. $e \in int\, C$ on $conv\,(K(N(\lambda_0)))$.

(H_4) For each $x, y \in K(N(\lambda_0)), f(x, y, \cdot)$ is $m.\gamma$-Hölder continuous around μ_0 on $N(\mu_0)$, θ uniformly in $K(N(\lambda_0))$ w.r.t. $e \in int\, C$ with $\theta < \alpha$.

Remark 3.1. In condition (H_1), a convex set is obviously a midpoint convex set, but converse is not true. An example is that the set of all rational numbers $\mathbb{Q}$. It can be proved that closed and midpoint convex set is a convex set. Since the union of convex sets may not be convex, condition (H_3) was assumed on convex hull of union of convex set.

The following two lemmas are useful for proving Hölder properties for the solution map $S(\xi, \lambda, \mu)$.

Lemma 3.2. *Under assumption* (H_3)*, the following properties hold*:

(i) *For each* $\xi \in B_e^*$, $\mu \in N(\mu_0)$, $x \in K(N(\lambda_0))$ *and for each* $y_1, y_2 \in conv\,(K(N(\lambda_0)))$

$$|\langle \xi, f(x, y_1, \mu)\rangle - \langle \xi, f(x, y_2, \mu)\rangle| \leq nd^\delta(y_1, y_2).$$

(ii) *For each* $t \in (0, 1)$

$$\begin{aligned}&\langle \xi, f\,(x, ty_1 + (1-t)y_2, \mu)\rangle \\ &\quad\leq t\,\langle \xi, f\,(x, y_1, \mu)\rangle + (1-t)\,\langle \xi, f\,(x, y_2, \mu)\rangle - t(1-t)d^\beta(x, y).\end{aligned}$$

Proof. (i) Since $f(x, \cdot, \mu)$ is $n.\delta$-Hölder continuous on $conv\,(K(N(\lambda_0)))$ w.r.t. $e \in int\, C$, for each $y_1, y_2 \in conv\,(K(N(\lambda_0)))$ and $\xi \in B_e^*$, we get

$$-nd^\delta(y_1, y_2) \leq \langle \xi, f(x, y_1, \mu)\rangle - \langle \xi, f(x, y_2, \mu)\rangle \leq nd^\delta(y_1, y_2).$$

This implies that

$$|\langle \xi, f(x, y_1, \mu)\rangle - \langle \xi, f(x, y_2, \mu)\rangle| \leq nd^\delta(y_1, y_2).$$

(ii) For each $t \in (0, 1)$, it follows from the strong convexity of f that

$$\begin{aligned}&\left\langle \xi, f\,(x, ty_1 + (1-t)y_2, \mu) + t(1-t)d^\beta(x, y)e\right\rangle \\ &\qquad\qquad\leq \langle \xi, tf\,(x, y_1, \mu) + (1-t)f\,(x, y_2, \mu)\rangle\,.\end{aligned}$$

By virtue of linearity of ξ and $e \in int\, C$, one has

$$\begin{aligned}&\langle \xi, f\,(x, ty_1 + (1-t)y_2, \mu)\rangle \\ &\quad\leq t\,\langle \xi, f\,(x, y_1, \mu)\rangle + (1-t)\,\langle \xi, f\,(x, y_2, \mu)\rangle - t(1-t)d^\beta(x, y).\end{aligned}$$

□

Lemma 3.3. *Suppose that assumption* (H$_4$) *is satisfied. Then, for each* $\xi \in B_e^*$, $\mu_1, \mu_2 \in N(\mu_0)$ *and* $x, y \in K(N(\lambda_0))$ *with* $x \neq y$,

$$|\langle \xi, f(x, y, \mu_1)\rangle - \langle \xi, f(x, y, \mu_2)\rangle| \leq md^{\gamma}(\mu_1, \mu_2)d^{\theta}(x, y).$$

Proof. Now, we show that the conclusion holds. Arguing as in the proof of Lemma 3.2, for each $\xi \in B_e^*$, we have

$$\begin{aligned} -md^{\gamma}(\mu_1, \mu_2)d^{\theta}(x, y) \\ \leq \langle \xi, f(x, y, \mu_1)\rangle - \langle \xi, f(x, y, \mu_2)\rangle \leq md^{\gamma}(\mu_1, \mu_2)d^{\theta}(x, y), \end{aligned}$$

that is,

$$|\langle \xi, f(x, y, \mu_1)\rangle - \langle \xi, f(x, y, \mu_2)\rangle| \leq md^{\gamma}(\mu_1, \mu_2)d^{\theta}(x, y).$$

□

We now present the Hölder continuity for the ξ-solution maps to the parametric weak vector equilibrium problem (**PWVEP**).

