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INITIAL COEFFICIENTS OF STARLIKE FUNCTIONS WITH REAL COEFFICIENTS

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Abstract. The sharp bounds for the third and fourth coefficients of Ma-Minda starlike functions having fixed second coefficient are determined. These results are proved by using certain constraint coefficient problem for functions with positive real part whose coefficients are real and the first coefficient is kept fixed. Analogous results are obtained for a general class of close-to-convex functions.

Keywords: Coefficient estimates, starlike functions, close to convex functions, functions with positive real part, real coefficient.
MSC(2010): Primary: 30C45; Secondary: 30C50, 30C80.

1. Introduction and main results

Let $\mathcal{S}$ be the class of all univalent analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined in the open unit disk $\mathbb{D}$. In 1916 Bieberbach conjectured that $|a_n| \leq n$ for $f \in \mathcal{S}$, equality holds for Koebe function $K(z) = z/(1-z)^2$ and its rotation $e^{-i\theta}K(e^{i\theta}z)$. After a long period of time, this conjecture finally proved by de Branges in 1985. In an attempt to resolve this conjecture, researchers pursued many directions. Several subclasses were introduced and investigated by imposing geometric properties on the image domain. Yet another option is to consider functions whose Taylor coefficients are real. This condition naturally implies that the image domain of such functions is symmetric with respect to real axis. Functions in the class $\mathcal{SR}$ of univalent analytic functions in $\mathbb{D}$ having the real coefficients satisfy $-n \leq a_n \leq n$ for all $n \geq 2$ [10, Theorem 1, p. 182]. In 1992, by using a Theorem of Dubins [8] related to the extreme points crossections of convex set, Al-Amiri and Bshouty [1] gave the sharp upper bounds for a_3 and a_4 of the functions in the subclass of $\mathcal{S}$ with real coefficient and fixed second coefficient. Further, Al-Amiri and Bshouty [2] determined the sharp upper bound for the fourth coefficients of close-to-convex functions with

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real coefficients under some restriction over the second coefficients. In 2000, by using Carathéodory-Toeplitz conditions Samaris and Koulorizos [30] obtained the sharp upper and lower bounds of the third and fourth coefficients of the starlike functions with real coefficients and for any fixed second coefficient in the interval $[-2, 2]$. Further, distortion results, Koebe and covering domains of certain classes of functions with real coefficients are investigated in [19, 23, 33, 34]. In [26] Nunokawa *et al.* investigated differential subordination results for functions with real coefficients. Recently, Kanas and Tatarczak [15] obtained coefficient bounds for the initial coefficients of the generalized typically real valued functions.

For two functions f and g analytic in $\mathbb{D}$, f is subordinate to g, written as $f \prec g$, if there exists a function $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$ such that $f(z) = g(w(z))$. If g is univalent in $\mathbb{D}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ (for details of differential subordination, we refer [25]). Let φ be a univalent analytic function with positive real part in $\mathbb{D}$ satisfying $\varphi(0) = 1$ and $\varphi'(0) > 0$. For such a function φ, Ma and Minda [22] and Ravichandran [27] introduced the subclasses $\mathcal{ST}(\varphi)$ and $\mathcal{STS}(\varphi)$ consisting of the functions $f \in \mathcal{S}$ satisfying $zf'(z)/f(z) \prec \varphi(z)$ and $2zf'(z)/(f(z) - f(-z)) \prec \varphi(z)$ respectively. If we take $\varphi(z) = (1 + z)/(1 - z)$, then the class $\mathcal{ST}(\varphi)$ reduces to the well known class $\mathcal{ST}$ of normalized starlike functions and similarly for different choices of φ, the class $\mathcal{ST}(\varphi)$ generates various subclasses studied in [14, 17, 24, 36]. Similarly, we can consider such subclasses of the class $\mathcal{STS}(\varphi)$. For $f \in \mathcal{ST}(\varphi)$, the sharp bound for the second and the third coefficients have been determined by Ma and Minda [22]. Later, Ali *et al.* [5] determined the sharp bound for the fourth coefficients of the functions in the class $\mathcal{ST}(\varphi)$. For the function $f \in \mathcal{STS}(\varphi)$, the sharp bound for the second and third coefficient are obtained in [31] by using the Fekete-Szegő coefficient functional. Determination of bounds on the coefficients a_n for $n \geq 5$ of the function $f \in \mathcal{ST}(\varphi)$ is still an open problem. For more information regarding coefficient bounds, we refer [3, 4, 6, 7, 12, 13, 16, 20, 21, 29, 32, 35]. In this paper, we determine initial coefficient bounds for Ma-Minda type univalent functions with real coefficients. We therefore, first consider such subclasses $\mathcal{ST}^x_R(\varphi)$ and $\mathcal{STS}^x_R(\varphi)$, which are defined as:

$$\mathcal{ST}^x_R(\varphi) := \left\{ f(z) = z + xz^2 + a_3z^3 + a_4z^4 + \cdots \in \mathcal{S} \ : \ \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{STS}^x_R(\varphi) := \left\{ f(z) = z + xz^2 + a_3z^3 + \cdots \in \mathcal{S} \ : \ \frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z) \right\}$$

for all $a_n \in \mathbb{R}$ and the function $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$ is a univalent analytic function with positive real part in $\mathbb{D}$ satisfying $B_1 > 0$ and $B_n \in \mathbb{R}$ $(n \in \mathbb{N})$. Let $\mathcal{P}_R$ be the class of analytic functions $p(z) = 1 + r_1z + r_2z^2 + \cdots$

with $\mathrm{Re}(p(z)) > 0$ ($z \in \mathbb{D}$) and for a fixed y with $|y| \leq 2$, let $\mathcal{P}_R^y$ be the subclass of $\mathcal{P}_R$ with $r_1 = y$. Motivated by the work done in [1,2], we first find for a fixed $w \in \mathbb{R}$, the minimum value of the coefficient functional $wr_2 + r_3$ associated with the function $p(z) = 1 + xz + r_2z^2 + r_3z^3 + \cdots \in \mathcal{P}_R^x$ and then by applying this minimum value of coefficient functional $wr_2 + r_3$ and the maximum value of coefficient functional $wr_2 + r_3$, given by Al-Amiri *et al.* [2], the sharp bounds for the third and the fourth coefficients of the functions in the classes $\mathcal{ST}_R^x(\varphi)$ and $\mathcal{STS}_R^x(\varphi)$ are obtained.

