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CLASSIFYING PENTAVALENT SYMMETRIC GRAPHS OF ORDER $24p$

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Abstract. A graph is said to be symmetric if its automorphism group is transitive on its arcs. A complete classification is given of pentavalent symmetric graphs of order $24p$ for each prime p. It is shown that a connected pentavalent symmetric graph of order $24p$ exists if and only if $p = 2$, 3, 5, 11 or 17, and up to isomorphism, there are only eleven such graphs.
Keywords: Symmetric graph, normal quotient, automorphism group.
MSC(2010): Primary: 05C25; Secondary: 05E18.

1. Introduction

Throughout this paper, all graphs are assumed to be finite, simple, connected and undirected.

Let Γ be a graph. We denote by $V\Gamma$, $E\Gamma$, $A\Gamma$ and $\mathsf{Aut}\Gamma$ its vertex set, edge set, arc set and full automorphism group respectively. We say Γ is *vertex-transitive* graph if $\mathsf{Aut}\Gamma$ is transitive on $V\Gamma$ and Γ is *arc-transitive* graph or *symmetric* graph if $\mathsf{Aut}\Gamma$ is transitive on $A\Gamma$. Let s be a positive integer. An *s-arc* in a graph Γ is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of $s+1$ vertices such that $(v_{i-1}, v_i) \in A\Gamma$ for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i \le s-1$. Let X be a subgroup of $\mathsf{Aut}\Gamma$. We say Γ is (X, s)-*arc-transitive* if X is transitive on the s-arcs of Γ and Γ is (X, s)-*transitive* if it is (X, s)-arc-transitive but not $(X, s+1)$-arc-transitive. In the case where $X = \mathsf{Aut}\Gamma$, we say an (X, s)-arc-transitive or (X, s)-transitive graph is an *s-arc-transitive* or *s-transitive* graph.

The study of symmetric graphs has a long history, beginning with a seminal work by Tutte [33, 34] on the cubic case. Since then the study of symmetric graphs with restricted order has been a current topic in the literature. For example, all symmetric graphs of order p, $2p$ or $3p$ were determined in [2,3,35], where p is a prime. For distinct primes p and q, Praeger et al. determined

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symmetric graphs of order pq in [30,31]. Li gave a characterization of symmetric graphs of prime-power order or odd order in [18, 19].

Recently, classifying symmetric graphs with certain valency and with restricted order has received considerable attention. For example, Conder and Dobcsańyi [4] determined all cubic symmetric graphs of orders up to 768. The classification of cubic symmetric graphs of order kp or kp^2 with $4 \leq k \leq 10$ was given in [8–10]. Cubic symmetric graphs of order $2p^2$, $14p$ or $16p$ were classified in [7, 26, 27]. For the tetravalent symmetric graphs, Zhou and Feng classified tetravalent 1-regular graphs of order $2pq$ in [38]. Tetravalent s-transitive graphs of order $4p$, $2p^2$ or $4p^2$ were classified in [11, 37, 39]. More recently, numerous papers of pentavalent symmetric graphs have been published. The stabilizers of pentavalent symmetric graphs were determined in [13, 40]. The classification of pentavalent symmetric graphs of order $8p$, $12p$, $18p$, $30p$, $2pq$ or $4pq$ were presented in [14–16, 24, 29, 36], where p and q are distinct primes. Li and Feng gave a classification of pentavalent one-regular graphs of square-free order in [22].

The main motivation for this paper arises from one result of Conder et al. [5] which proved that for any given positive integer k, there exist only finitely many connected d-valent 2-arc-transitive graphs whose order is kp or kp^2, where p is a prime and $d \geq 4$. In this paper, we classify pentavalent symmetric graphs of order $24p$ with p a prime. By using the Magma codes in Appendices, determining graphs in this paper is more simple than some related papers. Since the cases $p = 3$ and $p = 5$ have been treated in the classification of pentavalent symmetric graphs of order $36p$ or $40p$ in [21,23], we only consider the case when $p = 2$ or $p > 5$. The main result of this paper is the following theorem.

Theorem 1.1. *Let Γ be a pentavalent symmetric graph of order $24p$, where p is a prime. Then $p = 2$, 3, 5, 11 or 17. Furthermore, $\mathsf{Aut}\Gamma$, $(\mathsf{Aut}\Gamma)_v$ and Γ are described in Table 1, where $v \in V\Gamma$.*

The properties in Table 1 are determined with the help of the Magma system [1].

2. Preliminary Results

We give some necessary preliminary results in this section.