Lemma 3.4. *Assume that for each* $\xi \in B_e^*$, *the* ξ*-solution set* $S(\xi, \lambda, \mu)$ *for* (2.1) *exists in a neighborhood* $N(\lambda_0) \times N(\mu_0)$ *of the considered point* (λ_0, μ_0). *Furthermore, assume that assumptions* (H$_1$)-(H$_4$) *hold. Then, for any* $\bar{\xi} \in B_e^*$, *there exist open neighborhoods* $N'(\bar{\xi})$ *of* $\bar{\xi}$, $N'_{\bar{\xi}}(\lambda_0)$ *of* λ_0 *and* $N'_{\bar{\xi}}(\mu_0)$ *of* μ_0, *such that, the* ξ*-solution set* $S(\cdot, \cdot, \cdot)$ *on* $N'(\bar{\xi}) \times N'_{\bar{\xi}}(\lambda_0) \times N'_{\bar{\xi}}(\mu_0)$ *is a singleton and satisfies the following condition, for each* $\xi \in N'(\bar{\xi})$ *and* $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N'_{\bar{\xi}}(\lambda_0) \times N'_{\bar{\xi}}(\mu_0)$:

$$\begin{aligned} &d(x(\xi, \lambda_1, \mu_1), x(\xi, \lambda_2, \mu_2)) \\ &\leq \left(\frac{m}{h}\right)^{\frac{1}{\alpha-\theta}} d^{\frac{\gamma}{\alpha-\theta}}(\mu_1, \mu_2) + \left(\frac{2nl^{\delta}}{h}\right)^{\frac{1}{\alpha}} d^{\frac{\delta\beta}{\alpha}}(\lambda_1, \lambda_2), \end{aligned} \tag{3.1}$$

where $x(\xi, \lambda_1, \mu_1) \in S(\xi, \lambda_1, \mu_1)$ *and* $x(\xi, \lambda_2, \mu_2) \in S(\xi, \lambda_2, \mu_2)$.

Proof. For any $\bar{\xi} \in B_e^*$, let $N'(\bar{\xi}) \times N'_{\bar{\xi}}(\lambda_0) \times N'_{\bar{\xi}}(\mu_0) \subseteq B_e^* \times N(\lambda_0) \times N(\mu_0)$ be open (where $N'_{\bar{\xi}}(\lambda_0), N'_{\bar{\xi}}(\mu_0)$ depend on $\bar{\xi}$). Obviously, for each $(\xi, \lambda, \mu) \in N'(\bar{\xi}) \times N'_{\bar{\xi}}(\lambda_0) \times N'_{\bar{\xi}}(\mu_0)$, $S(\xi, \lambda, \mu)$ is nonempty. First, we want to show that

$$d(x(\xi, \lambda_1, \mu_1), x(\xi, \lambda_1, \mu_2)) \leq \left(\frac{m}{h}\right)^{\frac{1}{\alpha-\theta}} d^{\frac{\gamma}{\alpha-\theta}}(\mu_1, \mu_2), \tag{3.2}$$

for all $x(\xi, \lambda_1, \mu_1) \in S(\xi, \lambda_1, \mu_1)$ and $x(\xi, \lambda_1, \mu_2) \in S(\xi, \lambda_1, \mu_2)$.

Since $K(\lambda_1)$ is midpoint convex, one has $(x(\xi, \lambda_1, \mu_1) + x(\xi, \lambda_1, \mu_2))/2 \in K(\lambda_1)$. Then

$$\left\langle \xi, f\left(x(\xi, \lambda_1, \mu_2), \frac{x(\xi, \lambda_1, \mu_1) + x(\xi, \lambda_1, \mu_2)}{2}, \mu_2\right)\right\rangle \geq 0. \tag{3.3}$$

Thanks to Lemma 3.2(ii), one has

$$\begin{aligned}
&\frac{h}{4}d^{\alpha}(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2))\\
&\quad\leq -\left\langle \xi, f\left(x(\xi,\lambda_1,\mu_2),\frac{x(\xi,\lambda_1,\mu_1)+x(\xi,\lambda_1,\mu_2)}{2},\mu_2\right)\right\rangle\\
&\qquad+\frac{1}{2}\left\langle \xi, f(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_1,\mu_1),\mu_2)\right\rangle\\
&\qquad+\frac{1}{2}\left\langle \xi, f(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_1,\mu_2),\mu_2)\right\rangle.
\end{aligned}$$

By virtue of (3.3), we get that

$$\begin{aligned}
&\frac{h}{2}d^{\alpha}(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2))\\
&\quad\leq \left\langle \xi, f(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_1,\mu_1),\mu_2)\right\rangle+\left\langle \xi, f(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_1,\mu_2),\mu_2)\right\rangle.
\end{aligned}$$

