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DEFINING RELATIONS OF A GROUP $\Gamma = G^{3,4}(2, Z)$ AND ITS ACTION ON REAL QUADRATIC FIELD

M. ASHIQ*, T. IMRAN AND M.A. ZAIGHUM

(Communicated by Ali Reza Ashrafi)

Abstract. In this paper, we have shown that the coset diagrams for the action of a linear-fractional group Γ generated by the linear-fractional transformations $r : z \to \frac{z-1}{z}$ and $s : z \to \frac{-1}{2(z+1)}$ on the rational projective line is connected and transitive. By using coset diagrams, we have shown that $r^3 = s^4 = 1$ are defining relations for Γ. Furthermore, we have studied some important results for the action of group Γ on real quadratic field $Q(\sqrt{n})$. Also, we have classified all the ambiguous numbers in the orbit.

Keywords: Coset diagrams, modular group, linear-fractional transformations, real quadratic field, ambiguous numbers.
MSC(2010): Primary: 20F05; Secondary: 20G40, 20G15.

1. Introduction

It is well known [4,6,7] that the modular group $PSL(2, Z)$, where Z is the ring of integers, is generated by the linear-fractional transformations $x : z \longrightarrow \frac{-1}{z}$ and $y : z \longrightarrow \frac{z-1}{z}$ which satisfy the relations:

$$x^2 = y^3 = 1. \tag{1.1}$$

The group Γ is a proper subgroup of the modular group $PSL(2, Z)$, that is, it contains linear-fractional transformations of the form:

$$z \to \frac{az + b}{cz + d}, \tag{1.2}$$

where $a, b, c, d \in Z$ and $ad - bc = 1$ or 2. Specifically, the linear-fractional transformations of Γ are $r : z \to \frac{z-1}{z}$ and $s : z \to \frac{-1}{2(z+1)}$ which satisfy the relations:

$$r^3 = s^4 = 1. \tag{1.3}$$

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The group Γ^* is the group of transformations of the form:

$$z \to \frac{az+b}{cz+d}. \tag{1.4}$$

If t is the transformation $z \mapsto \frac{z+3}{4z-1}$ so that it belongs to Γ^* and not to Γ, then r, s, t satisfy:

$$r^3 = s^4 = t^2 = (rt)^2 = (st)^2 = 1. \tag{1.5}$$

Once we show that (1.3) are the defining relations of Γ, it is obvious that the relations (1.5) are defining relations for Γ^*. Let $PL(F_q)$ denote the projective line over the finite field F_q, where q is a prime power. The points of $PL(F_q)$ are the elements of F_q together with the additional point ∞, i.e., $PL(F_q) = F_q \cup \{\infty\}$.

A number is said to be square free if its prime decomposition contains no repeated factors. All primes are therefore trivially square free. The algebraic integer of the form $a + b\sqrt{n}$, where n is square free, forms a quadratic field and is denoted by $Q(\sqrt{n})$. If $n > 0$, the field is called real quadratic field, and if $n < 0$, it is called an imaginary quadratic field. The integers in $Q(\sqrt{1})$ are simply called the integers [5]. Consider a subset $Q^*(\sqrt{n}) = \{\frac{a+\sqrt{n}}{c} : a, c \in Z, c \neq 0\, b = \frac{a^2-n}{c}, (a,b,c) = 1\}$ of $Q(\sqrt{n})$. For a fixed non-square positive integer n, if the real quadratic irrational number $\beta = \frac{a+\sqrt{n}}{c}$ and its algebraic conjugate $\overline{\beta} = \frac{a-\sqrt{n}}{c}$ have different signs, such β is known as an ambiguous number [3, 4]. They play an important role in classifying the orbits of group Γ on $Q(\sqrt{n})$. If β and $\overline{\beta}$ are both positive (negative), β is called a totally positive (negative) number. In the action of Γ on $Q(\sqrt{n})$, $Stab_\beta(\Gamma)$ are the on non-trivial stabilizers and in the orbit $\Gamma(\beta))$. We have also classified all the ambiguous numbers in the orbit.