Let Γ be a graph and let X be a vertex-transitive subgroup of $\mathsf{Aut}\Gamma$. Let N be an intransitive normal subgroup of X on $V\Gamma$. Denote V_N the set of N-orbits in $V\Gamma$. The *normal quotient graph* Γ_N is the graph with vertex set V_N and two N-orbits $B, C \in V_N$ are adjacent in Γ_N if and only if some vertex of B is adjacent in Γ to some vertex of C. The following Lemma ([20, Lemma 2.5]) provides a basic reduction method for studying our pentavalent symmetric graphs.

Table 1. Pentavalent symmetric graphs of order $24p$

Γ	$\mathsf{Aut}\Gamma$	$(\mathsf{Aut}\Gamma)_v$	Girth	Diameter	Bipartite?	Cayley?
$\mathcal{C}_{48}$	$\mathrm{SL}(2,5){:}\mathrm{D}_8$	F_{20}	6	4	Yes	Yes
$\mathcal{C}_{72}^1$	$\mathrm{PGL}(2,9)$	D_{10}	4	4	No	Yes
$\mathcal{C}_{72}^2$	$\mathsf{Aut}(\mathrm{A}_6)\times\mathbb{Z}_2$	$\mathrm{F}_{20}\times\mathbb{Z}_2$	6	5	Yes	No
$\mathcal{C}_{120}^1$	$\mathrm{A}_5\times\mathrm{D}_{10}\times\mathbb{Z}_2$	D_{10}	6	6	Yes	Yes
$\mathcal{C}_{120}^2$	$\mathrm{S}_5\times\mathrm{D}_{10}$	D_{10}	4	6	Yes	Yes
$\mathcal{C}_{264}^1$	$\mathrm{PGL}(2,11)\times\mathbb{Z}_2$	D_{10}	4	7	Yes	No
$\mathcal{C}_{264}^2$	$\mathrm{PGL}(2,11)\times\mathbb{Z}_2$	D_{10}	6	6	Yes	No
$\mathcal{C}_{264}^3$	$\mathrm{PSL}(2,11){:}\mathrm{D}_8$	D_{20}	6	6	Yes	No
$\mathcal{C}_{264}^4$	$\mathrm{PGL}(2,11)\times\mathbb{Z}_2$	D_{10}	4	7	Yes	No
$\mathcal{C}_{408}^1$	$\mathrm{PSO}^-(4,4)$	D_{20}	6	6	No	No
$\mathcal{C}_{408}^2$	$\mathrm{PSL}(2,16)$	D_{10}	8	5	No	No

Lemma 2.1. *Let Γ be an X-arc-transitive graph of prime valency $p>2$, where $X\leq\mathsf{Aut}\Gamma$, and let $N\trianglelefteq X$ have at least three orbits on $V\Gamma$. Then the following statements hold.*

(i) *N is semiregular on $V\Gamma$, $X/N\leq\mathsf{Aut}\Gamma_N$, and Γ_N is an X/N-arc-transitive graph of valency p;*
(ii) *Γ is (X,s)-transitive if and only if Γ_N is $(X/N,s)$-transitive, where $1\leq s\leq 5$ or $s=7$.*

By [13, 40], we have the following lemma.

Lemma 2.2. *Let Γ be a pentavalent (G,s)-transitive graph for some $G\leq\mathsf{Aut}\Gamma$ and $s\geq 1$. Let $v\in V\Gamma$. Then the order of G_v equals one of the following values:* 5, 10, 20, 40, 60, 80, 120, 720, 960, 1440, 1920, 2880, 5760 *or* 23040. *In particular, the order of G_v is a divisor of $2^9\cdot 3^2\cdot 5$.*

From [12, pp. 12-14], one may obtain the following proposition by checking the 3-prime factor nonabelian simple groups.

Proposition 2.3. *Let G be a $\{2,3,5\}$-nonabelian simple group. Then $G=\mathrm{A}_5$, A_6 or $\mathrm{PSU}(4,2)$.*

By checking the orders of nonabelian simple groups, see [12, pp. 134-136] for example, we have the following proposition.

Proposition 2.4. *Let $p>5$ be a prime and let G be a $\{2,3,5,p\}$-nonabelian simple group such that $|G|$ divides $2^{12}\cdot 3^3\cdot 5\cdot p$ and $60p$ divides $|G|$. Then $G=\mathrm{A}_7$, A_8, M_{11}, M_{12}, $\mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,19)$, $\mathrm{PSL}(2,16)$, $\mathrm{PSL}(2,31)$ or $\mathrm{PSL}(3,4)$.*

By [16, 25], some information about pentavalent symmetric graphs of order $6p$ is given in the following lemma.