It follows from monotonicity of f that

$$\begin{aligned}
&\frac{h}{2}d^{\alpha}(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2))\\
&\quad\leq \left\langle \xi, f(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_1,\mu_1),\mu_2)\right\rangle\\
&\quad\leq -\left\langle \xi, f(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2),\mu_2)\right\rangle.
\end{aligned}\tag{3.4}$$

Notice that $(x(\xi,\lambda_1,\mu_1)+x(\xi,\lambda_1,\mu_2))/2\in K(\lambda_1)$ because of midpoint convexity. Then

$$\left\langle \xi, f\left(x(\xi,\lambda_1,\mu_1),\frac{x(\xi,\lambda_1,\mu_1)+x(\xi,\lambda_1,\mu_2)}{2},\mu_1\right)\right\rangle\geq 0.\tag{3.5}$$

Lemma 3.2(ii) give that

$$\begin{aligned}
&\frac{h}{4}d^{\alpha}(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2))\\
&\quad\leq -\left\langle \xi, f\left(x(\xi,\lambda_1,\mu_1),\frac{x(\xi,\lambda_1,\mu_1)+x(\xi,\lambda_1,\mu_2)}{2},\mu_1\right)\right\rangle\\
&\qquad+\frac{1}{2}\left\langle \xi, f(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_1),\mu_1)\right\rangle\\
&\qquad+\frac{1}{2}\left\langle \xi, f(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2),\mu_1)\right\rangle.
\end{aligned}$$

By virtue of (3.5) we get that

$$\begin{aligned}
&\frac{h}{2}d^{\alpha}(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2))\\
&\quad\leq \left\langle \xi, f(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_1),\mu_1)\right\rangle+\left\langle \xi, f(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2),\mu_1)\right\rangle.
\end{aligned}$$

It follows from condition (H_2) that

$$\frac{h}{2}d^{\alpha}(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2)) \tag{3.6}$$
$$\leq \langle \xi, f(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2),\mu_1)\rangle .$$

Adding (3.4) and (3.6) together with Lemma 3.3, we get that

$$\begin{aligned} &hd^{\alpha}(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2))\\ &\quad\leq \langle \xi, f(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2),\mu_1)\rangle - \langle \xi, f(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2),\mu_2)\rangle\\ &\quad\leq md^{\gamma}(\mu_1,\mu_2)d^{\theta}(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2)). \end{aligned}$$

Hence, (3.2) holds.

Next, we want to show that for all $x(\xi,\lambda_1,\mu_2)\in S(\xi,\lambda_1,\mu_2)$ and $x(\xi,\lambda_2,\mu_2)\in S(\xi,\lambda_2,\mu_2)$,

$$d(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2)) \leq \left(\frac{4nl^{\delta}}{h}\right)^{\frac{1}{\alpha}} d^{\frac{\delta\beta}{\alpha}}(\lambda_1,\lambda_2). \tag{3.7}$$

By the definition of $S(\xi,\lambda_2,\mu_2)$, we have

$$\langle \xi, f\left(x(\xi,\lambda_2,\mu_2),y,\mu_2\right)\rangle \geq 0, \quad \forall y\in K(\lambda_2).$$

By $l.\beta$-Hölder continuity of $K(\cdot)$, there exist $\sigma_1\in K(\lambda_1)$ and $\sigma_2\in K(\lambda_2)$ such that

$$d(x(\xi,\lambda_2,\mu_2),\sigma_1)\leq ld^{\beta}(\lambda_1,\lambda_2) \text{ and } d(x(\xi,\lambda_1,\mu_2),\sigma_2)\leq ld^{\beta}(\lambda_1,\lambda_2).$$

Also, we have that

$$\langle \xi, f(x(\xi,\lambda_1,\mu_2),\sigma_1,\mu_2)\rangle \geq 0 \quad \text{and} \quad \langle \xi, f(x(\xi,\lambda_2,\mu_2),\sigma_2,\mu_2\rangle \geq 0.$$

Lemma 3.2(ii) implies

$$\begin{aligned} &\frac{1}{4}hd^{\alpha}(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2)) \\ &\quad\leq -\left\langle \xi, f\left(x(\xi,\lambda_2,\mu_2),\frac{x(\xi,\lambda_2,\mu_2)+x(\xi,\lambda_1,\mu_2)}{2},\mu_2\right)\right\rangle\\ &\quad\quad+\frac{1}{2}\langle \xi, f(x(\xi,\lambda_2,\mu_2),x(\xi,\lambda_2,\mu_2),\mu_2)\rangle\\ &\quad\quad+\frac{1}{2}\langle \xi, f(x(\xi,\lambda_2,\mu_2),x(\xi,\lambda_1,\mu_2),\mu_2)\rangle. \end{aligned} \tag{3.8}$$