We state our first main result which yields the sharp bound for the third and the fourth coefficient of the function belonging to the class $\mathcal{ST}_R^x(\varphi)$.

Theorem 1.1. *Let the function $f \in \mathcal{ST}_R^x(\varphi)$ and $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$ be a univalent analytic function with positive real part, $B_1 > 0$ and $B_n \in \mathbb{R}$. Then*

(a) *For $-B_1 \leq x \leq B_1$, we have the following bound on the third coefficient:*

$$((B_1^2 + B_1 + B_2)x^2 - B_1^3)/2B_1^2 \leq a_3 \leq ((B_1^2 - B_1 + B_2)x^2 + B_1^3)/2B_1^2.$$

(b) *We have the following upper and lower bounds of the coefficient a_4:*

(i) *If $(3B_1^2 - 4B_1 + 4B_2)x + 4B_1^2 \in (-8B_1x,\, 8B_1^2]$, then*

$$\begin{aligned}48B_1^4a_4 \leq 16B_1^5 + B_1(9B_1^4 - 16B_1^2 + 24B_1^2B_2 + 16B_2^2)x^2 \\ + (-B_1^4 - 16B_2^2 + 16B_1B_3)x^3.\end{aligned}$$

(ii) *If $(3B_1^2 - 4B_1 + 4B_2)x + 4B_1^2 \notin (-8B_1x,\, 8B_1^2]$ and $(3B_1^2 + 4B_2)x > 0$, then*

$$\begin{aligned}6B_1^3a_4 \leq B_1^2(-2B_1 + 3B_1^2 + 4B_2)x \\ + (2B_1 - 3B_1^2 + B_1^3 - 4B_2 + 3B_1B_2 + 2B_3)x^3.\end{aligned}$$

(iii) *If $(3B_1^2 - 4B_1 + 4B_2)x + 4B_1^2 \notin (-8B_1x,\, 8B_1^2]$ and $(3B_1^2 + 4B_2)x < 0$, then*

$$\begin{aligned}6B_1^3a_4 \leq B_1^2(-2B_1 - 3B_1^2 - 4B_2)x \\ + (2B_1 + 3B_1^2 + B_1^3 + 4B_2 + 3B_1B_2 + 2B_3)x^3.\end{aligned}$$

(iv) *If $4B_1^2 + (-3B_1^2 + 4B_1 - 4B_2)x \in (8B_1x,\, 8B_1^2]$, then*

$$\begin{aligned}48B_1^4a_4 \geq -(16B_1^5 + B_1(-4B_1 + 3B_1^2 + 4B_2)(4B_1 + 3B_1^2 + 4B_2)x^2 \\ + (B_1^4 + 16B_2^2 - 16B_1B_3)x^3).\end{aligned}$$

(v) *If $4B_1^2 + (-3B_1^2 + 4B_1 - 4B_2)x \notin (8B_1x,\, 8B_1^2]$ and $(3B_1^2 + 4B_2)x < 0$, then*

$$\begin{aligned}6B_1^3a_4 \geq B_1^2(-2B_1 + 3B_1^2 + 4B_2)x \\ + (-4B_2 + B_1(2 - 3B_1 + B_1^2 + 3B_2) + 2B_3)x^3.\end{aligned}$$

(vi) *If* $4B_1^2+(-3B_1^2+4B_1-4B_2)x \notin (8B_1x,\, 8B_1^2]$ *and* $(3B_1^2+4B_2)x > 0$*, then*

$$6B_1^3a_4 \geq -B_1^2(2B_1+3B_1^2+4B_2)x + (B_1(2+3B_1+B_1^2+3B_2)+4B_2+2B_3)x^3.$$

The bounds are sharp.

On taking $\varphi(z)=\sqrt{1+z}$, $\varphi(z)=e^z$, and $\varphi(z)=(1+z)/(1-z)$ in the class $\mathcal{ST}_R^x(\varphi)$, we get the subclasses $\mathcal{ST}_R^x(\sqrt{1+z})=\mathcal{ST}_{R,L}^x$, $\mathcal{ST}_R^x(e^z)=\mathcal{ST}_{R,e}^x$ and $\mathcal{ST}_R^x((1+z)/(1-z))=\mathcal{ST}_R^x$ respectively. For more information regarding these classes, see [2, 14, 24, 36]. The following corollaries are the immediate consequence of Theorem 1.1.

Corollary 1.2. *Let the function* $f \in \mathcal{ST}_{R,e}^x$*. Then for* $|x| \leq 1$*, the sharp lower and upper bounds of the third coefficient are given by* $a_3 \geq -(1/2)+(5/4)x^2$ *and* $a_3 \leq (1/2)+(1/4)x^2$*. Sharp upper bound for fourth coefficient is given by* $a_4 \leq (-7x+(59/6)x^3)/6$ *for* $x \in [-1,\, -4/9]$ *and for* $x \in (-4/9, 1]$*,* $a_4 \leq 1/3+(3/16)x^2-(7/144)x^3$*. Further, sharp lower bound for fourth coefficient is given by* $a_4 \geq (-48-27x^2-7x^3)/144$ *for* $x \in [-1,\, 4/9)$ *and for* $x \in [4/9,\, 1]$*,* $a_4 \geq x(-42+59x^2)/36$*.*

Corollary 1.3. *Suppose that the function* f *belongs to the class* $\mathcal{ST}_{R,L}^x$*. Then we have sharp bounds for the third coefficient:* $a_3 \geq (5/4)x^2-(1/4)$ *and* $a_3 \leq -(3/4)x^2+(1/4)$ *for* $x \in [-1/2,\, 1/2]$*. The upper bound of the fourth coefficient is given as:* $a_4 \leq -(5/12)x+(7/4)x^3$ *for* $x \in [-1/2,\, -4/9]$ *and for* $x \in (-4/9,\, 1/2]$*,* $a_4 \leq (1/6)-(21/32)x^2+(1/16)x^3$*. The lower sharp bound of the fourth coefficient is given as:* $a_4 \geq (-16+63x^2+6x^3)/96$ *for* $x \in [-1/2,\, 4/9)$ *and for* $x \in [4/9,\, 1/2]$*,* $a_4 \geq x(-5+21x^2)/12$*.*

Corollary 1.4. *Let the function* $f \in \mathcal{ST}_R^x$*. Then we have the following sharp bound:* $x^2-1 \leq a_3 \leq x^2/2+1$ *for* $x \in [-2,\, 2]$*. The upper bound of the fourth coefficient is given as:* $a_4 \leq -2x+x^3$ *for* $x \leq -4/7$*,* $a_4 \leq (2/3)+(7/8)x^2-(1/48)x^3$ *for* $x \in (-4/7,\, 4/3]$ *and for* $x > 4/3$*,* $a_4 \leq (4/3)x+(1/6)x^3$*. The lower bound of the fourth coefficient is given as:* $a_4 \geq (4/3)x+(1/6)x^3$ *for* $x < -4/3$*,* $a_4 \geq -(2/3)-(7/8)x^2-(1/48)x^3$ *for* $x \in [-4/3,\, 4/7)$ *and for* $x \geq 4/7$*,* $a_4 \geq -2x+x^3$*.*

Remark 1.5. For $f \in \mathcal{ST}_R^x$, the upper bounds of the third coefficient for $-2 \leq x \leq 2$ and for $x \geq 4/3$, the upper bound of the fourth coefficient are precisely proved in [1, Theorem 1, p. 33].