Lemma 2.5. *Let Γ be a pentavalent symmetric graph. Let p be a prime. If $|V\Gamma| = 6p$, then Γ is isomorphic to one of the graphs in Table 2.*

TABLE 2. Pentavalent symmetric graphs of order 6p

Γ	$\mathsf{Aut}\Gamma$	Remark
Icosahedral Graph	$A_5 \times \mathbb{Z}_2$	$p = 2$
$\mathsf{K}_{6,6} - 6\mathsf{K}_2$	$A_6 \times \mathbb{Z}_2$	$p = 2$
$\mathcal{C}_{42}$	$\mathsf{Aut}(\mathrm{PSL}(3,4))$	$p = 7$
$\mathcal{C}_{66}$	$\mathrm{PGL}(2,11)$	$p = 11$
$\mathcal{C}_{114}$	$\mathrm{PGL}(2,19)$	$p = 19$

By [15] and with the help of Magma system [1], we give some information of pentavalent symmetric graphs of order $8p$ in the following lemma.

Lemma 2.6. *Let Γ be a pentavalent symmetric graph. Let p be a prime. If $|V\Gamma| = 8p$, then Γ is isomorphic to one of the graphs in Table 3.*

TABLE 3. Pentavalent symmetric graphs of order 8p

Γ	$\mathsf{Aut}\Gamma$	Remark
CL_{16}	$\mathbb{Z}_2^4{:}\mathrm{S}_5$	$p = 2$
$\mathrm{I}^{(2)}$	$(A_5 \times \mathbb{Z}_2^2){:}\mathbb{Z}_2$	$p = 3$
$\mathcal{C}_{248}$	$\mathrm{PSL}(2,31)$	$p = 31$

By [14], we give some information of pentavalent symmetric graphs of order $12p$ in the following lemma. In fact, in [14, Theorem 4.1], $\mathcal{C}_{66}^{(2)}$ is isomorphic to $\mathcal{C}_{132}^5$, $\mathsf{Aut}(\mathcal{C}_{132}^5) \cong \mathsf{Aut}(\mathcal{C}_{66}^{(2)}) \cong \mathrm{PGL}(2,11) \times \mathbb{Z}_2$, $\mathsf{Aut}(\mathbf{I}_{12}^{(2)}) \cong (A_5 \times \mathbb{Z}_2^2){:}\mathbb{Z}_2$ and $\mathsf{Aut}(\mathcal{C}_{60}) \cong A_5 \times D_{10}$ by Magma [1].

Lemma 2.7. *Let Γ be a pentavalent symmetric graph. Let p be a prime. If $|V\Gamma| = 12p$, then Γ is isomorphic to one of the graphs in Table 4.*

In the following, we need to introduce the concept of Schur multiplier. Let G be a perfect group, that is, $G' = G$. A *central extension* of G is a group H satisfying $H/N \cong G$ for $N \leq Z(H)$. If H is perfect, we call H a covering group of G. It was shown by Schur [32] that all covering groups of G are finite, and

TABLE 4. Pentavalent symmetric graphs of order 12p

Γ	$\mathsf{Aut}\Gamma$	Remark
$\mathbf{I}_{12}^{(2)}$	$(\mathrm{A}_5\times\mathbb{Z}_2^2){:}\mathbb{Z}_2$	$p=2$
$\mathcal{C}_{36}$	$\mathsf{Aut}(\mathrm{A}_6)$	$p=3$
$\mathcal{C}_{60}$	$\mathrm{A}_5\times\mathrm{D}_{10}$	$p=5$
$\mathcal{C}_{132}^1$	$\mathrm{PSL}(2,11)\times\mathbb{Z}_2$	$p=11$
$\mathcal{C}_{132}^i$	$\mathrm{PGL}(2,11)$	$p=11, 2\leq i\leq 4$
$\mathcal{C}_{132}^5$	$\mathrm{PGL}(2,11)\times\mathbb{Z}_2$	$p=11$

there is a unique maximal covering group M. This group M is called the full covering group of G, and define the *Schur multiplier* of G, written $\mathsf{Mult}(G)$, to be the center of M. The following lemma follows from a theorem of Schur (see [17]) and its proof can be seen in [28, Lemma 2.11].

Lemma 2.8. *Let $M = N.T^d$ be a central extension, where $d \geq 1$ and T is a nonabelian simple group. Then $M = NM'$ and $M' = Z.T^d$, where Z is a factor group of $\mathsf{Mult}(T)^d$ and $Z \leq N$.*

3. **The proof of Theorem 1.1**

In this section, we will prove Theorem 1.1 by giving some lemmas. Now let Γ be a pentavalent symmetric graph of order $24p$, where p is a prime. If $p=3$, then $|V\Gamma| = 72$, and Γ is isomorphic to $\mathcal{C}_{72}^1$ or $\mathcal{C}_{72}^2$ by [21]. If $p = 5$, then $|V\Gamma| = 120$, and Γ is isomorphic to $\mathcal{C}_{120}^1$ or $\mathcal{C}_{120}^2$ by [23]. Suppose $p = 2$ or $p > 5$ in the following. Let $A = \mathsf{Aut}\Gamma$ and let X be a subgroup of A. We say X is a *minimal arc-transitive subgroup* of A if X is arc-transitive on Γ and if a subgroup M of X is arc-transitive on Γ, then M equals X.