In view of C-monotonicity of f, one has

$$\langle \xi, f(x(\xi,\lambda_2,\mu_2),x(\xi,\lambda_2,\mu_2),\mu_2)\rangle = 0,$$

and

$$\langle \xi, f(x(\xi,\lambda_2,\mu_2),x(\xi,\lambda_1,\mu_2),\mu_2)\rangle \leq -\langle \xi, f(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2),\mu_2)\rangle.$$

It follows from (3.8) that

$$\frac{1}{4}hd^{\alpha}(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2)) \tag{3.9}$$
$$\leq -\frac{1}{2}\langle \xi, f(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2),\mu_2)\rangle$$
$$-\left\langle \xi, f\left(x(\xi,\lambda_2,\mu_2),\frac{x(\xi,\lambda_2,\mu_2)+x(\xi,\lambda_1,\mu_2)}{2},\mu_2\right)\right\rangle.$$

Since $\sigma_1 \in K(\lambda_1)$, we have that

$$\langle \xi, f(x(\xi,\lambda_1,\mu_2),\sigma_1,\mu_2)\rangle \geq 0. \tag{3.10}$$

And since $\frac{x(\xi,\lambda_2,\mu_2)+\sigma_2}{2} \in K(\lambda_2)$, we also have

$$\left\langle \xi, f\left(x(\xi,\lambda_2,\mu_2),\frac{x(\xi,\lambda_2,\mu_2)+\sigma_2}{2},\mu_2\right)\right\rangle \geq 0. \tag{3.11}$$

Then, by (3.9), (3.10) and (3.11) we see that

$$\frac{1}{4}hd^{\alpha}(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2))$$
$$\leq -\frac{1}{2}\langle \xi, f(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2),\mu_2)\rangle$$
$$-\left\langle \xi, f\left(x(\xi,\lambda_2,\mu_2),\frac{x(\xi,\lambda_2,\mu_2)+x(\xi,\lambda_1,\mu_2)}{2},\mu_2\right)\right\rangle$$
$$+\frac{1}{2}\langle \xi, f(x(\xi,\lambda_1,\mu_2),\sigma_1,\mu_2)\rangle$$
$$+\left\langle \xi, f\left(x(\xi,\lambda_2,\mu_2),\frac{x(\xi,\lambda_2,\mu_2)+\sigma_2}{2},\mu_2\right)\right\rangle$$
$$=\frac{1}{2}\left[\langle \xi, f(x(\xi,\lambda_1,\mu_2),\sigma_1,\mu_2)\rangle - \langle \xi, f(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2),\mu_2)\rangle\right]$$
$$+\Bigg[\left\langle \xi, f\left(x(\xi,\lambda_2,\mu_2),\frac{x(\xi,\lambda_2,\mu_2)+\sigma_2}{2},\mu_2\right)\right\rangle$$
$$-\left\langle \xi, f\left(x(\xi,\lambda_2,\mu_2),\frac{x(\xi,\lambda_2,\mu_2)+x(\xi,\lambda_1,\mu_2)}{2},\mu_2\right)\right\rangle\Bigg].$$

Thanks to 3.2(i), one has

$$\frac{1}{4}hd^{\alpha}(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2))$$
$$\leq \frac{1}{2}nd^{\delta}(\sigma_1,x(\xi,\lambda_2,\mu_2))+\frac{1}{2}nd^{\delta}(\sigma_2,x(\xi,\lambda_1,\mu_2))$$
$$\leq \frac{1}{2}nl^{\delta}d^{\delta\beta}(\lambda_1,\lambda_2)+\frac{1}{2}nl^{\delta}d^{\delta\beta}(\lambda_1,\lambda_2)$$
$$= nl^{\delta}d^{\delta\beta}(\lambda_1,\lambda_2).$$

Hence,

$$d(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2)) \le \left(\frac{4nl^\delta}{h}\right)^{\frac{1}{\alpha}} d^{\frac{\delta\beta}{\alpha}}(\lambda_1,\lambda_2).$$

Finally, we are ready to finish the proof. It follows from (3.2) and (3.7) that

$$\begin{aligned} &d(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_2,\mu_2)) \\ &\quad \le d(x(\xi,\lambda_1,\mu_1),x(\xi,\lambda_1,\mu_2)) + d(x(\xi,\lambda_1,\mu_2),x(\xi,\lambda_2,\mu_2)) \\ &\quad \le \left(\frac{m}{h}\right)^{\frac{1}{\alpha-\theta}} d^{\frac{\gamma}{\alpha-\theta}}(\mu_1,\mu_2) + \left(\frac{4nl^\delta}{h}\right)^{\frac{1}{\alpha}} d^{\frac{\delta\beta}{\alpha}}(\lambda_1,\lambda_2). \end{aligned}$$