In the next result, we determine the sharp bounds for the third and the fourth coefficients of the function belonging to the class $\mathcal{STS}_R^x(\varphi)$.

Theorem 1.6. *Suppose that the function* $f \in \mathcal{STS}_R^x(\varphi)$ *and* $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$ *is a univalent analytic function with positive real part,* $B_1>0$ *and* $B_n \in \mathbb{R}$*. Then,*

(a) *For* $|x| \leq B_1/2$, *we have the following estimate of the third coefficient:*
$(2(B_1+B_2)x^2 - B_1^3)/2B_1^2 \leq a_3 \leq (2(B_2-B_1)x^2 + B_1^3)/2B_1^2.$

(b) *We have the following upper and lower bound of the fourth coefficient:*

(i) *If* $x(-4B_1+B_1^2+4B_2) \in [-2B_1(4x+B_1),\, 2B_1^2)$, *then*

$$16B_1^4 a_4 \leq 4B_1^5 + B_1(B_1^4 + 8B_1^2(-2+B_2) + 16B_2^2)x^2 + 2((-B_1^4 - 16B_2^2 + 16B_1B_3))x^3.$$

(ii) *If* $x(-4B_1+B_1^2+4B_2) \notin [-2B_1(4x+B_1),\, 2B_1^2)$ *and* $(B_1^2+4B_2)x < 0$, *then*

$$4B_1^3 a_4 \leq (-2B_1 - B_1^2 - 4B_2)B_1^2 x + 4(B_1^2 + B_1(2+B_2) + 2(2B_2+B_3))x^3.$$

(iii) *If* $x(-4B_1+B_1^2+4B_2) \notin [-2B_1(4x+B_1),\, 2B_1^2)$ *and* $(B_1^2+4B_2)x > 0$, *then*

$$4B_1^3 a_4 \leq B_1^2(-2B_1 + B_1^2 + 4B_2)x - 4(B_1^2 + 4B_2 - B_1(2+B_2) - 2B_3)x^3.$$

(iv) *If* $x(-4B_1+B_1^2+4B_2) \in [\,-2B_1^2,\, -8B_1x + 2B_1^2\,)$, *then*

$$16B_1^4 a_4 \;\geq\; -4B_1^5 + B_1(16B_1^2 - B_1^4 - 8B_1^2B_2 - 16B_2^2)x^2 + 2(-B_1^4 - 16B_2^2 + 16B_1B_3)x^3.$$

(v) *If* $x(-4B_1+B_1^2+4B_2) \notin [\,-2B_1^2,\, -8B_1x+2B_1^2\,)$ *and* $(B_1^2+4B_2)x < 0$, *then*

$$4B_1^3 a_4 \geq -B_1^2(2B_1 - B_1^2 - 4B_2)x - (-8B_1 + 4B_1^2 + 16B_2 - 4B_1B_2 - 8B_3)x^3.$$

(vi) *If* $x(-4B_1+B_1^2+4B_2) \notin [\,-2B_1^2,\, -8B_1x+2B_1^2\,)$ *and* $(B_1^2+4B_2)x > 0$, *then*

$$4B_1^3 a_4 \geq -B_1^2(2B_1 + B_1^2 + 4B_2)x + (8B_1 + 4B_1^2 + 16B_2 + 4B_1B_2 + 8B_3)x^3.$$

The bounds are sharp.

On taking $\varphi(z) = \sqrt{1+z}$, $\varphi(z) = e^z$ and $\varphi(z) = (1+z)/(1-z)$, the class $\mathcal{STS}_R^x(\varphi)$ reduces to the subclasses $\mathcal{STS}_{R,L}^x$, $\mathcal{STS}_{R,e}^x$ and $\mathcal{STS}_R^x$ respectively. The following corollaries are the immediate consequence of Theorem 1.6.

Corollary 1.7. *Suppose that the function* $f \in \mathcal{STS}_{R,e}^x$. *Then we have* $3x^2 - (1/2) \leq a_3 \leq (1/2) - x^2$ *for* $x \in [-1/2, 1/2]$. *The upper bound of the fourth coefficient is given as:* $a_4 \leq (-5/4)x + (35/6)x^3$ *for* $-1/2 \leq x < -2/7$ *and for* $-2/7 \leq x \leq 1/2$, $a_4 \leq (16 - 21x^2 - 21x^3)/48$. *The lower bound of the fourth* *coefficient is given as:* $a_4 \geq (-16+21x^2-14x^3)/48$ *for* $-1/2 \leq x < 2/7$ *and for* $2/7 \leq x \leq 1/2$, $a_4 \geq (-5/4)x + (35/6)x^3$.

Corollary 1.8. *Suppose that the function* $f \in \mathcal{STS}^x_{R,L}$. *Then we have the sharp bounds of the third coefficient:* $3x^2 - (1/4) \leq a_3 \leq (1/4) - 5x^2$ *for* $x \in [-1/4, 1/4]$. *The upper bound of the fourth coefficient is given as:* $a_4 \leq (-3/8)x + (13/2)x^3$ *for* $-1/4 \leq x \leq -2/9$ *and for* $-2/9 < x \leq 1/4$, $a_4 \leq (4 - 63x^2 + 24x^3)/32$. *The lower bound of the fourth coefficient is given as:* $a_4 \geq (-4 + 63x^2 + 12x^3)/32$ *for* $-1/4 \leq x \leq 2/9$ *and for* $2/9 < x \leq 1/4$, $a_4 \geq (-5/8)x + (21/2)x^3$.