The next two simple lemmas are helpful to our argument.

Lemma 3.1. *Let $X \leq A$ be a subgroup of A which is arc-transitive on Γ. Let N be an insoluble normal subgroup of X. Then N has at most two orbits on $V\Gamma$. Furthermore, if N is not isomorphic to* $\mathrm{PSL}(2,7)$*, then the following statements hold.*

(1) *For each $v \in V\Gamma$, $5 \bigm| |N_v^{\Gamma(v)}|$.*
(2) *$60p$ divides the order of N.*

Proof. Suppose that N has at least three orbits on $V\Gamma$. Lemma 2.1 implies that $N_v = 1$ for each $v \in V\Gamma$. Hence $|N| \bigm| 24p$. If $p \neq 7$, then a group of order $24p$ is soluble, which follows that N is soluble, a contradiction. If $p = 7$, then $|N| \bigm| 24 \cdot 7 = 168$. It implies that $|N| = 168$ as N is insoluble, a contradiction with N has at least three orbits on $V\Gamma$. Hence N has at most two orbits on $V\Gamma$.

(1) For each $v \in V\Gamma$, if $N_v = 1$, then, arguing as the above paragraph, a contradiction occurs. Thus, $N_v \neq 1$. Since X is transitive on $V\Gamma$, we have $|N_v^{\Gamma(v)}| \neq 1$. It follows that $5 \bigm| |N_v^{\Gamma(v)}|$, since $N_v^{\Gamma(v)} \trianglelefteq X_v^{\Gamma(v)}$ and $X_v^{\Gamma(v)}$ acts primitively on $\Gamma(v)$.

(2) Since N has at most two orbits on $V\Gamma$, that is, $2^2 \cdot 3 \cdot p$ divides $|N : N_v|$ and by (1), $5 \bigm| |N_v|$, which implies that $60p \bigm| |N|$, as required. □

Lemma 3.2. *Let N be a minimal normal subgroup of A. Assume A has no soluble minimal normal subgroup. Then N is isomorphic to a nonabelian simple group. Furthermore, if N is not isomorphic to* $\mathrm{PSL}(2,7)$*, then $A \leq \mathsf{Aut}(N)$.*

Proof. Let N be an insoluble minimal normal subgroup of A. Then $N = T^d$ with T a nonabelian simple group. We first prove that $d = 1$. By Lemma 3.1, N has at most two orbits on $V\Gamma$, and so $12p$ divides $|N|$. It implies that $p \bigm| |T|$. Suppose that $d \geq 2$. Then $N = T_1 \times T_2 \times \cdots \times T_d$ and $p^d \bigm| |N|$. By Lemma 2.2, $|A_v| \bigm| 2^9 \cdot 3^2 \cdot 5$, we have $|N| \bigm| |A| \bigm| 2^{12} \cdot 3^3 \cdot 5 \cdot p$. Then the only possible case is $(d, p) = (2, 5)$, a contradiction with our assumption $p \neq 5$. Hence $d = 1$ and N is a nonabelian simple group. Let $C = \mathrm{C}_A(N)$. If $C \neq 1$, then C is insoluble, because A has no soluble minimal normal subgroup. By Lemma 3.1(2), we have $60p \bigm| |N|$. Since $|N| \bigm| |A| \bigm| 2^{12} \cdot 3^3 \cdot 5 \cdot p$ and $C \cap N = \mathbf{Z}(N) = 1$, we have $|C| \bigm| 2^{10} \cdot 3^2$. By the Burnside theorem, C is soluble, a contradiction. Hence $C = 1$. By 'N/C' theorem, $A \leq \mathsf{Aut}(N)$. □

Proof of Theorem 1.1. In the following, we prove Theorem 1.1 via a series of Lemmas.

Lemma 3.3. *If $p = 2$, then Γ is isomorphic to $\mathcal{C}_{48}$ in Table 1.*

Proof. Let N be a minimal normal subgroup of A. Suppose first that N is soluble. Then N is isomorphic to $\mathbb{Z}_r^d$ for some prime r. On the other hand, for each $v \in V\Gamma$, $|v^N|$ is a prime power and a divisor of 48, N has at least three orbits on $V\Gamma$. By Lemma 2.1, N is semiregular on $V\Gamma$. It follows that $|N| \bigm| |V\Gamma| = 2^3 \cdot 3$ and so $N \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^3$ or $\mathbb{Z}_3$. If $N \cong \mathbb{Z}_2^3$, then Lemma 2.1 implies that Γ_N is a pentavalent symmetric graph of odd order, a contradiction.