Taking $\lambda_2 = \lambda_1$ and $\mu_2 = \mu_1$ in the last inequality, we can get the diameter of $S(\xi,\lambda_1,\mu_1)$ is 0, i.e., the solution map of (**PWVEP**) is singleton in $N(\lambda_0) \times N(\mu_0)$. □

Remark 3.5. The estimation (3.1) indicates the solution mapping $S(\xi,\lambda,\mu)$ satisfies the Hölder property around (λ_0,μ_0). If $\gamma/(\alpha-\theta) = 1$ and $\delta\beta/\alpha = 1$, we say that $S(\xi,\lambda,\mu)$ satisfies Lipschitz property around (λ_0,μ_0).

The following example shows that the midpoint convexity of K is essential.

Example 3.6. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $e = (1,1) \in int\, C$, $\Lambda = M = [0,1]$, $K(\lambda) = [-1,-\lambda] \cup [\lambda,1]$ and $f(x,y,\lambda) = \big(\lambda(y^2-x^2),0\big)$. We see that $K(\Lambda) = [-1,1]$ which is convex. Clearly, for each $\lambda \in [0,1]$, $K(\cdot)$ is 1.1-Hölder continuous; for each $\lambda \in [0,1]$, $f(\cdot,\cdot,\lambda)$ is C-monotone on $[-1,1]\times[-1,1]$; for each $\lambda \in [0,1]$ and each $x \in [-1,1]$, $f(x,\cdot,\mu)$ is 2.1-Hölder continuous and 1.2-strongly C-convex w.r.t. $e = (1,1) \in int\, C$; for each $x,y \in [-1,1]$, $f(x,y,\cdot)$ is 2.1-Hölder continuous w.r.t. $e = (1,1) \in int\, C$. Then, all assumptions in Lemma 3.4 are satisfied, except $K(\lambda)$ has convex valued for all $\lambda \in [0,1]$. Putting $\xi \equiv (1,0)$, by direct computations, we see that

$$S(\xi,\lambda) = \begin{cases} [0,1], & \text{if } \lambda = 0, \\ \{-\lambda,\lambda\}, & \text{if } \lambda \ne 0. \end{cases}$$

Hence, $S(\xi,\cdot)$ is not lower semicontinuous at $\bar{\lambda} = 0$. The reason is that $K(\lambda)$ is not convex for all $\lambda \in (0,1]$.

Theorem 3.7. *Assume that for each $\xi \in B^*_e$, the ξ-solution set $S(\xi,\lambda,\mu)$ for (2.1) exists in a neighborhood $N(\lambda_0) \times N(\mu_0)$ of the considered point (λ_0,μ_0). Furthermore, assume that assumptions* (H_1)-(H_4) *hold. If for each $x \in K(N(\lambda_0))$ and $\mu \in N(\mu_0)$, $f(x,K(N(\lambda_0)),\mu) + C$ is a convex set, then there exist neighborhoods $\widetilde{N}(\lambda_0)$ of λ_0 and $\widetilde{N}(\mu_0)$ of μ_0, such that, the weak solution set $S_W(\cdot,\cdot)$ on $\widetilde{N}(\lambda_0) \times \widetilde{N}(\mu_0)$ is nonempty and satisfies the following* *condition, for each* (λ_1, μ_1), $(\lambda_2, \mu_2) \in \widetilde{N}(\lambda_0) \times \widetilde{N}(\mu_0)$:

$$S_W(\lambda_1, \mu_1) \subset S_W(\lambda_2, \mu_2) + \left(\left(\frac{m}{h}\right)^{\frac{1}{\alpha-\theta}} d^{\frac{\gamma}{\alpha-\theta}}(\mu_1, \mu_2) + \left(\frac{4nl^\delta}{h}\right)^{\frac{1}{\alpha}} d^{\frac{\delta\beta}{\alpha}}(\lambda_1, \lambda_2) \right) B(0,1). \tag{3.12}$$