Corollary 1.9. *Suppose that the function* $f \in \mathcal{STS}^x_R$. *Then we have the sharp bounds of the third coefficient:* $2x^2 - 1 \leq a_3 \leq 1$ *for* $x \in [-1, 1]$. *The upper bound of the fourth coefficient is given as:* $a_4 \leq -2x + 3x^2$ *for* $-1 \leq x < -2/5$ *and for* $-2/5 \leq x \leq 1$, $a_4 \leq (4 + 5x^2 - x^3)/8$. *The lower bound of the fourth coefficient is given as:* $a_4 \geq (-4 - 5x^2 - x^3)/8$ *for* $-1 \leq x < 2/5$ *and for* $2/5 \leq x \leq 1$, $a_4 \geq -2x + 3x^3$.

2. **Proof of main results**

The proof of the Theorem 1.1 and other results rely on some lemmas. We first present three important lemmas which play vital role in the proof of results. For $\alpha \in [0, 1)$, let $\mathcal{P}(\alpha)$ be the class of analytic functions $p(z) = 1 + r_1 z + r_2 z^2 + \cdots$ with real part greater than α on $\mathbb{D}$. Lecko [18] investigated the coefficient estimates of the functions in the class $\mathcal{P}(\alpha)$. Note that $\mathcal{P}(0) = \mathcal{P}$, the well known class of Carathéodory functions having positive real part in $\mathbb{D}$.

Lemma 2.1 ([9, Carathéodory Lemma, p. 41])**.** *For a Carathéodory function* $p(z) = 1 + \sum_{n=1}^{\infty} r_n z^n$, *we have a sharp inequality* $|r_n| \leq 2$ *for each* n.

Lemma 2.2 ([2, Lemma 1, p. 243])**.** *Let* $r(z) = 1 + yz + r_2 z^2 + \cdots$ *in* $\mathcal{P}^y_R$ *and let* w *be real. Then*

$$wr_2 + r_3 \leq \begin{cases} (8 + 4yw + w^2(2-y))/4, & \text{if } -(w+2)/2 \in [-2, y) \\ y^3 + wy^2 - 3y - 2w, & \text{if } -(w+2)/2 \notin [-2, y), y + w < 0 \\ y + 2w, & \text{if } -(w+2)/2 \notin [-2, y), y + w > 0. \end{cases}$$

The bounds are sharp for all w *and* $-2 \leq y \leq 2$.

Lemma 2.3. *For a real number* w *and* $r(z) = 1 + yz + r_2 z^2 + \cdots$ *in the class* $\mathcal{P}^y_R$, *we have*

$$wr_2 + r_3 \geq \begin{cases} -(8 + 4wy + w^2(y+2))/4, & \text{if } (2-w)/2 \in (y, 2] \\ y^3 + wy^2 - 3y - 2w, & \text{if } (2-w)/2 \notin (y, 2], y + w > 0 \\ y + 2w, & \text{if } (2-w)/2 \notin (y, 2], y + w < 0. \end{cases}$$

These estimates are sharp for all w *and* $-2 \leq y \leq 2$.

Proof of Lemma 2.3. The proof of this Lemma is essentially based on the proof of [2, Lemma 1, p. 243]. From [11], we observe that the subclass $\mathcal{P}_R$ is a closed convex set with respect to the topology of local uniform convergence and the set of extreme points $\mathrm{Ext}(\mathcal{P}_R)$ of the class $\mathcal{P}_R$ consists of all functions p_x given by

$$p_x(z) = \frac{1-z^2}{1-xz+z^2} \quad (-2 \le x \le 2). \tag{2.1}$$

The set $\mathcal{P}_R$ is closed convex hull of all its extreme points: $\mathcal{P}_R = \bar{\mathrm{C}}\mathrm{o}(\mathrm{Ext}(\mathcal{P}_R))$. In order to prove our lemma, we need to minimize the linear functional wr_2+r_3 on the intersection $\mathcal{P}_R$ with the hyperplane $r_1 = y$. The extreme points of the intersection of the linear functional $wr_2 + r_3$ with $r_1 = y$ are contained in the set of all convex combinations of two extreme points of $\mathcal{P}_R$ (see [8]). We need to find the extremum of the functional

$$wr_2 + r_3 = \lambda(s^3 + ws^2) + (1-\lambda)(t^3 + wt^2) - 3y - 2w \tag{2.2}$$

under the constraints $\lambda s + (1-\lambda)t = y$, $0 \le \lambda \le 1$ and $-2 \le s \le t \le 2$. We use Lagrange method of multipliers to find the minimum value of the $wr_2 + r_3$ and for this purpose we construct the auxiliary function

$$\begin{aligned} H(s,t,\lambda,\mu) &= (wr_2 + r_3) + \mu(\lambda s + (1-\lambda)t - y) \\ &= \lambda(s^3 + ws^2) + (1-\lambda)(t^3 + wt^2) - 3y - 2w + \mu(\lambda s + (1-\lambda)t - y). \end{aligned}$$

The necessary conditions $\partial H/\partial s = 0$, $\partial H/\partial t = 0$, $\partial H/\partial l = 0$ and $\partial H/\partial \mu = 0$ for the extreme value give the following equations

$$\lambda(2ws + 3s^2 + \mu) = 0, \qquad (1-\lambda)(3t^2 + 2wt + \mu) = 0,$$
$$ws^2 + s^3 - t^3 - wt^2 + \mu s - t\mu = 0, \qquad \lambda s + (1-\lambda)t - y = 0.$$

This system of equations has a solution in $-2 < s \le t < 2$ if and only if either $\lambda = 0$ or $\lambda = 1$ or $s = t$ and in all these cases, we have, from (2.2),

$$wr_2 + r_3 = y^3 + wy^2 - 3y - 2w. \tag{2.3}$$

Next we consider the boundary points of the interval $-2 \le s \le t \le 2$. For $s \ne t$, two cases arise:

Case(i) Let $t = 2$, $-2 \le s < 2$. In this case $\lambda = (2-y)/(2-s)$ for $-2 \le s \le y$. Since the function $wr_2+r_3 = -(2-y)s^2 - (2-y)(2+w)s + 2(w+2)y - 3y - 2w$ is concave in s, its minimum is attained at $s = -2$ or $s = y$. If $s = -2$, $t = 2$ then we get $wr_2 + r_3 = y + 2w$. The case $s = y$, $\lambda = 1$ has already been considered in (2.3).

