Bulletin of the Iranian Mathematical Society Vol. 31 *No. 1 (2005), pp 71-84.*

EXTENSION OF THE DOUADY-HUBBARD'S THEOREM ON CONNECTEDNESS OF THE MANDELBROT SET TO SYMMETRIC POLYNOMIALS

ARSALAN CHADEMAN* AND AHMAD ZIREH

Communicated by Freydoon Shahidi

ABSTRACT. We consider the complex dynamics of a one parametric family of polynomials $f_c(z) = z^{d+1}+cz$, where $d \geq 1$ is a given integer and $c \in \mathbb{C}$. The critical set of the *symmetric polynomial* f_c has d points and is stable under the symmetric group Σ_d. In the dynamics of quadratic polynomials $P_c(z) = z^2 + c$, Douady and Hubbard have proved that the Mandelbrot set is connected [5]. This result has been extended to the dynamics of uni-critical polynomials $g_c(z) = z^d+c$ [7, 11]. We extend the Douady-Hubbard's Theorem to the symmetric polynomials.

MSC(2000): Primary 37F45; Secondary 30D05, 30C10

Keywords: Mandelbrot set, Multibrot set, Connectedness locus, Polynomial dynamics, Symmetric polynomials, Douady-Hubbard's theorem

Received: 08 October 2005, Revised: 08 December 2005

*Corresponding author

1. **Introduction**

We first recall some terminology and definitions in holomorphic dynamics (see [1, 2, 9, 12]). Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial self-map of the complex plane. For each $z \in \mathbb{C}$, the orbit of z is

$$\mathrm{Orb}_f(z) = \{z, f(z), f(f(z)), \cdots, f^n(z), \cdots\}.$$

The dynamical plane $\mathbb{C}$ is decomposed into two complementary sets: the *filled Julia set*

$$K(f) = \{c \in \mathbb{C} : \ \mathrm{Orb}_f(z) \text{ is bounded}\},$$

and its complementary, the *basin of infinity*

$$A_f(\infty) = \mathbb{C} - K(f).$$

The boundary of $K(f)$, called the *Julia set*, is denoted by $J(f)$. When f is a quadratic polynomial, say $f(z) = P_c = z^2 + c$, the Mandelbrot set $\mathcal{M}_2$ is defined as the set of parameter values c, for which $K(P_c)$ is connected, that is

$$\mathcal{M}_2 = \{c \in \mathbb{C} : K(P_c) \quad \text{is connected}\},$$

or equivalently, as the set of parameters for which the orbit of 0 is bounded. In fact (see [1, 2]),

$$\mathcal{M}_2 = \{c \in \mathbb{C} : \ \forall n, \ |P_c^n(c)| \leq 2\}.$$

More generally, for the family $g_c(z) = z^d + c$, the connectedness locus, called the Mandelbrot set $\mathcal{M}_d$, is defined by

$$\mathcal{M}_d = \{c \in \mathbb{C} : K(g_c) \text{ is connected}\}.$$

In the last decades, the topological and measure theoretical properties of these sets have been extensively studied. Using techniques and notions based on the *quasi-conformal surgery, polynomial-like map, renormalizability, Yoccoz puzzle, Hubbard's tableau, etc.*, the following substantial results have been obtained.

Theorem A. (Douady-Hubbard [5]) *The Mandelbrot set $\mathcal{M}_2$ is connected.*

Theorem B. (Yoccoz [6]) *If P_c has no indifferent periodic points and is not infinitely renormalizable, then the Julia set $J(P_c)$ is locally connected.*

Theorem C. (Lyubich [8]) *If P_c has no irrational indifferent periodic points and is not infinitely renormalizable, then the Lebesgue measure of the Julia set $J(P_c)$ is equal to zero.*

The extension of these theorems to other classes of polynomials constitutes part of today's research in this area. For instance, Theorems A and B have been extended to uni-critical polynomials $g_c(z) = z^d + c$ (see [11, 7]).

The aim of this paper is to extend theorem A to the class of *symmetric polynomials* $f_c(z) = z(z^d + c)$. Theorems B and C are extended to this class in [4, 13].

