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EXISTENCE OF POSITIVE SOLUTIONS FOR A SECOND-ORDER p-LAPLACIAN IMPULSIVE BOUNDARY VALUE PROBLEM ON TIME SCALES

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Abstract. In this paper, we investigate the existence of positive solutions for a second-order multipoint p-Laplacian impulsive boundary value problem on time scales. Using a new fixed point theorem in a cone, sufficient conditions for the existence of at least three positive solutions are established. An illustrative example is also presented.
Keywords: Impulsive boundary value problems, p-Laplacian, positive solutions, fixed point theorem, time scales.
MSC(2010): Primary: 34B18; Secondary: 34B37, 34N05.

1. Introduction

Impulsive differential equations describe processes which experience a sudden change of state at certain moments. The theory of impulsive differential equations has undergone rapid development in recent years. The reason for this is the associated theory is richer than the corresponding theory of classical differential equations and impulsive differential equations are regarded as important mathematical tools for the better understanding of several real-world problems in applied sciences, such as population dynamics, ecology, biological systems, biotechnology and optimal control. For the general theory of impulsive differential equations, we refer the reader to the books [1, 12, 18].

This theory of dynamic equations on time scales was introduced in 1988 by Stefan Hilger in his Ph.D. thesis (see [9,10]). The theory unifies existing results in differential and finite difference equations and provides powerful new tools for exploring connections between the traditionally separated fields. We refer to the books by Bohner and Peterson [2, 3] and Lakshmikantham et al. [13].

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There are recently many studies focused on the boundary value problems (BVPs) for impulsive differential equations on time scales. For instance, see [6–8, 15, 20, 22]. However, the corresponding theory of such equations is still in the beginning stages of its development, especially the ones with p-Laplacian [4, 5, 11, 21]. There is not much work on m-point boundary value problems for the p-Laplacian impulsive dynamic equations on time scales, see [14, 16]. In particular, we would like to mention some results of Tian, Chen and Ge [19] and Ozen, Karaca and Tokmak [16].

In [19], Tian et al. studied the multiplicity of positive solutions to the multipoint one-dimensional p-Laplacian BVP with impulsive effects

$$\begin{cases} (\varphi_p(u'(t)))' + q(t)f(t, u(t), u'(t)) = 0, \ t \neq t_i, \ 0 < t < 1, \\ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, n, \\ \Delta \varphi_p(u'(t_i)) = -\bar{I}_i(u(t_i), u'(t_i)), \quad i = 1, 2, \ldots, n, \\ u(0)) = \displaystyle\sum_{j=1}^{m-2} \alpha_j u(\xi_j), \quad \varphi_p(u'(1)) = \displaystyle\sum_{j=1}^{m-2} \beta_j \varphi_p(u'(\eta_j)). \end{cases}$$

Applying the fixed point theorem due to Bai and Ge, they get the sufficient conditions for the existence of multiple positive solutions to the problem above.

In [16], Ozen et al. studied the following multipoint BVPs for p-Laplacian impulsive dynamic equation on time scales

$$\begin{cases} (\phi_p(u^{\Delta}(t)))^{\nabla} + q(t)f(t, u(t), u^{\Delta}(t)) = 0, \ t \in [0,1]_{\mathbb{T}}, \ t \neq t_k, \\ \Delta u(t_k) = -I_k(u(t_k)), \ k = 1, 2, \ldots, n, \\ \Delta \phi_p(u^{\Delta}(t_k)) = -\bar{I}_k(u(t_k), u^{\Delta}(t_k)), \ k = 1, 2, \ldots, n, \\ \phi_p(u^{\Delta}(0)) = \displaystyle\sum_{j=1}^{m-2} \alpha_j \phi_p(u^{\Delta}(\xi_j)), \quad u(1) = 2\displaystyle\sum_{j=1}^{m-2} \beta_j u(\eta_j). \end{cases}$$

Using the Bai-Ge's fixed point theorem, they obtained the existence of at least three positive solutions for the above problem.