Case(ii) Let $s = -2$, $-2 \le t < 2$. In this case $\lambda = (t-y)/(t+2)$, where $y \le t \le 2$. Since $wr_2 + r_3 = (y+2)t^2 + (2+y)(w-2)t + 2y(2-w) - 3y - 2w$ is convex in t, its minimum is attained at either $t = y$ or $t = 2$ or $t = (2-w)/2$. If $t = y$, then $\lambda = 0$ and if $t = 2$, $s = -2$. These two cases have already been considered. If $t = (2-w)/2 \in (y,\, 2)$, then $wr_2+r_3 = -(w^2y+4wy+2w^2+8)/4$

would give global minimum and in other cases, the minimum value is smaller than $y+2w$ and $y^3+wy^2-3y-2w$. Also we note that $y+2w > y^3+wy^2-3y-2w$ if and only if $y+w>0$, and the result follows. □

Proof of Theorem 1.1. Let $p(z) = zf'(z)/f(z) = 1 + b_1 z + b_2 z^2 + \cdots$. By a simple computation in this relation, we obtain $a_2 = b_1$, $2a_3 = (b_1^2 + b_2)$ and $2a_4 = b_1^3 + 3b_1 b_2 + 2b_3$. Since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ is univalent, $B_n \in \mathbb{R}$ and

$$p(z) = \frac{zf'(z)}{f(z)} \prec \varphi(z),$$

the function

$$p_1(z) = \frac{1+\varphi^{-1}(p(z))}{1-\varphi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + c_3 z^3 \cdots$$

is in $\mathcal{P}_R$. A simple calculation yields

$$p(z) = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right).$$

We now express the initial coefficients of the function $f(z) = z + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{ST}_R^x(\varphi)$ in terms of B_i and c_i $(i = 1,2,3)$. The last equation and the equation that expresses a_n in terms of b_n's yield the following expressions for the initial coefficients

$$2a_2 = B_1 c_1,$$

$$8a_3 = (B_1^2 - B_1 + B_2)c_1^2 + 2B_1 c_2, \tag{2.4}$$

$$\begin{aligned} 48a_4 = {} & (B_1^3 - 3B_1^2 + 3B_1 B_2 + 2B_1 - 4B_2 + 2B_3)\, c_1^3 \\ & + 2(3B_1^2 - 4B_1 + 4B_2)c_1 c_2 + 8B_1 c_3. \end{aligned} \tag{2.5}$$

For more details on the expressions a_3 and a_4 in terms of B_i and c_i $(i = 1,2,3)$, we refer [28]. Note that a_2 is fixed, namely, $a_2 = x$ and consequently $c_1 = 2x/B_1$. On substituting $c_1 = 2x/B_1$ in the equation (2.4), we get

$$a_3 = \frac{(B_1^2 - B_1 + B_2)}{2B_1^2}x^2 + \frac{1}{4}B_1 c_2. \tag{2.6}$$

(a) Using the Carathéodory Lemma (Lemma 2.1) in equation (2.6) yields the desired upper bound for a_3. Let the function $f_0 : \mathbb{D} \to \mathbb{C}$ be given by

$$f_0(z) = z \exp\left(\int_0^z \left(\varphi\left(\frac{q_0(t)-1}{q_0(t)+1}\right) - 1\right) t^{-1}\, dt\right), \tag{2.7}$$

where

$$q_0(t) = \frac{B_1 - x}{2B_1} p_{-2}(t) + \frac{B_1 + x}{2B_1} p_2(t)$$

and the function $p_x(z)$ is given by (2.1). The Taylor series expansion of f_0 is given by

$$\begin{aligned} f_0(z) = z + xz^2 &+ \frac{1}{2}\left(\frac{(B_1^2+B_2-B_1)x^2}{B_1^2}+B_1\right)z^3 \\ &+ \left(\frac{(-2B_1+3B_1^2+4B_2)}{6B_1}x \right. \\ &\left. + \frac{(2B_1-3B_1^2+B_1^3-4B_2+3B_1B_2+2B_3)}{6B_1^3}x^3\right)z^4 + \cdots . \end{aligned}$$

The upper bound on a_3 is clearly sharp for this function f_0.

On taking $y = c_1 = 2x/B_1$ and letting $w \to \infty$ in Lemma 2.3, we get the inequality $r_2 = c_2 \geq (4x^2/B_1^2) - 2$ and by using this minimum value of c_2 in (2.6), we get the desired lower bound of a_3. To show the sharpness of the lower bound on a_3, consider the function $g_0 : \mathbb{D} \to \mathbb{C}$ given by

$$g_0(z) = z \exp\left(\int_0^z \left(\varphi\left(\frac{l_0(t)-1}{l_0(t)+1}\right)-1\right)t^{-1}\,dt\right), \tag{2.8}$$

where

$$l_0(t) = \frac{(1-t^2)B_1}{(B_1-2xt+B_1t^2)}.$$

The Taylor's series expansion of g_0 given by

$$\begin{aligned} g_0(z) = z + xz^2 + \frac{1}{2}\left(\frac{(B_1^2+B_2+B_1)x^2}{B_1^2}-B_1\right)z^3 + \left(-\frac{(2B_1+3B_1^2+4B_2)}{6B_1}x\right. \\ \left. + \frac{2B_1+3B_1^2+B_1^3+3B_1B_2+4B_2+2B_3}{6B_1^3}x^3\right)z^4 + \cdots \end{aligned}$$

shows that the lower bound is sharp.

(b) On substituting $c_1 = 2x/B_1$ in (2.5), the coefficient a_4 is expressed as

$$\begin{aligned} a_4 &= \frac{1}{6B_1^3}(B_1^3-3B_1^2+3B_1B_2+2B_1-4B_2+2B_3)x^3 \\ &\quad + \frac{1}{12B_1}(3B_1^2-4B_1+4B_2)xc_2 + \frac{1}{6}B_1c_3 \\ &= \frac{1}{6B_1^3}(B_1^3-3B_1^2+3B_1B_2+2B_1-4B_2+2B_3)x^3 + \frac{B_1}{6}g(c_2,\, c_3), \end{aligned} \tag{2.9}$$

where

$$g(c_2,\, c_3) = \frac{(3B_1^2-4B_1+4B_2)x}{2B_1^2}c_2 + c_3.$$

We apply Lemma 2.2 to the function $g(c_2,\, c_3)$. The upper bound on a_4 is discussed in the following three cases (i)-(iii):