If $N \cong \mathbb{Z}_2$, then Γ_N is a pentavalent symmetric graph of order 24. By Lemma 2.7, Γ_N is isomorphic to $\mathrm{I}^{(2)}$ with $\mathsf{Aut}\Gamma_N \cong (\mathrm{A}_5 \times \mathbb{Z}_2^2){:}\mathbb{Z}_2$. By Magma [1], a minimal arc-transitive subgroup of $\mathsf{Aut}\Gamma_N$ is isomorphic to S_5 or $\mathrm{A}_5 \times \mathbb{Z}_2$. By Lemma 2.1, A/N contains a subgroup isomorphic to S_5 or $\mathrm{A}_5 \times \mathbb{Z}_2$, which implies that A contains an arc-transitive subgroup isomorphic to $\mathbb{Z}_2.\mathrm{S}_5$ or $\mathbb{Z}_2.(\mathbb{Z}_2 \times \mathrm{A}_5)$. By Magma [1] (see our Magma codes in Appendix), Γ is isomorphic to $\mathcal{C}_{48}$ in Table 1.

If $N \cong \mathbb{Z}_2^2$, then Γ_N is a pentavalent symmetric graph of order 12. By Lemma 2.5, Γ_N is isomorphic to I with $\mathsf{Aut}\Gamma_N \cong \mathrm{A}_5 \times \mathbb{Z}_2$ or $\mathsf{K}_{6,6} - 6\mathsf{K}_2$ with $\mathsf{Aut}\Gamma_N \cong \mathrm{S}_6 \times \mathbb{Z}_2$. For the former case, since A/N is arc-transitive on Γ_N, we have $60 \bigm| |A/N|$. Thus, by Magma [1], A/N contains an arc-transitive subgroup

$H/N \cong \mathrm{A}_5$. Since the Schur multiplier of A_5 is $\mathbb{Z}_2$, we have $H \cong \mathbb{Z}_2^2{\times}\mathrm{A}_5$ or $\mathbb{Z}_2{\times}\mathrm{SL}(2,5)$ and H is arc-transitive on Γ. By Magma [1], no pentavalent symmetric graphs of order 48 appears for this case. For the latter case, by Magma [1], a minimal arc-transitive subgroup of $\mathsf{Aut}\Gamma_N$ is isomorphic to S_5 or $\mathrm{A}_5{\times}\mathbb{Z}_2$. Lemma 2.1 implies that A contains an arc-transitive subgroup isomorphic to $\mathbb{Z}_2^2.\mathrm{S}_5$ or $\mathbb{Z}_2^2.(\mathrm{A}_5{\times}\mathbb{Z}_2)$. By Magma [1], we have Γ is isomorphic to $\mathcal{C}_{48}$.

If $N \cong \mathbb{Z}_3$, then Γ_N is a pentavalent symmetric graph of order 16. By Lemma 2.6, $\Gamma_N \cong \mathrm{CL}_{16}$ and $\mathsf{Aut}\Gamma \cong \mathbb{Z}_2^4{:}\mathrm{S}_5$. By Magma [1], the minimal arc-transitive subgroup of $\mathsf{Aut}\Gamma_N$ is isomorphic to $\mathbb{Z}_2^4{:}\mathbb{Z}_5$. Therefore, Lemma 2.1 implies that A/N contains $H/N \cong \mathbb{Z}_2^4{:}\mathbb{Z}_5$, that is, A contains an arc-transitive subgroup $H \cong \mathbb{Z}_3.(\mathbb{Z}_2^4{:}\mathbb{Z}_5)$. By Magma [1] (see our Magma codes in Appendices), $H \cong \mathbb{Z}_3{\times}(\mathbb{Z}_2^4{:}\mathbb{Z}_5)$ and no pentavalent symmetric graph of order 48 appears for this case.