Motivated by the above mentioned works, in this paper we consider the existence of positive solutions of the following second order multipoint p-Laplacian impulsive BVP on time scales

$$\begin{cases} (\phi_p(u^{\Delta}(t)))^{\nabla} + q(t)f(t, u(t)) = 0, \ t \in [0,1]_{\mathbb{T}}, \quad t \neq t_k, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1, 2, \ldots, n, \\ \Delta \phi_p(u^{\Delta}(t_k)) = -\bar{I}_k(u(t_k)), \quad k = 1, 2, \ldots, n, \\ u(0) = \displaystyle\sum_{j=1}^{m-2} \alpha_j \phi_p(u^{\Delta}(\xi_j)) + \displaystyle\sum_{j=1}^{m-2} \theta_j u(\zeta_j), \\ \phi_p(u^{\Delta}(1)) = \displaystyle\sum_{j=1}^{m-2} \beta_j \phi_p(u^{\Delta}(\eta_j)), \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $0, 1 \in \mathbb{T}$, $[0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}$, $t_k \in (0,1)_{\mathbb{T}}$, $k = 1, 2, \ldots, n$ with $0 < t_1 < t_2 < \cdots < t_n < 1$, ξ_j, ζ_j, $\eta_j \in (0,1)_{\mathbb{T}}$, $(j = 1, 2, \ldots, m-2)$ with

$0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ 0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{m-2} < 1,\ 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ and $\xi_j,\ \zeta_j,\ \eta_j \neq t_k,\ j = 1, 2, \ldots, m-2,\ k = 1, 2, \ldots, n$. $\phi_p(s)$ is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, and $\Delta \phi_p(u^\Delta(t_k)) = \phi_p(u^\Delta(t_k^+)) - \phi_p(u^\Delta(t_k^-))$ where $u(t_k^+)$, $u^\Delta(t_k^+)$ and $u(t_k^-)$, $u^\Delta(t_k^-)$ represent the right and the left limits of the functions $u(t)$ and $u^\Delta(t)$ at $t = t_k,\ k = 1, 2, \ldots, n$, respectively.

In this paper we assume that

(C1) $f \in \mathcal{C}([0,1]_{\mathbb{T}} \times \mathbb{R}^+, \mathbb{R}^+)$, $q \in \mathcal{C}([0,1]_{\mathbb{T}}, \mathbb{R}^+)$,

(C2) $\alpha_j \in [0,\infty),\ \theta_j \in [0,\infty),\ \beta_j \in [0,\infty), j = 1, 2, \ldots, m-2$ with $0 < \displaystyle\sum_{j=1}^{m-2} \alpha_j < 1$, $0 < \displaystyle\sum_{j=1}^{m-2} \beta_j < 1$ and $0 < \displaystyle\sum_{j=1}^{m-2} \theta_j < 1$,

(C3) $I_k \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$, $\bar{I}_k \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $k = 1, 2, \ldots, n$.

In this study, utilizing a new fixed point theorem due to Ren et al. [17], we get the existence of at least three positive solutions for the impulsive BVP (1.1). In fact, our result is also new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Therefore, the result can be considered as a contribution to this field.

This paper is organized as follows. In Section 2, we give some preliminary lemmas which are key tools for our proof. The main result is given in Section 3. Finally, in Section 4, we give an example to demonstrate our main result.

2. **Preliminaries**

In this section, we give some lemmas which are useful for our main results.

Throughout the rest of this paper, we assume that the points of impulse t_k are right dense for each $k = 1, 2, \ldots, n$. Let $J = [0,1]_{\mathbb{T}}$, $J_0 = [0, t_1]_{\mathbb{T}}$, $J_1 = (t_1, t_2]_{\mathbb{T}}, \ldots, J_{n-1} = (t_{n-1}, t_n]_{\mathbb{T}}$, $J_n = (t_n, 1]_{\mathbb{T}}$, $J' = J \setminus \{t_1, t_2, \ldots, t_n\}$.

Set

$$\begin{aligned} PC(J) &= \Big\{u : [0,1]_{\mathbb{T}} \to \mathbb{R};\ u \in C(J'),\ u(t_k^+) \text{ and } u(t_k^-) \text{ exist, and} \\ &\qquad u(t_k^-) = u(t_k),\ 1 \le k \le n\Big\}, \end{aligned}$$

$$\begin{aligned} PC^1(J) &= \Big\{u \in PC(J) :\ u^\Delta \in C(J'),\ u^\Delta(t_k^+) \text{ and } u^\Delta(t_k^-) \text{ exist, and} \\ &\qquad u^\Delta(t_k^-) = u^\Delta(t_k),\ 1 \le k \le n\Big\}. \end{aligned}$$

Obviously, $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|u\|_{PC} = \max_{t \in [0,1]_{\mathbb{T}}} |u(t)|, \quad \|u\|_{PC^1} = \max\left\{\|u\|_{PC}, \|u^\Delta\|_{PC}\right\},$$

respectively.