Proof. For system of $\{N'(\bar{\xi})\}_{\bar{\xi} \in B_e^*}$, which are given by Lemma 3.4, we have $B_e^* \subseteq \bigcup_{\bar{\xi} \in B_e^*} N'(\bar{\xi})$. Since B_e^* weak*-compact set, there exists $\{\xi_1, \xi_2, \dots, \xi_k\} \subset B_e^*$ such that

$$B_e^* \subset \bigcup_{i=1}^{k} N'(\xi_i). \tag{3.13}$$

We put $\widetilde{N}(\lambda_0) := \cap_{i=1}^k N'_{\xi_i}(\lambda_0)$ and $\widetilde{N}(\mu_0) := \cap_{i=1}^k N'_{\xi_i}(\mu_0)$. Then $\widetilde{N}(\lambda_0)$ and $\widetilde{N}(\mu_0)$ are desired neighborhoods of λ_0 and μ_0, respectively. Let $(\lambda, \mu) \in \widetilde{N}(\lambda_0) \times \widetilde{N}(\mu_0)$ be given arbitrarily. For any $\xi \in B_e^*$, by virtue of (3.13), there exists $i_0 \in \{1, 2, \dots, k\}$ such that $\xi \in N'(\xi_{i_0})$. From the construction of the neighborhoods $\widetilde{N}(\lambda_0)$ and $\widetilde{N}(\mu_0)$, one has $(\lambda, \mu) \in N'_{\xi_{i_0}}(\lambda_0) \times N'_{\xi_{i_0}}(\mu_0)$. It follows from Lemma 3.4 that the ξ-solution $S(\xi, \lambda, \mu)$ is a nonempty singleton set. By Lemma 2.3, $S_W(\lambda, \mu) = \cup_{\xi \in B_e^*} S(\xi, \lambda, \mu)$ is nonempty. We show that (3.12) holds. Indeed, taking any $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \widetilde{N}(\lambda_0) \times \widetilde{N}(\mu_0)$, we need to show that for any $x_1 \in S_W(\lambda_1, \mu_1)$, there exists $x_2 \in S_W(\lambda_2, \mu_2)$ such that

$$d(x_1, x_2) \le \left(\frac{m}{h}\right)^{\frac{1}{\alpha-\theta}} d^{\frac{\gamma}{\alpha-\theta}}(\mu_1, \mu_2) + \left(\frac{4nl^\delta}{h}\right)^{\frac{1}{\alpha}} d^{\frac{\delta\beta}{\alpha}}(\lambda_1, \lambda_2). \tag{3.14}$$

Since $x_1 \in S_W(\lambda_1, \mu_2) = \bigcup_{\xi \in B_e^*} S(\xi, \lambda_1, \mu_1)$, there exists $\bar{\xi} \in B_e^*$ such that

$$x_1 := x(\bar{\xi}, \lambda_1, \mu_1) \in S(\bar{\xi}, \lambda_1, \mu_1).$$

By (3.13), there exists $i_0 \in \{1, 2, \dots, k\}$ such that $\bar{\xi} \in N'(\xi_{i_0})$. Thus, by the construction of the neighborhoods $\widetilde{N}(\lambda_0)$ and $\widetilde{N}(\mu_0)$, we have

$$(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in N_{\xi_{i_0}}(\lambda_0) \times N_{\xi_{i_0}}(\mu_0).$$

Then, it follows from Lemma 3.4 that

$$d(x(\bar{\xi}, \lambda_1, \mu_1), x(\bar{\xi}, \lambda_2, \mu_2)) \le \left(\frac{m}{h}\right)^{\frac{1}{\alpha-\theta}} d^{\frac{\gamma}{\alpha-\theta}}(\mu_1, \mu_2) + \left(\frac{4nl^\delta}{h}\right)^{\frac{1}{\alpha}} d^{\frac{\delta\beta}{\alpha}}(\lambda_1, \lambda_2).$$

Putting $x_2 = x(\bar{\xi}, \lambda_2, \mu_2) \in S(\bar{\xi}, \lambda_2, \mu_2)$, then (3.14) holds.

□

Remark 3.8. If $K(N(\lambda_0))$ in (H_4) of Theorem 3.7 is bounded, then we can take $\theta = 0$ because $d(x, y) \le L$ for some $L > 0$ for all $x, y \in K(N(\lambda_0))$.

Then, the condition $\alpha > \theta$ can be removed and we have that, for each (λ_1, μ_1), $(\lambda_2, \mu_2) \in \widetilde{N}(\lambda_0) \times \widetilde{N}(\mu_0)$:

$$S_W(\lambda_1, \mu_1) \subset S_W(\lambda_2, \mu_2) + \left(\left(\frac{m}{h}\right)^{\frac{1}{\alpha}} d^{\frac{\gamma}{\alpha}}(\mu_1, \mu_2) + \left(\frac{4nl^\delta}{h}\right)^{\frac{1}{\alpha}} d^{\frac{\delta\beta}{\alpha}}(\lambda_1, \lambda_2) \right) B(0,1).$$

Remark 3.9. Theorem 3.7 adapts the corresponding results of Li and Li [17] in the following two aspects.