2. **Coset diagrams**

Let G be a group generated by the elements $g_1, g_2, g_3, \ldots, g_k$ acting on a set S. Then the elements of S may be represented by the vertices of a diagram, with edge of 'colour j' directed from vertex e to vertex f whenever $eg_j = f$.

$$y \mapsto eg_j = f.$$

The resulting diagram is a graph whose vertices can be identified with the right cosets in G of the stabiliser N of any given point of S. Hence an edge of 'colour j' joins the coset Nh to the coset Nhg_j, for each h in G, and the resulting diagram is called a *coset diagram*.

This is very similar to the notion of a Schreier's coset graph whose vertices represent the cosets of any given subgroup in a finitely-generated group, and also to that of a Cayley graph whose vertices are the group elements themselves [1, 2], with trivial stabilizer.

We use coset diagrams for the group Γ^* and study its action on the projective line over finite field $PL(F_q)$, where q is a prime power. The coset diagrams defined for the actions of Γ^* are special in a number of ways [4]. First, they are defined for a particular group, namely, Γ^*, which has a presentation in terms of three generators t, r and s. Since there are only three generators, it is possible to avoid using colours as well as the orientation of edges associated with the involution t. For r having order 3 and s having order 4, there is a need to distinguish r from r^2 and also s from s^2 and s^3 . The three cycles (clockwise) of the transformation r are denoted by three green (unbroken) edges of a r-triangle and the four cycles (anti-clockwise) of the transformation s are denoted by four black (broken) edges of a s-square. The action of t is depicted by the symmetry about vertical axis. Fixed points of t, r and s, if they exist, are denoted by heavy dots. For example, the following diagram depicts a permutation representation of Γ^* on twelve points in which:

r acts as: $(1\ 0\ \infty)(2\ 6\ 10)(3\ 8\ 5)(4\ 9\ 7)$
s acts as: $(0\ 5\ 10\ \infty)(1\ 8\ 3\ 4)(2\ 9\ 6\ 7)$, and
t acts as: $(0\ 3)(1\ 10)(2\ 6)(4\ \infty)(5\ 8)(7)(9)$

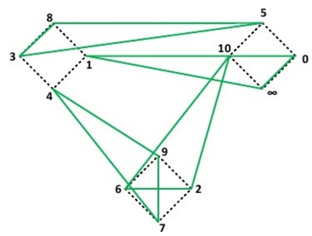

FIGURE 1. Action of Γ^* on $PL(F_{12})$

3. **Observations**

(i) If $z \neq 1, 0, \infty$ then of the vertices z, $r(z)$, $r^2(z)$ of a r-triangle, in a coset diagram for the action of Γ on any subset of the real projective line, one vertex is negative and two are positive.

(ii) If $z \neq -1/2, -1, 0, \infty$ then of the vertices z, $s(z)$, $s^2(z)$, $s^3(z)$ of a s-square, in a coset diagram for the action of Γ on any subset of the real projective line, one vertex is positive and three are negative.

(iii) Let $z = \pm\frac{m}{n}$ where m, n are positive integers with no common factor. For $z \neq 0$, ∞ we define $\|z\| = \max(|m|, |n|)$, then

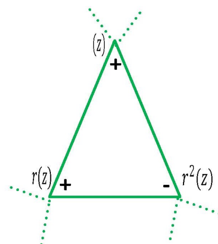

FIGURE 2.

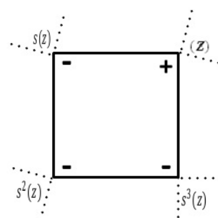

FIGURE 3. Representation of s

(a) If z is positive, then $\|z\| < \|s(z)\|$, $\|z\| < \left\|s^2(z)\right\|$ and $\|z\| < \left\|s^3(z)\right\|$.
(b) If z is negative with $n < 0$, then $\|z\| < \|r(z)\|$ and $\|z\| < \left\|r^2(z)\right\|$.

(iv) Let $\beta = \frac{a+\sqrt{n}}{c}$ be a totally positive quadratic number, then $\beta\overline{\beta} = \frac{a^2-n}{c^2} > 0$ implying that $\frac{b}{c} > 0$. Therefore, either $b, c > 0$ or $b, c < 0$. if $b, c > 0$, then as $\frac{a-\sqrt{n}}{c} > 0$ implies $a - \sqrt{n} > 0$ or $a > \sqrt{n}$ and so $a > 0$. Now if $b, c < 0$, then as $\frac{a+\sqrt{n}}{c} > 0$ implies $a + \sqrt{n} < 0$ or $a < -\sqrt{n}$ and so $a < 0$. Thus β is a totally positive quadratic number either $a, b, c > 0$ or $a, b, c < 0$.