(i) If $(3B_1^2 - 4B_1 + 4B_2)x + 4B_1^2 \in (-8B_1x, 8B_1^2]$, then Lemma 2.2 shows that

$$g(c_2, c_3) \leq 2 + \frac{(-16B_1^2 + 9B_1^4 + 24B_1^2B_2 + 16B_2^2)}{8B_1^4}x^2 + \frac{(-16B_1^2 + 24B_1^3 - 9B_1^4 + 32B_1B_2 - 24B_1^2B_2 - 16B_2^2)}{8B_1^5}x^3. \tag{2.10}$$

Using (2.10) in (2.9), we get the desired bound for the fourth coefficient. To prove the sharpness of the bound, consider the function $f_1 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_1(z) = z \exp\left(\int_0^z \left(\varphi\left(\frac{q_1(t) - 1}{q_1(t) + 1}\right) - 1\right) t^{-1}\, dt\right),$$

where

$$q_1(z) = \frac{8B_1(B_1 - x)}{-4B_1x + 4B_2x + 3B_1^2(4 + x)}\, p_s(z) + \frac{4B_1x + 4B_2x + B_1^2(4 + 3x)}{-4B_1x + 4B_2x + 3B_1^2(4 + x)}\, p_2(z),$$

$s = ((4B_1 - 3B_1^2 - 4B_2)x - 4B_1^2)/4B_1^2$ and the function $p_x(z)$ is given by (2.1). The Taylor series expansion of f_1 is given by

$$f_1(z) = z + xz^2 + \left(\frac{(3B_1^2 + 4B_2)}{8B_1}x + \frac{1}{8}x^2\right) z^3 + \left(\frac{B_1}{3} + \frac{(-4B_1 + 3B_1^2 + 4B_2)(4B_1 + 3B_1^2 + 4B_2)}{48B_1^3}x^2 + \frac{(-B_1^4 - 16B_2^2 + 16B_1B_3)}{48B_1^4}x^3\right) z^4 + \cdots$$

and it clearly shows that the bound is sharp.

(ii) If $(3B_1^2 - 4B_1 + 4B_2)x + 4B_1^2 \notin (-8B_1x, 8B_1^2]$ and $(3B_1^2 + 4B_2)x > 0$, then Lemma 2.2 shows that

$$g(c_2, c_3) \leq \frac{(-2B_1 + 3B_1^2 + 4B_2)}{B_1^2}x. \tag{2.11}$$

Use of (2.11) in (2.9) gives the required upper bound on a_4. The bound is clearly sharp for the function f_0 defined by (2.7).

(iii) If $(3B_1^2 - 4B_1 + 4B_2)x + 4B_1^2 \notin (-8B_1x, 8B_1^2]$ and $(3B_1^2 + 4B_2)x < 0$, then Lemma 2.2 yields

$$g(c_2, c_3) \leq \frac{(-2B_1 - 3B_1^2 - 4B_2)}{B_1^2}x + \frac{2(3B_1^2 + 4B_2)}{B_1^4}x^3. \tag{2.12}$$

Using (2.12) in (2.9), the desired bound on a_4 is obtained and equality is attained for the function g_0 defined by (2.8).

The lower bound for the fourth coefficients given in (iv)-(vi) are proved by applying Lemma 2.3 to the function $g(c_2, c_3)$. As before, the construction of the extremal function is the important step in the proof.

(iv) If $4B_1^2+(-3B_1^2+4B_1-4B_2)x \in (8B_1x, 8B_1^2]$, then Lemma 2.3 shows

$$\begin{aligned} g(c_2, c_3) \geq -2 &+ \frac{(16B_1^3-9B_1^5-24B_1^3B_2-16B_1B_2^2)}{8B_1^5}x^2 \\ &+ \frac{(-16B_1^2+24B_1^3-9B_1^4+32B_1B_2-24B_1^2B_2-16B_2^2)}{8B_1^5}x^3. \end{aligned} \tag{2.13}$$

The required lower bound of a_4 follows from the equation (2.10) upon using (2.13). Consider the function $g_1 : \mathbb{D} \to \mathbb{C}$ defined by

$$g_1(z) = z \exp\left(\int_0^z \left(\varphi\left(\frac{l_1(t)-1}{l_1(t)+1} \right) - 1 \right) t^{-1}\, dt \right),$$

where

$$\begin{aligned} l_1(z) = &\frac{4B_1x+4B_2x+B_1^2(-4+3x)}{3B_1^2(-4+x)-4B_1x+4B_2x}\, p_{-2}(z) \\ &- \frac{8B_1(B_1+x)}{3B_1^2(-4+x)-4B_1x+4B_2x}\, p_s(z), \end{aligned}$$

$s = (4B_1^2+4B_1x-3B_1^2x-4B_2x)/4B_1^2$ and the function $p_x(z)$ is given by (2.1). The Taylor series expansion of g_1 is given by

$$\begin{aligned} g_1(z) = &\; z + xz^2 + \left(-\frac{(3B_1^2+4B_2)}{8B_1}x + \frac{x^2}{8} \right) z^3 \\ &- \left(\frac{(B_1^4+16B_2^2-16B_1B_3)}{48B_1^4}x^3 \right. \\ &\left. + \frac{(-4B_1+3B_1^2+4B_2)(4B_1+3B_1^2+4B_2)}{48B_1^3}x^2 + \frac{B_1}{3} \right) z^4 + \cdots \end{aligned}$$

The bound is clearly sharp for the function g_1.

(v) If $4B_1^2+(-3B_1^2+4B_1-4B_2)x \notin (8B_1x, 8B_1^2]$ and $(3B_1^2+4B_2)x < 0$, then Lemma 2.3 shows that

$$g(c_2, c_3) \geq \frac{(-2B_1+3B_1^2+4B_2)}{B_1^2}x. \tag{2.14}$$

The equation (2.9) and the inequality (2.14) together yield the required result. The bound is sharp for the function f_0 defined by (2.7).