(i) The strong Hölder monotonicity assumption (H_1) in [17, Theorem 3.1] is relaxed to the monotonicity assumption (H_2) in Theorem 3.7.
(ii) The assumption (H_4) in [17, Theorem 3.1] is omitted.

In order to obtain the Hölder continuity for the solution map to a parametric weak vector equilibrium problem, the strong convexity (H_3) was assumed. However, the advantages of our results is that we can derive the Hölder continuity of a solution mapping to a parametric weak vector optimization problem (see Section 4).

The following example give some situations that Theorem 3.7 is applicable while [17, Theorem 3.1] is not.

Example 3.10. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $e = (1,1) \in int\, C$, $\Lambda = M = [0,1]$. Define $K(\lambda) := \{(x_1, x_2) \in [0,2] \times [0,2] : x_1 + x_2 \geq 1 + \lambda\}$ and

$$f(x, y, \lambda) = (1+\lambda)\left((y_1^2 - x_1^2), (y_2^2 - x_2^2)\right).$$

Direct computations show that $K(\Lambda) = K(0)$ which is convex. Obviously, for each $\lambda \in \Lambda$, $K(\cdot)$ is 1.1-Hölder continuous with convex valued; for each $\lambda \in [0,1]$ and each $x \in K(0)$, $f(x, \cdot, \lambda)$ is 4.1-Hölder continuous and 1.2-strongly C-convex w.r.t. $e = (1,1) \in int\, C$ on $K(0)$; for each $x, y \in K(0)$, $f(x, y, \cdot)$ is 4.1-Hölder continuous on $[0,1]$, 1 uniformly on K(0) w.r.t. $e = (1,1) \in int\, C$. Assumption (H_2) is clear. So all assumptions of Theorem 3.7 hold. Then, Theorem 3.7 is applicable. However, for each $x, y \in K(0)$ and $\lambda \in [0,1]$,

$$\begin{aligned} &f(x, y, \lambda) + f(y, x, \lambda) \\ &\quad = (1+\lambda)\left((y_1^2 - x_1^2) + (x_1^2 - y_1^2), (y_2^2 - x_2^2) + (x_2^2 - y_2^2)\right) \\ &\quad = 0 \in -C, \end{aligned}$$

which implies that f is not strongly monotone in the sense of Li and Li [17] and so [17, Theorem 3.1] is not applicable. It follows from the direct computations that

$$S_W(\lambda) = A(\lambda) \cup V(\lambda) \cup H(\lambda),$$

where $A(\lambda) := \{(x_1, x_2) \in [0,2] \times [0,2] : x_1 + x_2 = 1 + \lambda\}$, $V(\lambda) = \{(0, v) : v \in [1+\lambda, 2]\}$ and $H(\lambda) = \{(h, 0) : h \in [1+\lambda, 2]\}$.

4. **Application**

Since the parametric weak vector equilibrium problem contains the parametric weak vector otimization problem, we can derive from Theorem 3.7 direct consequences.

Let $g : X \times M \to Y$ be a vector valued mapping. For each $\mu \in M$, the parametric weak optimization problem is to find $\bar{x} \in K(\lambda)$ such that

$$g(y,\mu) - g(\bar{x},\mu) \notin -int\, C, \ \forall y \in K(\lambda). \tag{4.1}$$

Setting

$$f(x,y,\mu) = g(y,\mu) - g(x,\mu), \tag{4.2}$$

then the parametric weak optimization problem becomes a special case of the parametric weak equilibrium problem.

For each $\mu \in M$, the efficient solution set of (4.1) is denoted by

$$S_O(\lambda,\mu) := \{x \in K(\lambda) : g(y,\mu) - g(x,\mu) \notin -int\, C, \quad \forall y \in K(\lambda)\}.$$

The ξ-efficient solution set of (4.1) is

$$S_O(\xi,\lambda,\mu) := \{x \in K(\lambda) : \xi(g(y,\mu)) \geq \xi(g(x,\mu)), \ \forall y \in K(\lambda)\}.$$

We directly obtain the following corollary from Theorem 3.7.