Lemma 2.1. *Assume that* (C1)-(C3) *hold. Then* $u \in PC^1(J) \cap C^2(J')$ *is a solution to problem* (1.1) *if and only if* $u \in PC^1(J)$ *is a solution to the integral equation:*

$$
\begin{aligned}
u(t) = {} & \int_0^t \phi_q \left(\int_s^1 q(\tau) f(\tau, u(\tau)) \nabla\tau + \sum_{s<t_k<1} \bar{I}_k(u(t_k)) + A \right) \Delta s \\
& + \sum_{0<t_k<t} I_k(u(t_k)) + \frac{\sum_{j=1}^{m-2} \alpha_j}{1 - \sum_{j=1}^{m-2} \theta_j} \left(\int_{\xi_j}^1 q(s) f(s, u(s)) \nabla s + \sum_{\xi_j<t_k<1} \bar{I}_k(u(t_k)) + A \right) \\
& + \frac{\sum_{j=1}^{m-2} \theta_j}{1 - \sum_{j=1}^{m-2} \theta_j} \Bigg(\sum_{0<t_k<\zeta_j} I_k(u(t_k)) + \int_0^{\zeta_j} \phi_q \Bigg(\int_s^1 q(\tau) f(\tau, u(\tau)) \nabla\tau \\
& + \sum_{s<t_k<1} \bar{I}_k(u(t_k)) + A \Bigg) \Delta s \Bigg),
\end{aligned}
\tag{2.1}
$$

where

$$
A = \frac{\sum_{j=1}^{m-2} \beta_j}{1 - \sum_{j=1}^{m-2} \beta_j} \left(\int_{\eta_j}^1 q(s) f(s, u(s)) \nabla s + \sum_{\eta_j<t_k<1} \bar{I}_k(u(t_k)) \right).
$$

Proof. First, suppose that $u \in PC^1(J) \cap C^2(J')$ is a solution to problem (1.1). Then

$$
(\phi_p(u^\Delta(t)))^\nabla + q(t) f(t, u(t)) = 0, \ t \neq t_k, \ k = 1, 2, \ldots, n.
$$

So,

$$
\phi_p(u^\Delta(t_n^-)) - \phi_p(u^\Delta(t)) = -\int_t^{t_n} q(s) f(s, u(s)) \nabla s,
$$

$$
\phi_p(u^\Delta(1)) - \phi_p(u^\Delta(t_n^+)) = -\int_{t_n}^1 q(s) f(s, u(s)) \nabla s, \ t \in J_{n-1}.
$$

Thus,

$$
\phi_p(u^\Delta(t)) = \phi_p(u^\Delta(1)) + \int_t^1 q(s) f(s, u(s)) \nabla s + \bar{I}_n(u(t_n)), \ t \in J_{n-1}.
$$

Repeating the above process, for $t \in [0,1]_{\mathbb{T}}$ we have

$$\phi_p(u^{\Delta}(t)) = \phi_p(u^{\Delta}(1)) + \int_t^1 q(s)f(s,u(s))\nabla s + \sum_{t<t_k<1} \bar{I}_k(u(t_k)), \tag{2.2}$$

and taking $t = \eta_j$ in (2.2), we obtain

$$\phi_p(u^{\Delta}(\eta_j)) = \phi_p(u^{\Delta}(1)) + \int_{\eta_j}^1 q(s)f(s,u(s))\nabla s + \sum_{\eta_j<t_k<1} \bar{I}_k(u(t_k)).$$

So, we get

$$\begin{aligned}\sum_{j=1}^{m-2} \beta_j \phi_p(u^{\Delta}(\eta_j)) \quad = \quad & \sum_{j=1}^{m-2} \beta_j \phi_p(u^{\Delta}(1)) + \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 q(s)f(s,u(s))\nabla s \\ & + \sum_{j=1}^{m-2} \beta_j \sum_{\eta_j<t_k<1} \bar{I}_k(u(t_k)).\end{aligned}$$