Now we suppose that A has no soluble minimal normal subgroup. Then, by Lemma 3.2, $N \trianglelefteq A$, where N is a $\{2,3,5\}$-nonabelian simple group. By Proposition 2.3, N is isomorphic to A_5, A_6 or $\mathrm{PSU}(4,2)$. If $N \cong \mathrm{A}_5$, then Lemma 3.1 implies that N has at most two orbits on $V\Gamma$, that is, $2^3 \cdot 3 \mid |N|$, a contradiction with $|N| = 2^2 \cdot 3 \cdot 5$. If $N \cong \mathrm{A}_6$, then $|N_v| = \frac{|N|}{24} = 15$, a contradiction with A_6 has no subgroup of order 15. If $N \cong \mathrm{PSU}(4,2)$, then $|N_v| = \frac{|N|}{24} = 540$ or $|N_v| = \frac{|N|}{48} = 1080$, a contradiction with $\mathrm{PSU}(4,2)$ has no subgroup of order 540 or 1080. □

Now we consider the case when $p > 5$. First we suppose that A contains a soluble minimal normal subgroup N. Then we have the following lemma.

Lemma 3.4. *If A has a soluble minimal normal subgroup N, then Γ is isomorphic to $\mathcal{C}_{264}^i$ in Table 1, where $1 \le i \le 4$.*

Proof. Let N be a soluble minimal normal subgroup. Then $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_3$ or $\mathbb{Z}_p$. If $N \cong \mathbb{Z}_2^3$, then Γ_N is a pentavalent symmetric graph of odd order, which is impossible.

If $N \cong \mathbb{Z}_2$, then Γ_N is a pentavalent symmetric graph of order $12p$. By Lemma 2.7, we have $\Gamma_N \cong \mathcal{C}_{132}^i$, where $1 \le i \le 5$. Furthermore, $A/N \le \mathsf{Aut}(\mathcal{C}_{132}^i)$ and $p = 11$. By Magma [1], a minimal arc-transitive subgroup of $\mathsf{Aut}(\mathcal{C}_{132}^i)$ is isomorphic to $\mathrm{PSL}(2,11)$ or $\mathrm{PGL}(2,11)$. Since A/N is arc-transitive on Γ_N, we have A/N has an arc-transitive subgroup H/N isomorphic to $\mathrm{PSL}(2,11)$ or $\mathrm{PGL}(2,11)$. Since $\mathbb{Z}_2.\mathrm{PSL}(2,11) \cong \mathbb{Z}_2{\times}\mathrm{PSL}(2,11)$ or $\mathrm{SL}(2,11)$ and $\mathbb{Z}_2.\mathrm{PGL}(2,11) \cong \mathbb{Z}_2{\times}\mathrm{PGL}(2,11)$ or $\mathrm{SL}(2,11){:}\mathbb{Z}_2$, we have A has an arc-transitive subgroup H isomorphic to $\mathbb{Z}_2{\times}\mathrm{PSL}(2,11)$, $\mathrm{SL}(2,11)$, $\mathbb{Z}_2{\times}\mathrm{PGL}(2,11)$ or $\mathrm{SL}(2,11){:}\mathbb{Z}_2$. By Magma [1], Γ is isomorphic to $\mathcal{C}_{264}^i$ in Table 1, where $1 \le i \le 4$.

If $N \cong \mathbb{Z}_2^2$, then Γ_N is a pentavalent symmetric graph of order $6p$. By Lemma 2.5, Γ_N is isomorphic to $\mathcal{C}_{42}$, $\mathcal{C}_{66}$ or $\mathcal{C}_{114}$. Assume $\Gamma_N \cong \mathcal{C}_{42}$. Then

$A/N \leq \mathsf{Aut}\Gamma_N \cong \mathsf{Aut}(\mathrm{PSL}(3,4))$ and $p = 7$. By Lemma 2.1, A/N is arc-transitive on Γ_N. By Magma [1], a minimal arc-transitive subgroup of $\mathsf{Aut}\Gamma_N$ is isomorphic to $\mathrm{PSL}(3,4){:}\mathbb{Z}_2$. Hence A/N contains an arc-transitive subgroup H/N isomorphic to $\mathrm{PSL}(3,4){:}\mathbb{Z}_2$. Then $H \cong \mathbb{Z}_2^2.(\mathrm{PSL}(3,4){:}\mathbb{Z}_2)$ is arc transitive on Γ. By Magma [1], no pentavalent symmetric graph of order $24\cdot 7$ appears for this case.

Assume Γ_N is isomorphic to $\mathcal{C}_{66}$. Then $A/N \leq \mathsf{Aut}\Gamma_N \cong \mathrm{PGL}(2,11)$ and $p = 11$. By Lemma 2.1, A/N is arc-transitive on Γ_N. By Magma [1], A/N contains an arc-transitive subgroup H/N isomorphic to $\mathrm{PSL}(2,11)$. Since the Schur multiplier of $\mathrm{PSL}(2,11)$ is isomorphic to $\mathbb{Z}_2$ (see Atlas [6] for example), we have $H \cong \mathbb{Z}_2^2{\times}\mathrm{PSL}(2,11)$ or $\mathbb{Z}_2{\times}\mathrm{SL}(2,11)$. Furthermore, H is arc-transitive on Γ. By Magma [1], Γ is isomorphic to $\mathcal{C}_{264}^1$, $\mathcal{C}_{264}^2$ or $\mathcal{C}_{264}^3$ in Table 1.