(v) Let $\beta = \frac{a+\sqrt{n}}{c}$ be a totally negative quadratic number, then β and $\overline{\beta}$ both are negative. Thus, $\beta\overline{\beta} = \frac{a^2-n}{c^2} > 0$ implying that $\frac{b}{c} > 0$. Therefore, either $b, c > 0$ or $b, c < 0$. If $b, c > 0$, then as $\frac{a+\sqrt{n}}{c} < 0$ but $c > 0$. Thus $a + \sqrt{n} < 0$ or $a < -\sqrt{n}$ and so $a < 0$. Now if $b, c < 0$, then as $\frac{a-\sqrt{n}}{c} < 0$ but $c < 0$. Thus $a - \sqrt{n} > 0$ or $a > \sqrt{n}$ and so $a > 0$. This proves that β is a totally negative quadratic number either $a < 0$ and $b, c > 0$ or $a > 0$ and $b, c < 0$.

(vi) Let $\beta = \frac{a+\sqrt{n}}{c}$ be an ambiguous number, then β and $\overline{\beta}$ both have opposite signs. Therefore $\beta\overline{\beta} = \frac{a^2-n}{c^2} < 0$ implying that $\frac{b}{c} < 0$ and so

b and c have different signs and so $bc < 0$. Thus α is an ambiguous number if $bc < 0$.

4. Main Results

Theorem 4.1. *The coset diagrams for the action of the group* Γ *on the rational projective line is connected.*

Proof. To prove this we need only to show that for any rational number z there is a path joining z to ∞.

Let $z = z_\circ = \frac{m}{n}$ be a positive rational number. Then $s^i(z_\circ) = \frac{-n}{(m+n)}$ and $\frac{-(m+n)}{m}$ for $i = 1, 2$ or 3. Then, by observation (iii), $\|s(z_\circ)\| = (m+n)$ and $\left\|s^2(z_\circ)\right\| = (m+n)$, so, $\left\|s^i(z_\circ)\right\| > \|z_\circ\|$ for $i = 1, 2$ or 3 respectively. Similarly, if $z_\circ = \frac{m}{n}$ is a negative rational number with $n < 0$, then $r^j(z_\circ) = \frac{m-n}{m}$ and $\frac{-m}{m-n}$ for $j = 1$ or 2 respectively. That is $\|r(z_\circ)\| = (m-n)$ and $\left\|r^2(z_\circ)\right\| = (m-n)$. Hence $\left\|r^j(z_\circ)\right\| > \|z_\circ\|$ for $j = 1$ or 2.

If $z_\circ$ is positive then one of $r^i(z_\circ)$ is negative. If we let this negative number to be z_1 then $\|z_\circ\| > \|z_1\|$. As z_1 is negative one of $s^i(z_1)$ is positive. Let it be z_2, that is, $z_2 = s^i(z_1)$ where $i = 1, 2$ or 3. This implies that $\|z_1\| > \|z_2\|$. If we continue in this way, we obtain a unique alternating sequence of positive and negative rational numbers $z_\circ, z_1, z_2, \ldots$ such that $\|z_\circ\| > \|z_1\| > \|z_2\| \ldots$.

The decreasing sequence of positive integers must terminate, and it can terminate only because ultimately the directed path leads us to a r-triangle with the vertices $1, 0, \infty$ or s-square with the vertices $-1, 1, 0, \infty$.

An alternating sequence of positive and negative rational numbers $z_\circ, z_1, z_2, \ldots$ such that $\|z_\circ\| > \|z_1\| > \|z_2\| > \cdots$ shows that there is a directed graph joining $z_\circ$ to ∞. This implies that every rational number occurs in the diagram and that the diagram for the action of Γ on the rational projective line is connected. □

Theorem 4.2. *The coset diagrams for the action of* $\Gamma = G^{3,4}(2, Z)$ *on the rational projective line is transitive.*

Proof. We shall prove the transitivity of this action, by showing that, if there is a path from a rational number u to a rational number v then there exists some w in Γ such that $uw = v$.