Since $\phi_p(u^{\Delta}(1)) = \sum_{j=1}^{m-2} \beta_j \phi_p(u^{\Delta}(\eta_j))$, we have

$$\begin{aligned}\phi_p(u^{\Delta}(1)) &= \frac{\sum_{j=1}^{m-2} \beta_j \left(\int_{\eta_j}^1 q(s)f(s,u(s))\nabla s + \sum_{\eta_j<t_k<1} \bar{I}_k(u(t_k)) \right)}{1 - \sum_{j=1}^{m-2} \beta_j} \\ &= A.\end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we get

$$\phi_p(u^{\Delta}(t)) = \int_t^1 q(s)f(s,u(s))\nabla s + \sum_{t<t_k<1} \bar{I}_k(u(t_k)) + A,$$

which implies that

$$u^{\Delta}(t) = \phi_q \left(\int_t^1 q(s)f(s,u(s))\nabla s + \sum_{t<t_k<1} \bar{I}_k(u(t_k)) + A \right). \tag{2.4}$$

On the other hand, note that

$$\begin{aligned} u(t_1^-) - u(0) &= \int_0^{t_1} u^{\Delta}(s)\Delta s, \\ u(t) - u(t_1^+) &= \int_{t_1}^t u^{\Delta}(s)\Delta s, \ t \in J_1. \end{aligned}$$

So that we have

$$u(t) = u(0) + \int_0^t u^{\Delta}(s)\Delta s + I_1(u(t_1)),\ t \in J_1.$$

Repeating the above process for $t \in [0,1]_{\mathbb{T}}$, one can verify that

$$u(t) = u(0) + \int_0^t u^{\Delta}(s)\Delta s + \sum_{0<t_k<t} I_k(u(t_k)). \tag{2.5}$$

Substituting (2.4) into (2.5), we obtain that

$$\begin{aligned} u(t) &= u(0) + \sum_{0<t_k<t} I_k(u(t_k)) \\ &+ \int_0^t \phi_q\left(\int_s^1 q(\tau)f(\tau,u(\tau))\nabla\tau + \sum_{s<t_k<1} \bar{I}_k(u(t_k)) + A\right)\Delta s, \end{aligned} \tag{2.6}$$

and taking $t = \zeta_j$ in (2.6), we get

$$\begin{aligned} u(\zeta_j) &= u(0) + \sum_{0<t_k<\zeta_j} I_k(u(t_k)) \\ &\quad + \int_0^{\zeta_j} \phi_q\left(\int_s^1 q(\tau)f(\tau,u(\tau))\nabla\tau + \sum_{s<t_k<1} \bar{I}_k(u(t_k)) + A\right)\Delta s. \end{aligned}$$

So,

$$\begin{aligned} &\sum_{j=1}^{m-2} \theta_j u(\zeta_j) = u(0)\sum_{j=1}^{m-2}\theta_j + \sum_{j=1}^{m-2}\theta_j \sum_{0<t_k<\zeta_j} I_k(u(t_k)) \\ &+ \sum_{j=1}^{m-2}\theta_j \int_0^{\zeta_j} \phi_q\left(\int_s^1 q(\tau)f(\tau,u(\tau))\nabla\tau + \sum_{s<t_k<1} \bar{I}_k(u(t_k)) + A\right)\Delta s. \end{aligned}$$

Since $u(0)=\sum_{j=1}^{m-2}\alpha_j\phi_p(u^\Delta(\xi_j))+\sum_{j=1}^{m-2}\theta_j u(\zeta_j)$,

$$
\begin{aligned}
u(0)=&\frac{\sum_{j=1}^{m-2}\alpha_j}{1-\sum_{j=1}^{m-2}\theta_j}\left(\int_{\xi_j}^{1}q(s)f(s,u(s))\nabla s+\sum_{\xi_j<t_k<1}\bar{I}_k(u(t_k))+A\right)\\
&+\frac{\sum_{j=1}^{m-2}\theta_j}{1-\sum_{j=1}^{m-2}\theta_j}\Bigg(\sum_{0<t_k<\zeta_j}I_k(u(t_k))+\int_0^{\zeta_j}\phi_q\bigg(\int_s^1 q(\tau)f(\tau,u(\tau))\nabla\tau\\
&+\sum_{s<t_k<1}\bar{I}_k(u(t_k))+A\bigg)\Delta s\Bigg).
\end{aligned}
\tag{2.7}
$$

Substituting (2.7) into (2.6), we get (2.1), which completes the proof of sufficiency.