For the family $f_c(z) = z^{d+1} + cz$, the *connectedness locus* $\mathcal{C}_d$, or what is the same, the *Mandelbrot set*, is defined by

$$\mathcal{C}_d = \{c \in \mathbb{C} : K(f_c) \quad \text{is connected}\}.$$

Our main result now reads as:

Theorem 1.1. (Extension of Douady-Hubbard's Theorem) *The Mandelbrot set $\mathcal{C}_d$ is connected.*

The techniques used to prove it, are similar to those used in the quadratic case, repeated in the uni-critical one, with a slight difference. In the uni-critical case g_c, using the Böttcher function ϕ_c, a conformal isomorphism $\Phi_{M_d} : \mathbb{C} - \mathcal{M}_d \to \mathbb{C} - \overline{D}, \Phi(c) = \phi_c(c)$ is defined, from which one concludes the connectedness of $\mathcal{M}_d$. This method works because the parameter c in g_c coincides with the critical value. In the case of symmetric polynomials f_c, we apply the Böttcher function to one of the critical values $f_c(c_0)$, which is a function of c, to get a new function $\Phi(c) = \phi_c(f_c(c_0))$. Then it is shown that $\Phi : \mathbb{C} - \mathcal{C}_d \to \mathbb{C} - \overline{D}$ is a conformal map (see the proof of Theorem 1, §4).
Another analogy (see §3, Proposition 2) consists of the characterization of $\mathcal{C}_d$ by a boundedness condition:

$$\mathcal{C}_d = \{c \in \mathbb{C}; \quad |f_c^n(c_0)| \leq (1 + \frac{d+1}{d}\alpha)^{1/d} \quad \text{for every} \quad n \in \mathbb{N}\}.$$

2. **Symmetric polynomials**

For convenience, in this section we provide necessary preliminaries. We first give a symmetry condition on the family of polynomials of degree ≥ 3. This condition is similar to the one investigated by Milnor [10]. Instead of the general family of all quadratic rational maps, Milnor introduced quadratic rational maps with symmetries and derived the one-parameter family $f_k(z) = k(z + z^{-1})$, which depends on the complex parameter $k \in \mathbb{C}$.

From now on, $d \geq 2$ will be a fixed integer. In the dynamical study of monic centered polynomials $f(z) = z^{d+1} + a_{d-1}z^{d-1} + \cdots + a_0$, with the parameter space $\mathbb{C}^d$, at a first step we can restrict ourselves to a one-parameter family, just as Milnor did. For this purpose, by an *automorphism of* $f(z) = z^{d+1} + a_{d-1}z^{d-1} + \cdots + a_0$, we mean a non-constant affine map $R(z) = az + b$, with $a \neq 0, a, b \in \mathbb{C}$, satisfying $R \circ f \circ R^{-1} = f$. An easy calculation shows that the collection of all automorphisms of f, denoted by Aut (f), forms a subgroup of the finite rotation group $\Sigma_d = \{\gamma \in \mathbb{C} : \gamma^d = 1\}$. In some particular cases, the groups $\mathrm{Aut}(f)$ and Σ_d are equal. All such polynomials are characterized in the following proposition.

Proposition 2.1. *For a monic centered polynomial* $f(z) = z^{d+1} + a_{d-1}z^{d-1} + \cdots + a_0$, *the following are equivalent.*

(i) $\mathrm{Aut}(f) = \Sigma_d$;

(ii) *There exists* $c \in \mathbb{C}$ *such that* $f(z) = f_c(z) = z(z^d + c)$ for all $z \in \mathbb{C}$.

(iii) *The polynomial* f *vanishes at the origin and the critical set* $Z(f') = \{z \in \mathbb{C} : f'(z) = 0\}$ *is stable under the action of* Σ_d.

The proof is straightforward and is omitted. □

Definition 2.2. A $d-$*symmetric polynomial*, or *symmetric polynomial* if there is no confusion, is a polynomial of the form $f_c(z) = z(z^d + c), c \in \mathbb{C}$.

Remark 2.3. A $d-$symmetric polynomial $f_c(z) = z(z^d + c), c \in \mathbb{C}$ has d symmetric critical points (counting with multiplicity when $c = 0$):

$$c_0,\ c_1 = \omega c_0,\ \cdots,\ c_{d-1} = \omega^{d-1}c_0, \tag{2.1}$$

where $\omega = e^{2\pi i/d}$ and c_0 is one of the solutions of $(d+1)z^d + c = 0$, i.e.,

$$c_0 = (\frac{-c}{d+1})^{1/d}.$$

The property (i), or direct calculation, shows that also f_c has d symmetric critical values:

$$v_0,\ v_1 = \omega v_0,\ \cdots,\ v_{d-1} = \omega^{d-1} v_0, \tag{2.2}$$

where $v_0 = f_c(c_0) = \frac{dcc_0}{d+1}$.