As we have shown in Theorem 4.1 that there exists a path joining $z_\circ$ to ∞, that is, there exists an element $w_1 = r^{\alpha_1}s^{\delta_1}r^{\alpha_2}s^{\delta_2}r^{\alpha_3}s^{\delta_3}\cdots r^{\alpha_m}s^{\delta_m}$ of Γ such that $\infty = uw_1 = u(r^{\alpha_1}s^{\delta_1}r^{\alpha_2}s^{\delta_2}r^{\alpha_3}s^{\delta_3}\cdots r^{\alpha_m}s^{\delta_m})$ where $\alpha_i = 0, 1$ or 2 for $i = 1, 2, \ldots, m$ and $\delta_j = 0, 1, 2$ or 3, where $j = 1, 2, \ldots, m$. Similarly we can find another element w_2 in Γ such that $\infty = vw_2$. Hence $uw_1 = vw_2$ or $uv_1v_2^{-1} = s$. That is, the action of Γ on the rational projective line is transitive. □

Theorem 4.3. $r^3 = s^4 = 1$ *are defining relations for* $\Gamma = G^{3,4}(2, Z)$.

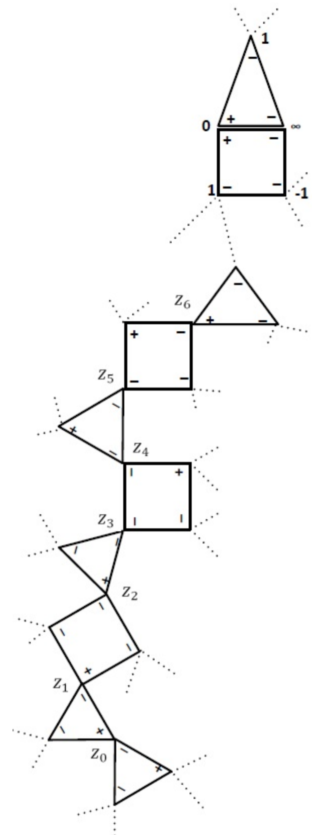

FIGURE 4. Defining relations of Γ

Proof. Suppose $r^3 = s^4 = 1$ are not defining relations of Γ. Then there is a relation of the form $r^{\alpha_1}s^{\delta_1}r^{\alpha_2}s^{\delta_2}r^{\alpha_3}s^{\delta_3}\cdots r^{\alpha_m}s^{\delta_m} = 1$ where $m \geq 1$, $\alpha_i = 1$ or 2, $\delta_j = 1, 2$ or 3 and $i, j = 1, 2, \ldots, m$. We know that neither r nor s can be 1.

The coset diagram (Figure 4) depicts that it does not contain any closed path [7]. For if it contains a closed path and $z_1, z_2, \ldots, z_m$ are the vertices of the triangles in the diagram such that $z_\circ > 0$, then this leads to a contradiction $\|z_\circ\| > \|z_1\| \cdots > \|z_m\| > \|z_\circ\|$. So the coset diagram (Figure 4) does not contain any closed path.

This shows that there are points in the diagram whose 'distance' from the point ∞ is arbitrarily large. Choose $z > 0$, so that the 'distance' from the point z to the point ∞ is greater than m. Define $z_i = r^{\alpha_1}s^{\delta_1}r^{\alpha_2}s^{\delta_2}r^{\alpha_3}s^{\delta_3}\cdots r^{\alpha_j}s^{\delta_j}(z)$ where $i, j = 1, 2, \ldots, m$. Then $\|z_\circ\| > \|z_1\| > \cdots > \|z_m\|$ and in particular