Conversely, if $u(t)\in PC^1(J)$ is a solution to (2.1), apparently

$$\Delta u(t_k)=I_k(u(t_k)),\quad k=1,2,\ldots,n.$$

The Δ-derivative of (2.1) implies that for $t\neq t_k$,

$$u^\Delta(t)=\phi_q\left(\int_t^1 q(s)f(s,u(s))\nabla s+\sum_{t<t_k<1}\bar{I}_k(u(t_k))+A\right).$$

$$(\phi_p(u^\Delta(t)))^\nabla=-q(t)f(t,u(t)).$$

Hence $u\in C^2(J')$, and

$$\Delta\phi_p(u^\Delta(t_k))=-\bar{I}_k(u(t_k)),\ k=1,2,\ldots,n,$$

$$u(0)=\sum_{j=1}^{m-2}\alpha_j\phi_p(u^\Delta(\xi_j))+\sum_{j=1}^{m-2}\theta_j u(\zeta_j),$$

$$\phi_p(u^\Delta(1))=\sum_{j=1}^{m-2}\beta_j\phi_p(u^\Delta(\eta_j)).$$

The proof is completed. □

Define the cone $\mathcal{P} \subset PC(J)$ by

$$\mathcal{P} = \Big\{ u \in PC(J) : u(t) \text{ is nonnegative, nondecreasing on } [0,1]_{\mathbb{T}} \text{ and}$$
$$u^{\Delta}(t) \text{ is nonincreasing on } [0,1]_{\mathbb{T}} \Big\},$$

and define the operator $T : \mathcal{P} \to PC(J)$ by

$$\begin{aligned} Tu(t) = {} & \int_0^t \phi_q\left(\int_s^1 q(\tau) f(\tau, u(\tau))\nabla\tau + \sum_{s<t_k<1} \bar{I}_k(u(t_k)) + A\right)\Delta s \\ & + \sum_{0<t_k<t} I_k(u(t_k)) + \frac{\sum_{j=1}^{m-2}\alpha_j}{1-\sum_{j=1}^{m-2}\theta_j}\Bigg(\int_{\xi_j}^1 q(s) f(s,u(s))\nabla s \\ & + \sum_{\xi_j<t_k<1} \bar{I}_k(u(t_k)) + A\Bigg) + \frac{\sum_{j=1}^{m-2}\theta_j}{1-\sum_{j=1}^{m-2}\theta_j}\Bigg(\sum_{0<t_k<\zeta_j} I_k(u(t_k)) \\ & + \int_0^{\zeta_j}\phi_q\Big(\int_s^1 q(\tau) f(\tau,u(\tau))\nabla\tau + \sum_{s<t_k<1}\bar{I}_k(u(t_k)) + A\Big)\Delta s\Bigg), \end{aligned} \tag{2.8}$$

where

$$A = \frac{\sum_{j=1}^{m-2}\beta_j}{1-\sum_{j=1}^{m-2}\beta_j}\left(\int_{\eta_j}^1 q(s) f(s,u(s))\nabla s + \sum_{\eta_j<t_k<1}\bar{I}_k(u(t_k))\right).$$

Lemma 2.2. *Assume that* (C1)-(C3) *hold. Then $T : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator.*

Proof. From the definition of T, it is clear that $T(\mathcal{P}) \subset \mathcal{P}$. On the other hand, by the conditions (C1)-(C3) and the definition of $Tu(t)$, it is clear that $T : \mathcal{P} \to \mathcal{P}$ is continuous. By Arzela-Ascoli theorem, one can easily prove that operator T is completely continuous. □

3. Main result

The following fixed point theorem is fundamental and important for the proof of our main result.

Lemma 3.1 ([17]). *Let $\mathcal{P}$ be a cone in a real Banach space $\mathbb{B}$. Let α, β and γ be three increasing, nonnegative and continuous functionals on $\mathcal{P}$, satisfying for some $c > 0$ and $M > 0$ such that*

$$\gamma(x) \leq \beta(x) \leq \alpha(x), \ \ \|x\| \leq M\gamma(x)$$

for all $x \in \overline{\mathcal{P}(\gamma, c)}$. Suppose there exists a completely continuous operator $T : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$ and $0 < a < b < c$ such that