3. Boundedness and symmetry

In this section, we first observe that the Mandelbrot set $\mathcal{C}_d$ is bounded. Then we show that it is symmetric with respect to the action of Σ_d.

Proposition 3.1. *For each $d \geq 2$, there exists a real number $1 < \alpha < 2$ such that*

$$\mathcal{C}_d = \{c \in \mathbb{C} : \quad |f_c^n(c_0)| \leq (1 + \frac{d+1}{d}\alpha)^{1/d} \quad \text{for every} \quad n \in \mathbb{N}\}.$$

Proof. For $|z| > (1+|c|)^{1/d}$, we have $|f_c(z)| \geq |z|(|z|^d - |c|) > |z|$. It follows that

$$\{z; |z| > (1+|c|)^{1/d}\} \subset A_c(\infty).$$

Now if α is the unique positive root of the polynomial $g(t) = t^{d+1} - (d+1)t - d$, then for $|c| > \frac{d+1}{d}\alpha$, we have

$$|f_c(c_0)| = \frac{d|c|}{d+1}(\frac{|c|}{d+1})^{1/d} > (1+|c|)^{1/d}.$$

Henceforth, $c_0 \in A_c(\infty)$. □

Corollary 3.2. The Mandelbrot set $\mathcal{C}_d$ is a compact subset of the disk $\{c : |c| \leq \frac{d+1}{d}\alpha\}$.

Proposition 3.3. *The Mandelbrot set $\mathcal{C}_d$ is invariant under the action of the group Σ_d.*

Proof. Let $c \in \mathcal{C}_d, \omega = e^{2\pi i/d}$ and let c_0 be a critical point of f_c. Then $\omega c \in \mathcal{C}_d$. Indeed, $\omega^{1/d} c_0$ is a critical point of $f_{\omega c}$, the corresponding critical values being related by

$$f_{\omega c}(\omega^{1/d} c_0) = \omega^{1/d} c_0(\omega c_0^d + \omega c) = \omega^{\frac{1}{d}+1} c_0(c_0^d + c) = \omega^{\frac{1}{d}+1} f_c(c_0).$$

Hence $|f_{\omega c}(\omega^{1/d} c_0)| = |f_c(c_0)|$. In view of the Proposition 3.1, the value α depends only on the degrees of the polynomials f_c and $f_{\omega c}$ which are of the same degree. This completes the proof. □

Proposition 3.4. *The Mandelbrot set $\mathcal{C}_d$ has the following properties:*

(a) *$\mathcal{C}_d$ contains the unit closed disk $\{c : |c| \leq 1\}$.*

(b) *The components of the interior of $\mathcal{C}_d$ are simply connected domains. In other words, $\mathcal{C}_d$ is full.*

(c) *The open set $\mathbb{C} - \mathcal{C}_d$ is connected.*

Proof. (a) It is enough to consider the behavior of one of the critical points in (1). For $|c| < 1$, $z_0 = 0$ is an attractive fixed point and its basin domain contains at least one critical point. Similarly for $|c| = 1$, there is a critical point in the basin domain or on the Julia set. Therefore, no critical point belongs to the unbounded Fatou domain.

(b) Assume that one of the components of the $\text{int}(\mathcal{C}_d)$, say D, is not

simply connected. Then D has a bounded complementary component X. Since the boundary ∂X is a subset of $\mathcal{C}_d$, by Proposition 3.1, the sequence $\{|f_c^n(c_0)|\}_{n\in\mathbb{N}}$ is bounded by $(1+\frac{d+1}{d}\alpha)^{1/d}$. Now we can apply the Maximum Principle to the bounded domain X. The sequence $\{|f_c^n(c_0)|\}_{n\in\mathbb{N}}$ is therefore bounded by $(1+\frac{d+1}{d}\alpha)^{1/d}$ for any $c \in X$. From the Proposition 3.1, we have $X \subset \mathcal{C}_d$, which is a contradiction.
(c) If the assertion is false, then $\mathbb{C} - \mathcal{C}_d$ has some bounded component X, to which we can apply the same reasoning as in (b) and get a contradiction. □

Remark 3.5. If the integer d converges to infinity, the value of α converges to 1, and the Mandelbrot set $\mathcal{C}_d$ converges to the unit disk.