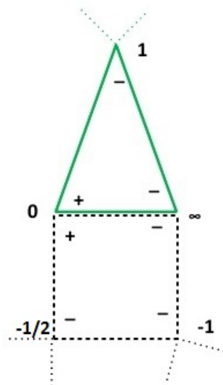

Figure 5. Connection of r and s

Table 1. Actions of r on β

β	a	b	c
$r(\beta)$	$b-a$	$-2a+b+c$	b
$r^2(\beta)$	$-a+c$	c	$-2a+b+c$

$z_m \neq z_\circ$. Thus $r^{\alpha_1}s^{\delta_1}r^{\alpha_2}s^{\delta_2}r^{\alpha_3}s^{\delta_3}\cdots r^{\alpha_m}s^{\delta_m}(z) \neq 1$ and so $r^3 = s^4 = 1$ are defining relations for $\Gamma = G^{3,4}(2, Z)$. □

This of course shows that $r^3 = s^4 = t^2 = (rt)^2 = (st)^2 = 1$ are defining relations for $\Gamma^* =< r, s, t >$.

Theorem 4.4.

(a) *If β is totally negative quadratic number then $r(\beta)$ and $r^2(\beta)$ both are totally positive quadratic numbers.*
(b) *If β is totally positive quadratic number then $s(\beta)$, $s^2(\beta)$ and $s^3(\beta)$ both are totally negative quadratic numbers.*

Proof. (a) Let β be a totally negative quadratic number. Then by Observation (v) there are two possibilities either $a < 0$ and $b, c > 0$ or $a > 0$ and $b, c < 0$. Let $a < 0$ and $b, c > 0$. We can easily tabulate the following information:

From the above information we see that the new values of a, b and c for $r(\beta)$ and $r^2(\beta)$ are positive. Therefore $r(\beta)$ and $r^2(\beta)$ are totally positive quadratic numbers. Now let $a > 0$ and $b, c < 0$, then the new values of a, b and c for $r(\beta)$ and $r^2(\beta)$ are negative. Therefore, $r(\beta)$ and $r^2(\beta)$ are totally positive quadratic numbers.

TABLE 2. Actions of s on β

β	a	b	c
$s(\beta)$	$-a-c$	$\frac{c}{2}$	$2(2a+b+c)$
$s^2(\beta)$	$-3a-2b-c$	$2a+b+c$	$4a+4b+c$
$s^3(\beta)$	$-a-2b$	$2a+2b+\frac{c}{2}$	$2b$

TABLE 3. Ambiguous numbers of r

β	$r(\beta)$	$r^2(\beta)$	$\overline{\beta}$	$\overline{r(\beta)}$	$\overline{r^2(\beta)}$
$+$	$-$	$+$	$-$	$+$	$+$
$+$	$+$	$-$	$-$	$+$	$+$

(b) Let β be a totally positive quadratic number. Then by Observation (iv), there are two possibilities either $a, b, c > 0$ or $a, b, c < 0$. Let $a, b, c > 0$ then we can easily tabulate the following information:

From the above information we see that the new value of a for $s(\beta)$, $s^2(\beta)$ and $s^3(\beta)$ is negative and the new values of b, c for $s(\beta)$, $s^2(\beta)$ and $s^3(\beta)$ are positive. Therefore $s(\beta)$, $s^2(\beta)$ and $s^3(\beta)$ are totally negative quadratic numbers. Now let $a, b, c < 0$ then new value of a for $s(\beta)$, $s^2(\beta)$ and $s^3(\beta)$ is positive and the new values of b, c for $s(\beta)$, $s^2(\beta)$ and $s^3(\beta)$ are negative. Therefore $s(\beta)$, $s^2(\beta)$ and $s^3(\beta)$ are totally negative quadratic numbers. □

5. **Existence of Ambiguous Numbers**

Remark 5.1. The coset diagrams depicting an orbit of the action of Γ on $Q^*(\sqrt{n})$ do not contain a closed path unless there is an ambiguous number in the orbit. A closed path, if it exists, will evolve the element $g = r^{\alpha_1}s^{\delta_1}r^{\alpha_2}s^{\delta_2}\cdots r^{\alpha_{1n}}s^{\delta_n}$ of Γ, where $\alpha_i = 0, 1, 2$ fixing the elements of s_1 of $Q^*(\sqrt{n})$. Let $\beta \in Q^*(\sqrt{n})$ and $\Gamma(\beta)$ denote the orbits of β in Γ. The existence of ambiguous numbers in $\Gamma(\beta)$ is related to the stabilization of Γ. We describe the action of Γ on $Q^*(\sqrt{n})$ in the following theorems.