(vi) If $4B_1^2+(-3B_1^2+4B_1-4B_2)x \notin (8B_1x,\, 8B_1^2]$ and $(3B_1^2+4B_2)x>0$, then Lemma 2.3 shows

$$g(c_2,\, c_3) \geq \frac{-(2B_1+3B_1^2+4B_2)}{B_1^2}x + \frac{2(3B_1^2+4B_2)}{B_1^4}x^3. \tag{2.15}$$

The equation (2.9) and the inequality (2.15) together yield the required result. The bound is sharp for the function g_0 defined by (2.8). ☐

Proof of Theorem 1.6. As in the proof of Theorem 1.1, we first determine the initial three coefficients of the function $f(z) = z + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{STS}_R^x(\varphi)$ in terms of B_i and c_i $(i = 1, 2, 3, 4)$, where c_i's are the coefficients of a suitably defined function with positive real part. Let $p(z) = 2zf'(z)/(f(z)-f(-z)) = 1 + b_1 z + b_2 z^2 + b_3 z^2 + \cdots$. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $2zf'(z) = (1+\sum_{k=1}^{\infty} b_k z^k)(f(z)-f(-z))$ readily gives

$$\begin{aligned} z + \sum_{n=2}^{\infty} n a_n z^n &= \left(1 + \sum_{k=1}^{\infty} b_k z^k\right)\left(z + \sum_{n=2}^{\infty} a_{2n+1} z^{2n+1}\right) \\ &= \sum_{n=1}^{\infty} a_{2n-1} z^{2n-1} + \sum_{p=1}^{\infty}\left(\sum_{n=1}^{\lfloor \frac{p}{2} \rfloor} b_{p-2n+1} a_{2n-1}\right) z^p. \end{aligned} \tag{2.16}$$

On equating the coefficients of z^2, z^3 and z^4 on both sides, we obtain $a_2 = b_1/2$, $a_3 = b_2/2$ and $a_4 = (b_1 b_2 + 2b_3)/8$. Since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ is univalent and $2zf'(z)/(f(z)-f(-z)) \prec \varphi(z)$, the function

$$p_2(z) = \frac{1+\varphi^{-1}(p(z))}{1-\varphi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathcal{P}_R.$$

A simple calculation gives

$$p(z) = \varphi\left(\frac{p_2(z)-1}{p_2(z)+1}\right),$$

and by power series expansion, we get

$$\begin{aligned} 2b_1 &= B_1 c_1, \\ 4b_2 &= (B_2 - B_1)c_1^2 + 2B_1 c_2, \\ 8b_3 &= (B_1 - 2B_2 + B_3)c_1^3 + 4(B_2 - B_1)c_1 c_2 + 4B_1 c_3, \\ 16b_4 &= (-B_1 + 3B_2 - 3B_3 + B_4)c_1^4 + 6(B_3 - 2B_2 + B_1)c_1^2 c_2 + 4(B_2 - B_1)c_2^2 \\ &\quad + 8(B_2 - B_1)c_1 c_3 + 8B_1 c_4. \end{aligned}$$

The coefficients a_n $(n = 2, 3, 4)$ of the function $f \in \mathcal{STS}_R^x(\varphi)$ in terms of B_i and c_i are given by

$$a_2 = x = \frac{B_1 c_1}{4}, \tag{2.17}$$

$$a_3 = \frac{1}{8}((B_2 - B_1)c_1^2 + 2B_1 c_2), \tag{2.18}$$

$$\begin{aligned} a_4 = &\frac{1}{64}(2B_1 - B_1^2 - 4B_2 + B_1B_2 + 2B_3)c_1^3 \\ &+ \frac{1}{32}(-4B_1 + B_1^2 + 4B_2)c_1c_2 + \frac{1}{8}B_1c_3. \end{aligned} \tag{2.19}$$

(a) On using $c_1 = 4x/B_1$ in the equation (2.18), we get

$$a_3 = \frac{2(B_2 - B_1)}{B_1^2}x^2 + \frac{1}{4}B_1c_2. \tag{2.20}$$

Using the Carathéodory Lemma (Lemma 2.1) in (2.20) yields the desired result. Consider the function $f_0 : \mathbb{D} \to \mathbb{C}$ given by

$$\frac{2zf_0'(z)}{f_0(z) - f_0(-z)} = \varphi\left(\frac{q_0(z) - 1}{q_0(z) + 1}\right), \tag{2.21}$$

where

$$q_0(z) = \frac{B_1 - 2x}{2B_1}p_{-2}(z) + \frac{B_1 + 2x}{2B_1}p_2(z)$$

and the function $p_x(z)$ is given by (2.1). The Taylor series expansion of f_0 is given by

$$\begin{aligned} f_0(z) =& z + xz^2 + \left(\frac{2(B_2 - B_1)}{B_1^2}x^2 + \frac{B_1}{2}\right)z^3 + \left(\frac{(-2B_1 + B_1^2 + 4B_2)}{4B_1}x\right. \\ &\left. - \frac{(B_1^2 + 4B_2 - 2B_1 - B_1B_2 - 2B_3)}{B_1^3}x^3\right)z^4 + \cdots . \end{aligned}$$

The upper bound on a_3 is clearly sharp for the function f_0.
Further, from Lemma 2.3, we have $c_2 \geq 16x^2/B_1^2 - 2$. By using the inequality $c_2 \geq 16x^2/B_1^2 - 2$ in (2.20), the required lower bound for a_3 follows. Consider the function $g_0 : \mathbb{D} \to \mathbb{C}$ given by

$$\frac{2zg_0'(z)}{g_0(z) - g_0(-z)} = \varphi\left(\frac{l_0(z) - 1}{l_0(z) + 1}\right), \tag{2.22}$$

where

$$l_0(z) = \frac{(1 - z^2)B_1}{(1 + z^2)B_1 - 4xz}.$$

The Taylor series expansion of g_0 is given by

$$\begin{aligned} g_0(z) =& z + xz^2 + \left(\frac{2(B_1 + B_2)}{B_1^2}x^2 - \frac{B_1}{2}\right)z^3 + \left(-\frac{(2B_1 + B_1^2 + 4B_2)}{4B_1}x\right. \\ &\left. + \frac{(2B_1 + B_1^2 + 4B_2 + B_1B_2 + 2B_3)}{B_1^3}x^3\right)z^4 + \cdots . \end{aligned}$$

The lower bound is clearly sharp for the function g_0.