Assume $\Gamma_N \cong \mathcal{C}_{114}$. Then $A/N \leq \mathsf{Aut}\Gamma_N \cong \mathrm{PGL}(2,19)$ and $p = 19$. By Lemma 2.1, A/N is arc-transitive on Γ_N. By Magma [1], the minimal arc-transitive subgroup of $\mathsf{Aut}\Gamma_N$ is isomorphic to $\mathrm{PGL}(2,19)$. Hence A/N is isomorphic to $\mathrm{PGL}(2,19)$. Then $A = \mathbb{Z}_2^2.\mathrm{PGL}(2,19) \cong \mathbb{Z}_2^2{\times}\mathrm{PGL}(2,19)$ or $\mathbb{Z}_2{\times}(\mathrm{SL}(2,19){:}\mathbb{Z}_2)$. By Magma [1], no pentavalent symmetric graph of order $24\cdot 19$ appears for this case.

If $N \cong \mathbb{Z}_3$, then Γ_N is a pentavalent symmetric graph of order $8p$. Since $p > 5$, Lemma 2.6 implies that $\Gamma_N \cong \mathcal{C}_{248}$ and $p = 31$. By Lemma 2.1, $A/N \lesssim \mathsf{Aut}\Gamma_N \cong \mathrm{PSL}(2,31)$ and A/N is arc-transitive on Γ_N. Hence $5 \cdot 248 \mid |A/N|$. By checking the maximal subgroup of $\mathrm{PSL}(2,31)$, we have $A/N \cong \mathrm{PSL}(2,31)$. On the other hand, by Atlas [6], the Schur multiplier of $\mathrm{PSL}(2,31)$ is isomorphic to $\mathbb{Z}_2$, Lemma 2.8 implies that $A = \mathbb{Z}_3{\times}\mathrm{PSL}(2,31)$. By Magma [1], no pentavalent symmetric graph of order $24\cdot 31$ appears for this case.

If $N \cong \mathbb{Z}_p$, then Γ_N is a pentavalent symmetric graph of order 24. By Lemma 2.7, Γ_N is isomorphic to $\mathrm{I}^{(2)}$. By Lemma 2.1, $A/N \leq \mathsf{Aut}\Gamma_N \cong (\mathrm{A}_5{\times}\mathbb{Z}_2^2){:}\mathbb{Z}_2$. Since A/N is arc-transitive on Γ_N, we have $120 \mid |A/N|$. It implies that A/N contains a normal subgroup H/N isomorphic to A_5. Since $p > 5$ and the Schur multiplier of A_5 isomorphic to $\mathbb{Z}_2$, Lemma 2.8 implies that $H \cong \mathbb{Z}_p{\times}\mathrm{A}_5$ and $H' = \mathrm{A}_5$. Since H' is a characteristic subgroup of H and $H \trianglelefteq A$, we have $H' \trianglelefteq A$. By Lemma 3.1, H' has at most two orbits on $V\Gamma$, which implies that $|V\Gamma| \leq 120$, a contradiction with $p > 5$. □

Now we may treat the case when A has no soluble minimal normal subgroup and the next lemma completes the proof of Theorem 1.1.

Lemma 3.5. *If A has no soluble minimal normal subgroup, then Γ is isomorphic to $\mathcal{C}_{408}^1$ or $\mathcal{C}_{408}^2$ in Table 1.*

Proof. Let N be an insoluble minimal normal subgroup of A. By Lemma 3.2, $d = 1$ and $N \trianglelefteq A$. Assume that N is isomorphic to $\mathrm{PSL}(2,7)$. Let $C = \mathrm{C}_A(N)$. If $C \neq 1$, then $C \cap N = \mathbf{Z}(N) = 1$, and so $CN = C{\times}N$. Since A has no soluble minimal normal subgroup, we have C is insoluble. Since $|A| \mid 2^{12}\cdot 3^3\cdot 5\cdot 7$ and