(i) $\gamma(Tx) < c$, *for all $x \in \partial\mathcal{P}(\gamma, c)$;*
(ii) $\beta(Tx) > b$, *for all $x \in \partial\mathcal{P}(\beta, b)$;*
(iii) $\mathcal{P}(\alpha, a) \neq \emptyset$, *and $\alpha(Tx) < a$, for all $x \in \partial\mathcal{P}(\alpha, a)$.*

Then T has at least three fixed points, $x_1, \ x_2$ and $x_3 \in \overline{\mathcal{P}(\gamma, c)}$ such that

$$0 \leq \alpha(x_1) < a < \alpha(x_2), \quad \beta(x_2) < b < \beta(x_3), \quad \gamma(x_3) < c.$$

Now we consider the existence of at least three positive solutions for the impulsive boundary value problem (1.1) by the fixed point theorem in [17].

We define the increasing, nonnegative, continuous functionals $\gamma, \ \beta$, and α on $\mathcal{P}$ by

$$\begin{aligned}
\gamma(u) &= \max_{t\in[0,\xi_1]_{\mathbb{T}}} u(t) = u(\xi_1), \\
\beta(u) &= \min_{t\in[\xi_1,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_1), \\
\alpha(u) &= \max_{t\in[0,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_{m-2}).
\end{aligned}$$

It is obvious that for each $u \in \mathcal{P}, \ \gamma(u) = \beta(u) \leq \alpha(u)$. Additionally, for each $u \in \mathcal{P}$, since $u^{\triangle}$ is nonincreasing on $[0, 1]_{\mathbb{T}}$, we have $\gamma(u) = u(\xi_1) \geq \xi_1 u(1)$. Thus, $\|u\| \leq \dfrac{1}{\xi_1}\gamma(u), \ \ \forall u \in \mathcal{P}$.

For convenience, we denote

$$\Omega = \frac{\sum_{j=1}^{m-2} \alpha_j}{1 - \sum_{j=1}^{m-2} \theta_j} \int_{\xi_{m-2}}^{1} q(s)\nabla s,$$

$$B = \frac{1}{1 - \sum_{j=1}^{m-2} \beta_j} \left(n + \int_0^1 q(\tau)\nabla\tau \right),$$

$$\Lambda = \xi_{m-2}\phi_q(B) + \frac{\sum_{j=1}^{m-2} \theta_j}{1 - \sum_{j=1}^{m-2} \theta_j}(n + \phi_q(B)) + \frac{\sum_{j=1}^{m-2} \alpha_j}{1 - \sum_{j=1}^{m-2} \theta_j} B + n.$$

Theorem 3.2. *Suppose the assumptions of* (C1)-(C3) *are satisfied. Let there exist positive numbers $a < b < c$ such that*

$$a < \xi_1 b < \frac{\xi_1 \Omega}{\Lambda} c < c,$$

and assume that f, I_k and $\bar{I}_k$ satisfy the following conditions:

(C4) $f(t,u) < \min\left\{\frac{c}{\Lambda},\ \phi_p\left(\frac{c}{\Lambda}\right)\right\},\ I_k(u(t_k)) \le \frac{c}{\Lambda},\ \bar{I}_k(u(t_k)) \le \min\left\{\frac{c}{\Lambda},\ \phi_p\left(\frac{c}{\Lambda}\right)\right\}$ *for all* $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{c}{\xi_1}\right],\ k = 1,2,\ldots,n,$

(C5) $f(t,u) > \phi_p\left(\frac{b}{\Omega}\right)$, *for all* $(t,u) \in [\xi_1, 1]_{\mathbb{T}} \times \left[b, \frac{b}{\xi_1}\right],$

(C6) $f(t,u) < \min\left\{\frac{a}{\Lambda},\ \phi_p\left(\frac{a}{\Lambda}\right)\right\},\ I_k(u(t_k)) \le \frac{a}{\Lambda},\ \bar{I}_k(u(t_k)) \le \min\left\{\frac{a}{\Lambda},\ \phi_p\left(\frac{a}{\Lambda}\right)\right\}$ *for all* $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{a}{\xi_1}\right],\ k = 1,2,\ldots,n.$

Then the boundary value problem (1.1) *has at least three positive solutions u_1, u_2 and u_3 which belong to $\overline{\mathcal{P}(\gamma, c)}$ such that*

$$0 \le \alpha(u_1) < a < \alpha(u_2),\ \beta(u_2) < b < \beta(u_3),\ \gamma(u_3) < c.$$

Proof. We define the completely continuous operator T by (2.8). So, it is easy to check that $T : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$.