(b) Putting $c_1 = 4x/B_1$ in equation (2.19), the coefficient a_4 is expressed as

$$a_4 = \frac{(2B_1 - B_1^2 - 4B_2 + B_1B_2 + 2B_3)}{B_1^3}x^3 + \frac{(-4B_1 + B_1^2 + 4B_2)}{8B_1}xc_2 + \frac{1}{8}B_1c_3$$

(2.23)

$$= \frac{(2B_1 - B_1^2 - 4B_2 + B_1B_2 + 2B_3)}{B_1^3}x^3 + \frac{B_1}{8}h(c_2,\, c_3),$$

where

$$h(c_2,\, c_3) = \frac{(-4B_1 + B_1^2 + 4B_2)x}{B_1^2}c_2 + c_3.$$

We apply Lemma 2.2 to the function $h(c_2,\, c_3)$. The upper bound on a_4 is discussed in following three cases (i)-(iii):

(i) If $x(-4B_1 + B_1^2 + 4B_2) \in [-2B_1(4x + B_1),\, 2B_1^2)$, then Lemma 2.2 shows

$$h(c_2,\, c_3) \leq 2 + \frac{(-16B_1^2 + B_1^4 + 8B_1^2B_2 + 16B_2^2)}{2B_1^4}x^2$$

$$+ \frac{(-16B_1^2 + 8B_1^3 - B_1^4 + 32B_1B_2 - 8B_1^2B_2 - 16B_2^2)}{B_1^5}x^3 \tag{2.24}$$

On using the estimate of $h(c_2,\, c_3)$ from (2.24) in (2.23), the required result follows. Consider the function $f_1 : \mathbb{D} \to \mathbb{C}$ given by

$$\frac{2zf_1'(z)}{f_1(z) - f_1(-z)} = \varphi\left(\frac{q_1(z) - 1}{q_1(z) + 1}\right),$$

where

$$q_1(z) = \frac{4B_1(B_1 - 2x)}{-4B_1x + 4B_2x + B_1^2(6 + x)}p_s(z) + \frac{4B_1x + 4B_2x + B_1^2(2 + x)}{-4B_1x + 4B_2x + B_1^2(6 + x)}p_2(z),$$

$s = (-2B_1^2 + 4B_1x - B_1^2x - 4B_2x)/(2B_1^2)$ and $p_x(z)$ is given by (2.1). The Taylor series expansion of function f_1 is

$$f_1(z) = z + xz^2 + \left(\frac{(B_1^2 + 4B_2)}{4B_1}x - \frac{x^2}{2}\right)z^3 + \left(-\frac{(B_1^4 + 16B_2^2 - 16B_1B_3)}{8B_1^4}x^3 + \frac{(B_1^4 + 8B_1^2B_2 - 16B_1^2 + 16B_2^2)}{16B_1^3}x^2 + \frac{B_1}{4}\right)z^4 + \cdots.$$

The result is clearly sharp for the function f_1.

(ii) If $x(-4B_1+B_1^2+4B_2) \notin [-2B_1(4x+B_1),\, 2B_1^2)$ and $(B_1^2+4B_2)x < 0$, then Lemma 2.2 shows that

$$h(c_2,\, c_3) \leq \frac{-2(2B_1+B_1^2+4B_2)}{B_1^2}x + \frac{16(B_1^2+4B_2)}{B_1^4}x^3 \tag{2.25}$$

On putting the value of $h(c_2,\, c_3)$ from (2.25) in (2.23), we get the desired result. The bound is clearly sharp for the function g_0 defined by (2.22).

(iii) If $x(-4B_1+B_1^2+4B_2) \notin [-2B_1(4x+B_1),\, 2B_1^2)$ and $(B_1^2+4B_2)x > 0$, then Lemma 2.2 shows that

$$h(c_2,\, c_3) \leq \frac{2(-2B_1+B_1^2+4B_2)}{B_1^2}x. \tag{2.26}$$

The required bound of a_4 follows from the equation (2.23) upon using (2.26). The bound is sharp for the function f_0 given by (2.21).

The lower bound for the fourth coefficient given in (iv)-(vi) are proved by applying Lemma 2.3 to the function $h(c_2,\, c_3)$.

(iv) If $x(-4B_1+B_1^2+4B_2) \in [-2B_1^2,\, -8B_1x+2B_1^2)$, then Lemma 2.3 gives

$$\begin{aligned} h(c_2,\, c_3) \geq {} & -2 + \frac{(16B_1^2 - B_1^4 - 8B_1^2B_2 - 16B_2^2)}{2B_1^4}x^2 \\ & + \frac{(-16B_1^2 + 8B_1^3 - B_1^4 + 32B_1B_2 - 8B_1^2B_2 - 16B_2^2)}{B_1^5}x^3 \end{aligned} \tag{2.27}$$

The required bound of a_4 follows from the equation (2.23) upon using (2.27). Consider the function $g_1 : \mathbb{D} \to \mathbb{C}$ given by

$$\frac{2zg_1'(z)}{g_1(z)-g_1(-z)} = \varphi\left(\frac{l_1(z)-1}{l_1(z)+1}\right),$$

where

$$\begin{aligned} l_1(z) = {} & \frac{-4B_1(B_1+2x)}{-6B_1^2+(B_1^2-4B_1+4B_2)x}p_s(z) \\ & + \frac{-2B_1^2+(B_1^2+4B_1+4B_2)x}{-6B_1^2+(B_1^2-4B_1+4B_2)x}p_{-2}(z), \end{aligned}$$

$s = (2B_1^2 - (-4B_1+B_1^2+4B_2)x)/(2B_1^2)$ and $p_x(z)$ is given by (2.1). In fact, the Taylor series expansion of function g_1 is given by

$$\begin{aligned} g_1(z) = {} & z + xz^2 - \frac{x(B_1^2+4B_2+2B_1x)}{4B_1}z^3 \\ & + \left(\frac{(-B_1^4-16B_2^2+16B_1B_3)}{8B_1^4}x^3 \right. \\ & \left. + \frac{(16B_1^2-B_1^4-8B_1^2B_2-16B_2^2)}{16B_1^3}x^2 - \frac{B_1}{4}\right)z^4 + \cdots, \end{aligned}$$

which shows that the bound is sharp.

(v) If $x(-4B_1+B_1^2+4B_2)\notin[\,-2B_1^2,\,-8B_1x+2B_1^2\,)$ and $(B_1^2+4B_2)x<0$, then Lemma 2.3 shows that

$$h(c_2,\,c_3)\geq\frac{2(-2B_1+B_1^2+4B_2)}{B_1^2}x \tag{2.28}$$

Using (2.28) in (2.23) gives the required bound on a_4. The bound is sharp for the function f_0 defined by (2.21)

(vi) If $x(-4B_1+B_1^2+4B_2)\notin[\,-2B_1^2,\,-8B_1x+2B_1^2\,)$ and $(B_1^2+4B_2)x>0$, then Lemma 2.3 shows that

$$h(c_2,\,c_3)\geq-\frac{2(2B_1+B_1^2+4B_2)}{B_1^2}x+\frac{2(8B_1^2+32B_2)}{B_1^4}x^3 \tag{2.29}$$

Using (2.29) in (2.23) gives the required bound on a_4. The bound is sharp for the function g_0 defined by (2.22). □

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