4. **Connectedness**

As we know, the Mandelbrot set $\mathcal{M}_d$ is full, compact and connected. We have proved in §3 that the Mandelbrot set $\mathcal{C}_d$ is also compact and full. Now we are going to prove that it is connected. For this purpose, we need slight modifications of some well known results and machinaries.

If we apply the Böttcher theorem ([2, 12]) to the symmetric polynomial f_c, for a sufficiently large positive number R_c, we get a *Böttcher function* $\phi_c(z)$ on $D_c = \{z \in \mathbb{C}; |z| \geq R_c\}$, such that $\phi_c(f_c(z)) = (\phi_c(z))^{d+1}$.

Proposition 4.1. *For the symmetric polynomial* $f_c(z) = z(z^d + d)$ *the Böttcher function* ϕ_c *satisfies*

$$\phi_c(f_c(\omega z)) = \phi_c(f_c(z)),$$

where $\omega = e^{\frac{2\pi i}{d}}$.

Proof. We have seen that $f_c(\omega z) = \omega f_c(z)$. Therefore,

$$\phi_c(f_c(\omega z)) = \lim_{n\to\infty} \sqrt[(d+1)^n]{f_c^n(\omega z)} = \lim_{n\to\infty} \sqrt[(d+1)^n]{\omega f_c^n(z)}.$$

It follows that $\phi_c(f_c(\omega z)) = \phi_c(f_c(z))$. □

Let us recall (see [2, 12]) that in general a Böttcher function $\phi_c(z)$ cannot be continued analytically to the whole $A_c(\infty)$. However, since $A_c(\infty) = \bigcup_{n=1}^{\infty}(f_c^n)^{-1}(D_c)$, we can extend the harmonic function $G_c(z) = \log|\phi_c(z)|$ to the whole $A_c(\infty)$ by setting

$$G_c(z) = \frac{1}{(d+1)^{n+1}} G_c(f_c^n(z)) \quad \text{if} \quad z \in (f_c^n)^{-1}(D_c).$$

This extension is well defined and $G_c(z)$ is clearly harmonic on $A_c(\infty)$. Moreover, we set $G_c(z) = 0$ for every $z \in K_c$. Then G_c is a continuous subharmonic function on $\mathbb{C}$, which is called *the Green function* associated to $K(f_c)$.

Proposition 4.2. (Sibony's Theorem) *For every* $A \geq 0$ *there exists* $\alpha = \alpha(A) \geq 0$ *such that the restriction of the Green function* G_c *to the disk* $|c| \leq A$ *is* $\alpha-$*Hölder.*

Proof. We repeat the arguments given in the proof of Theorem 3.2, page 138 of [2] by making necessary modifications. Assume that $A \geq 10$. Let $z \in \mathbb{C} - K(f_c)$ and let $\delta(z) = \text{dist}(z, K(f_c))$. Let us take the closest point $z_0 \in K_c$ to z, and let $S = \{z_0 + t(z - z_0) : \ 0 \leq t \leq 1\}$. Take $N = N(z)$ to satisfy $|f_c^n(w)| < A$ for all $w \in S$ and all $n < N$, while $|f_c^N(z_1)| \geq A$ for some $z_1 \in S$. Using $f_c'(z) = (d+1)z^d + c$ and the chain rule, we see that $|(f_c^n)'(w)| \leq ((d+1)A^d + A)^n$ for all $n < N$ and $w \in S$. Also, $|f_c^n(z_0)| < 2\sqrt[d]{A}$ for all n, or else, the iterates

$$|f_c^{n+1}(z_0)| \geq |f_c^n(z_0)|(|f_c^n(z_0)|^d - |c|) \geq |f_c^n(z_0)|(2^d - 1)A > |f_c^n(z_0)|$$