Corollary 4.1. *Assume that for each $\xi \in B_e^*$, the ξ-solution set $S_O(\xi,\lambda,\mu)$ for (4.1) exists in a neighborhood $N(\lambda_0) \times N(\mu_0)$ of the considered point (λ_0,μ_0). Furthermore, assume that the following conditions hold.*

(O_1) *$K(\cdot)$ is $l_1.\beta_1$-Hölder continuous around μ_0 on $N(\lambda_0)$ and has midpoint convex valued.*

(O_2) *For each $\mu \in N(\mu_0)$ and $x \in K(N(\lambda_0)), g(\cdot,\mu)$ is $n_1.\delta_1$-Hölder continuous as well as $h_1.\alpha_1$-strongly C-convex w.r.t. $e \in int\, C$ on $conv\,(K(N(\lambda_0)))$.*

(O_3) *For each $x \in K(N(\lambda_0)), g(x,\cdot)$ is $m_1.\gamma_1$-Hölder continuous around μ_0 on $N(\mu_0)$, θ_1 uniformly in $K(N(\lambda_0))$ w.r.t. $e \in int\, C$ with $\theta_1 < \alpha_1$.*

If for each $\mu \in N(\mu_0)$, $g(K(N(\lambda_0)),\mu) + C$ is convex set, then there exist neighborhoods $\widetilde{N}(\lambda_0)$ of λ_0 and $\widetilde{N}(\mu_0)$ of μ_0, such that, the solution set $S_O(\cdot,\cdot)$ on $\widetilde{N}(\lambda_0) \times \widetilde{N}(\mu_0)$ is singleton and satisfies the following condition, for each (λ_1,μ_1), $(\lambda_2,\mu_2) \in \widetilde{N}(\lambda_0) \times \widetilde{N}(\mu_0)$:

$$\begin{aligned} &S_O(\lambda_1,\mu_1) \subset S_O(\lambda_2,\mu_2) \\ &+\left(\left(\frac{m_1}{h_1}\right)^{\frac{1}{\alpha_1-\theta_1}} d^{\frac{\gamma_1}{\alpha_1-\theta_1}}(\mu_1,\mu_2) + \left(\frac{4n_1 l^{\delta_1}}{h_1}\right)^{\frac{1}{\alpha_1}} d^{\frac{\delta_1\beta_1}{\alpha_1}}(\lambda_1,\lambda_2)\right)B(0,1). \end{aligned}$$

Proof. Setting f as in (4.2), we see that assumptions (H_1) and (H_2) are obviously fulfilled. It sufficies to show that (H_3) and (H_4) are satisfied. Indeed, for any $x \in K(N(\lambda_0)), \mu \in N(\mu_0), y_1, y_2 \in conv\,(K(N(\lambda_0)))$ and $t \in (0,1)$, we have

$$\begin{aligned} f(x,(1-t)y_1 + ty_2,\mu) - (1-t)f(x,y_1,\mu) - tf(x,y_2,\mu) + h_1t(1-t)d^{\beta_1}(y_1,y_2) \\ = g((1-t)y_1 + ty_2,\mu) - g(x,\mu) - (1-t)g(y_1,\mu) + (1-t)g(x,\mu) \\ - tg(y_2,\mu) + tg(x,\mu) + h_1t(1-t)d^{\beta_1}(y_1,y_2) \\ = g((1-t)y_1 + ty_2,\mu) - (1-t)g(y_1,\mu) - tg(y_2,\mu) \\ + h_1t(1-t)d^{\beta_1}(y_1,y_2) \in C. \end{aligned}$$

It is not hard to verify the Hölder continuity (H_3), in fact,

$$\begin{aligned} f(x,y_1,\mu) - f(x,y_2,\mu) &= g(y_1,\mu) - g(x,\mu) - g(y_2,\mu) + g(x,\mu) \\ &= g(y_1,\mu) - g(y_2,\mu) \in n_1 d^{\delta_1}(y_1,y_2)[-e,e]. \end{aligned}$$

Hence (H_3) is fulfilled. Finally, we need to check (H_4) is satisfied. For any $\mu_1, \mu_2 \in N(\mu_0)$ and $x, y \in K(N(\lambda_0))$ with $x \neq y$,

$$\begin{aligned} f(x,y,\mu_1) - f(x,y,\mu_2) &= g(y,\mu_1) - g(x,\mu_1) - g(y,\mu_2) + g(x,\mu_2) \\ &= (g(y,\mu_1) - g(y,\mu_2)) + (g(x,\mu_2) - g(x,\mu_1)) \\ &= 2m_1 d^{\gamma_1}(\mu_1,\mu_2)[-e,e]. \end{aligned}$$

Hence, (H_4) is satisfied with $m = 2m_1$ and $\theta = 0$. □

5. Conclusions

In this paper, we consider a parametric weak vector equilibrium problem in the case of the solution mapping is general set-valued. By using a linear scalarization technique, we establish sufficient conditions for the Hölder continuity of the set-valued mapping for the weak vector equilibrium problem under the assumptions of strong convexity and Hölder continuity with respect to an interior point of a fixed cone. As an application, we derive this Hölder continuity of solution maps to parametric weak vector optimization problems.

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