$C \cap N = 1$, we have $|C| \bigm| 2^9 \cdot 3^2 \cdot 5$. Hence C contains a normal subgroup of A which is isomorphic to A_5 or A_6. It implies that A has a normal subgroup isomorphic to $A_5 \times PSL(2,7)$ or $A_6 \times PSL(2,7)$. Let $H_1 = A_5 \times PSL(2,7)$ and $H_2 = A_6 \times PSL(2,7)$. If A has a normal subgroup isomorphic to H_1, then, with similar discussion as above, we have $C_A(H_1) = 1$. By 'N/C' theorem, $A \leq \mathsf{Aut}(H_1) \cong S_5 \times PGL(2,7)$. By Magma [1], no pentavalent symmetric graph of order $24 \cdot 7$ appears for this case. A similar argument shows that no pentavalent symmetric graph of order $24 \cdot 7$ appears when A has a normal subgroup isomorphic to H_2. Hence N is not isomorphic to $PSL(2,7)$. By Lemma 3.1, $60p \bigm| |N|$. Since $|A| \bigm| 2^{12} \cdot 3^3 \cdot 5 \cdot p$, we have $|N|$ is a divisor of $2^{12} \cdot 3^3 \cdot 5 \cdot p$. By Proposition 2.4, N is isomorphic to A_7, A_8, M_{11}, M_{12}, $PSL(2,11)$, $PSL(2,19)$, $PSL(2,16)$, $PSL(2,31)$ or $PSL(3,4)$.

Assume $N \cong A_7$. Then $p = 7$, and by Lemma 3.2, $A_7 \leq A \leq S_7$. It implies that $|A_v| = \frac{|A_7|}{24 \cdot 7} = 15$ or $|A_v| = \frac{|S_7|}{24 \cdot 7} = 30$, this is impossible by Lemma 2.2. Similarly, N is not isomorphic to M_{11} or $PSL(2,19)$.

Assume $N \cong PSL(2,31)$. Then $p = 31$, and Lemma 3.2 implies that $PSL(2,31) \leq A \leq PGL(2,31)$. It follows that $|A_v| = \frac{|PSL(2,31)|}{24 \cdot 31} = 20$ or $|A_v| = \frac{|PGL(2,31)|}{24 \cdot 31} = 40$. However, by Atlas [6], $PSL(2,31)$ has no subgroup of order 20 and $PGL(2,31)$ has no subgroup of order 40, a contradiction.

Assume $N \cong M_{12}$. Then $p = 11$ and $|V\Gamma| = 264$. By Lemma 3.2, $A \leq \mathsf{Aut}(M_{12}) = M_{12}.\mathbb{Z}_2$. If $A = M_{12}$, then $|A_v| = \frac{|M_{12}|}{24 \cdot 11} = 360$. This is impossible by Lemma 2.2. Hence $A = M_{12}.\mathbb{Z}_2$. By Magma [1] (see the Magma codes in Appendices), no pentavalent symmetric graph of order $24 \cdot 11$ satisfies our condition. With similar discussion, we have N is not isomorphic to A_8, $PSL(2,11)$ or $PSL(3,4)$.

Finally, assume $N \cong PSL(2,16)$. Then $p = 17$, and by Lemma 3.2, $PSL(2,16)$
$\leq A \leq P\Sigma L(2,16)$. By Magma [1], Γ is isomorphic to $\mathcal{C}^1_{408}$ or $\mathcal{C}^2_{408}$ in Table 1. □

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Appendix

Magma codes

```
/*
Input : a positive integer n and two finite groups G, N
```

```
Output: all groups X of order n, satisfying that X has a
normal subgroup isomorphic to N and the quotient group
X/N isomorphic to G
*/
f:=function(n,G,N);
P:=SmallGroupProcess(n);
X:=[];
repeat GG:=Current(P);
NN:=NormalSubgroups(GG);
for i in [1..#NN] do
 if IsIsomorphic(NN[i]`subgroup,N) eq true then
  F:=quo<GG|NN[i]`subgroup>;
  if IsIsomorphic(F,G) eq true then
   _,a:=CurrentLabel(P);
   Append(~X,SmallGroup(n,a));
  end if;
 end if;
end for;
Advance(~P);
until IsEmpty(P);
return X;
end function;


/*
Input : a finite group G and a positive integer n
Output: all graphs of order |G|/n, which admit G as an
arc-transitive automorphism group
*/
Graph:=function(G,n);
graph:=[];
i:=0;
H:=Subgroups(G:OrderEqual:=n);
for j in [1..#H] do
 HH:=H[j]`subgroup;
 CA:=CosetAction(G,HH);
 O:=Orbits(CA(HH));
  for k in [1..#O] do
   OO:=SetToSequence(O[k]);
   GR:=OrbitalGraph(CA(G),1,{OO[1]});
    if (IsConnected(GR) eq true) and (Valence(GR) eq 5) and
     (not exists{t:t in graph|IsIsomorphic(GR,t) eq true}) then
```

```
     Append(~graph,GR);
     i:=i+1;
    end if;
  end for;
end for;
return i,graph;
end function;
```

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