would escape to ∞. The mean value theorem then implies that

$$|f_c^N(z_1) - f_c^N(z_0)| \leq ((d+1)A^d + A)^N \delta(z_1).$$

Thus we have

$$|f_c^N(z_1)| \leq 2\sqrt[d]{A} + ((d+1)A^d + A)^N \delta(z_1).$$

But $|f_c^N(z_1)| \geq A$, hence $((d+1)A^d + A)^N \delta(z_1) \geq 1$, and $((d+1)A^d + A)^N \delta(z) \geq 1$. For $\alpha = \frac{\log(d+1)}{\log((d+1)A^d+A)}$ we have $\delta(z)^\alpha > (d+1)^{-N}$, so that

$$G_c(z) = G_c(f_c^N(z))(d+1)^{-N} \leq M\delta(z)^\alpha,$$

where M depends only on A. Consider two points z_1, z_2 and suppose $\delta(z_1) \geq \delta(z_2)$. We would like to prove

$$|G_c(z_1) - G_c(z_2)| \leq C|z_1 - z_2|^\alpha.$$

If $|z_1 - z_2| > \frac{1}{2}\delta(z_1)$, this follows from the above estimate. If $|z_1 - z_2| \leq \frac{1}{2}\delta(z_1)$, we use the Harnack's inequality for the positive harmonic function $G_c(z)$ in the disk $D(z_1, \delta(z_1))$ to conclude

$$|G_c(z_1) - G_c(z_2)| \leq C_0 M\delta(z_1)^\alpha \frac{|z_1 - z_2|}{\delta(z_1)} \leq C|z_1 - z_2|^\alpha.$$

□

Proposition 4.3. *If $c_n \to c$, then the corresponding Green functions G_{c_n} converge uniformly on $\mathbb{C}$ to G_c. Thus $G_c(z)$ is jointly continuous in c and z.*

Proof. Proposition 4.2 guarantees the equicontinuity of the sequence $G_{c_n}(z)$ and the arguments of [2], page 139, can be reproduced. □

Finally we have:

Theorem 4.4. *The Mandelbrot set $\mathcal{C}_d$ is connected.*

Proof. As $\hat{\mathbb{C}} - \overline{D}$ is simply connected, it is enough to find a homeomorphism from $\hat{\mathbb{C}} - \mathcal{C}_d$ onto $\hat{\mathbb{C}} - \overline{D}$. We will indeed find a conformal map

$$\Phi : \mathbb{C} - \mathcal{C}_d \to \mathbb{C} - \overline{D}.$$

Let $\Omega = \{(z, c) \in \mathbb{C} \times \mathbb{C}; c \in \mathbb{C} - \mathcal{C}_d, G_c(z) > G_c(c_0)\}$. For every $n \geq 0$ the pre-critical points, $(f_c^n)^{-1}(c_0)$, and the pre-images of the origin, $(f_c^n)^{-1}(0)$, do not belong to Ω. Hence the Böttcher function $\phi(z, c) := \phi_c(z)$ is well defined and analytic on Ω. It has the following representation

$$\phi(z, c) = \lim_{n\to\infty} (f_c^n(z))^{\frac{1}{(d+1)^n}} = z \prod_{n=1}^{\infty} \frac{(f_c^n(z))^{\frac{1}{(d+1)^n}}}{(f_c^{n-1}(z))^{\frac{1}{(d+1)^{n-1}}}} =$$

$$z \prod_{n=1}^{\infty} \Big(\frac{(f_c^{n-1}(z)((f_c^{n-1}(z))^d + c))}{(f_c^{n-1}(z))^{d+1}}\Big)^{\frac{1}{(d+1)^n}} = z \prod_{n=0}^{\infty} \Big(1 + \frac{c}{f_c^n(z)}\Big)^{\frac{1}{(d+1)^{n+1}}}.$$

We obviously have $(f_c(c_0), c) \in \Omega$. Now we define

$$\Phi : \mathbb{C} - \mathcal{C}_d \to \mathbb{C} - \overline{D},$$

$$\Phi(c) = \phi(f_c(c_0), c),$$

and prove that the function Φ is well defined, onto and conformal. By Proposition 4.1, $\phi_c(f_c(\omega c_0)) = \phi_c(f_c(c_0))$. Therefore, $\phi(z, c)$ does not depend on the choice of c_0 among the critical points, and Φ is well defined.