Theorem 5.2.

(a) *If β is an ambiguous number the one of the $r(\beta)$ and $r^2(\beta)$ is ambiguous and the other is totally positive.*

(b) *If β is an ambiguous number the $s^2(\beta)$ is totally negative while one of the $s(\beta)$, and $s^3(\beta)$ is ambiguous and the other is totally negative.*

Proof. (a) We first assume that β is a positive number, then we have following information:

Similarly, if β is negative number, then we have following information:

Therefore, from above tables we can easily deduce that one of $r(\beta)$ and $r^2(\beta)$ is ambiguous and the other is totally positive.

Table 4. Ambiguous numbers of r

β	$r(\beta)$	$r^2(\beta)$	$\overline{\beta}$	$\overline{r(\beta)}$	$\overline{r^2(\beta)}$
$-$	$+$	$+$	$+$	$-$	$+$
$-$	$+$	$+$	$+$	$+$	$-$

Table 5. Ambiguous numbers of s

β	$s(\beta)$	$s^2(\beta)$	$s^3(\beta)$	$\overline{\beta}$	$\overline{s(\beta)}$	$\overline{s^2(\beta)}$	$\overline{s^3(\beta)}$
$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$
$+$	$-$	$-$	$-$	$-$	$+$	$-$	$-$

Table 6. Ambiguous numbers of s

β	$s(\beta)$	$s^2(\beta)$	$s^3(\beta)$	$\overline{\beta}$	$\overline{s(\beta)}$	$\overline{s^2(\beta)}$	$\overline{s^3(\beta)}$
$+$	$+$	$-$	$-$	$+$	$-$	$-$	$-$
$+$	$-$	$-$	$-$	$+$	$-$	$-$	$-$

(b) First we suppose that β is a positive number, then we have the following information:

Similarly, if β is a negative number, then we have the following information:

Therefore from above information we can easily deduce that if β is an ambiguous number the $s^2(\beta)$ is totally negative while one of the $s(\beta)$ and $s^3(\beta)$ is ambiguous and the other is totally negative. □

Theorem 5.3. *The ambiguous numbers in the coset diagram for the orbit* $\Gamma(\beta)$, *where* $\beta = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$, *form a closed path and it is the only closed path contained in it.*

Proof. If z_0 is an ambiguous number in $\Gamma(\beta)$, then either $r^i(z_0)$ is ambiguous or $s^j(z_0)$ is ambiguous for $i = 1$ or 2 and $j = 1, 2$ or 3. We may therefore assume that $s^j(z_0)$ is an ambiguous number. Due to Theorem 5.2, each triangle representing three edges of r and each square representing four edges of s contains two ambiguous numbers, so within the $z-th$ triangle, we successively apply r (or s) to reach the next ambiguous number in the $(z+1)th$ triangle or square. Suppose the $z-th$ triangle (square) depicting three (four) cycles of r (s) contains two ambiguous numbers, namely δ_1 and δ_2. Then, $\delta_2^{(z-1)} = \delta_1^{(z-1)} s^{\varepsilon_1}$, $\delta_2^{(z)} = \delta_1^{(z)} r^{\varepsilon_2}$ and $\delta_2^{(z+1)} = \delta_1^{(z+1)} s^{\varepsilon_3}$ where $\varepsilon_1, \varepsilon_2, \varepsilon_3 = 1, 2$ or 3. Also since $\delta_2^{(z-1)} = \delta_1^{(z)}$ and $\delta_2^{(z)} = \delta_1^{(z+1)}$, therefore, $\delta_1^{(z-1)} s^{\varepsilon_1} s^{\varepsilon_2} s^{\varepsilon_3} = \delta_2^{(z+1)}$. We can continue in this way and since by [3, Theorem 3] there are only a finite number of ambiguous numbers, so after a finite number of steps we reach the vertex (ambiguous number) $\delta_2^{(z+m)} = \delta_1^{(z-1)}$. Hence the ambiguous number s form a path in coset diagram. The path is closed because there are only a finite number of ambiguous numbers in a coset diagram. Since only ambiguous numbers form a closed path and these are the only ambiguous numbers therefore they form a single closed path in the coset diagram of the orbit $\Gamma(\beta)$. □

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