From the representation

$$\phi(z, c) = z \prod_{n=0}^{\infty} (1 + \frac{c}{f_c^n(z)})^{\frac{1}{(d+1)^{n+1}}},$$

the Böttcher function $\phi(z, c)$ is an analytic function of two variables in Ω. It follows that Φ is analytic on $\mathbb{C} - \mathcal{C}_d$. Furthermore, for $c \in \mathbb{C} - \mathcal{C}_d$ we have

$$\log |\Phi(c)| = \log |\phi_c(f_c(c_0))| = G_c(f_c(c_0)) = (d + 1)G_c(c_0) > 0,$$

and $|\Phi(c)| > 1$. On the other hand, from Proposition 4.3, the Green function G_c is continuous. Since $G_c|_{\mathcal{C}_d} = 0$, we conclude that $G_c(f_c(c_0)) \to 0$ as c tends to the boundary of $\mathbb{C} - \mathcal{C}_d$. Consequently, $|\Phi(c)| \to 1$ as c tends to the boundary of $\mathbb{C} - \mathcal{C}_d$. Near infinity we have $\phi_c(f_c(c_0)) = (\phi_c(c_0))^{d+1} \sim \frac{-c}{d+1}$, which implies that Φ is injective near ∞, has a simple pole at ∞, and has no zero in $\mathbb{C} - \mathcal{C}_d$. The argument principle can apply: Φ assumes each value in $\mathbb{C} - \overline{D}$ exactly once on $\mathbb{C} - \mathcal{C}_d$. Hence Φ maps $\hat{\mathbb{C}} - \mathcal{C}_d$ conformally onto $\hat{\mathbb{C}} - \overline{D}$. In particular, $\hat{\mathbb{C}} - \mathcal{C}_d$ is simply connected, which implies the assertion. □

Acknowledgment

The original version of this paper has been subject to substantial improvements after the comments of the anonymous referee.

The authors would like to thank her/him for invaluable suggestions. This work has been supported by the University of Teheran while the authors were visiting Purdue University and University of Warwick, respectively.

References

[1] A. F. Beardon, *Iteration of Rational Functions*, Grad. Texts Math. **132**, Springer-Verlag, New York, 1991.

[2] L. Carleson, and T. W. Gamelin, *Complex Dynamics*, Springer-Verlag, New York, 1993.

[3] A. Chademan, Bounded algebraic critical values, *J. Sci. Univ. of Tehran International Edition* **1** (1996), 43-50.

[4] A. Chademan, and A. Zireh, Extension of the Lyubich theorem on Lebesgue measure of the Julia set to symmetric polynomials, *submitted*, 2005.

[5] A. Douady, and J. Hubbard, Itération des polynômes quadratiques, *C. R. Acad. Sci. Paris* **294** (1982), 123-126.

[6] J. H. Hubarrd, *Local connectivity of Julia sets and bifurcation loci: three theorems of J. C. Yoccoz*, 467-511 of "Topological Methods in Modern Mathematics", ed. Goldberg and Phillips, Publish or Perish, 1993.

[7] J. Kahn, and M. Lyubich, *Local connectivity of Julia sets for unicritical polynomial. IMS at Stony Brook Preprint 2005/03*, April 2005 (arXiv:math.DS/0505194v1).

[8] M. Yu. Lyubich, *On the Lebesgue measure of the Julia set of a quadratic polynomial. IMS at Stony Brook Preprint 1991/10*, 1991.

[9] J. Milnor, *Dynamics in one complex variable*, American Mathematical Society, 2000.

[10] J. Milnor, *Geometry and dynamics of quadratic rational maps.* With an appendix by the author and Tan Lei, *Experiment. Math.* **2** (1993) **1**, 37-83.

[11] D. Schleicher, On Fibers and Local Connectivity of Mandelbrot and Multibrot Sets, *Preprint IMS at Stony Brook 1998/13a.*

[12] N. Steinmetz, *Rational Iteration. Complex Analytic Dynamical System,* de Gruyter Studies in Mathematics, **16**. Walter de Gruyter Co., Berlin, 1993.

[13] A. Zireh, Extension of Yoccoz theorem on local connectivity of Julia sets to symmetric polynomials, to appear in *Indian Journal of Pure and Applied Mathematics.*

Arsalan Chademan
Ahmad Zireh
Department of Mathematics
Statistics and Computer Science
University of Tehran
P.O. Box 14155/6455
Tehran, Iran.
e-mail:chademan@khayam.ut.ac.ir
e-mail:zireh@khayam.ut.ac.ir