We now show that all conditions of Lemma 3.1 are satisfied. In order to verify condition (i) of Lemma 3.1, we choose $u \in \partial\mathcal{P}(\gamma, c)$. Then $\gamma(u) = \max_{t \in [0,\xi_1]_{\mathbb{T}}} u(t) = u(\xi_1) = c$, and this implies that $0 \le u(t) \le c$ for $t \in [0, \xi_1]_{\mathbb{T}}$. If we recall that $\|u\| \le \frac{1}{\xi_1}\gamma(u) = \frac{1}{\xi_1}c$, then we have

$$0 \le u(t) \le \frac{c}{\xi_1},\ t \in [0,1]_{\mathbb{T}}.$$

Then assumption (C4) implies for all $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{c}{\xi_1}\right],\ k = 1,2,\ldots,n,$

$$f(t,u) < \min\left\{\frac{c}{\Lambda},\ \phi_p\left(\frac{c}{\Lambda}\right)\right\},\ I_k(u(t_k)) \le \frac{c}{\Lambda},$$
$$\bar{I}_k(u(t_k)) \le \min\left\{\frac{c}{\Lambda},\ \phi_p\left(\frac{c}{\Lambda}\right)\right\}.$$

Therefore,

$$
\begin{aligned}
\gamma(Tu) &= \max_{t\in[0,\xi_1]_{\mathbb{T}}}(Tu)(t) = (Tu)(\xi_1)\\
&< \frac{c}{\Lambda}\Bigg(\int_0^{\xi_{m-2}}\phi_q\Bigg(\int_0^1 q(\tau)\nabla\tau + n + \frac{\left(\int_0^1 q(s)\nabla s + n\right)\sum_{j=1}^{m-2}\beta_j}{1-\sum_{j=1}^{m-2}\beta_j}\Bigg)\Delta s + n\\
&\quad + \frac{\sum_{j=1}^{m-2}\alpha_j}{1-\sum_{j=1}^{m-2}\theta_j}\Bigg(\int_0^1 q(s)\nabla s + n + \frac{\left(\int_0^1 q(s)\nabla s + n\right)\sum_{j=1}^{m-2}\beta_j}{1-\sum_{j=1}^{m-2}\beta_j}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\sum_{j=1}^{m-2}\theta_j}{1-\sum_{j=1}^{m-2}\theta_j}\Bigg(n + \int_0^1\phi_q\Bigg(\int_0^1 q(\tau)\nabla\tau + n + \frac{\left(\int_0^1 q(s)\nabla s + n\right)\sum_{j=1}^{m-2}\beta_j}{1-\sum_{j=1}^{m-2}\beta_j}\Bigg)\Delta s\Bigg)\Bigg)\\
&= \frac{c}{\Lambda}\Bigg(\xi_{m-2}\phi_q(B) + \frac{\sum_{j=1}^{m-2}\theta_j}{1-\sum_{j=1}^{m-2}\theta_j}(n+\phi_q(B)) + \frac{\sum_{j=1}^{m-2}\alpha_j}{1-\sum_{j=1}^{m-2}\theta_j}B + n\Bigg)\\
&= c.
\end{aligned}
$$

Hence, condition (i) is satisfied.

Secondly, we show that (ii) of Lemma 3.1 is satisfied. For that purpose, we take $u \in \partial\mathcal{P}(\beta, b)$. Then,

$$\beta(u) = \min_{t\in[\xi_1,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_1) = b,$$

this means that $u(t) \geq b$, for all $t \in [\xi_1, 1]_{\mathbb{T}}$. Noticing that $\|u\| \leq \dfrac{1}{\xi_1}\gamma(u) \leq \dfrac{1}{\xi_1}\beta(u) = \dfrac{b}{\xi_1}$, we get

$$b \leq u(t) \leq \frac{b}{\xi_1}, \text{ for } t \in [\xi_1, 1]_{\mathbb{T}}.$$

Then, assumption (C5) implies $f(t,u) > \dfrac{b}{\Omega}$. Therefore

$$\begin{aligned}
\beta(Tu) &= \min_{t\in[\xi_1,\xi_{m-2}]_{\mathbb{T}}} (Tu)(t) = (Tu)(\xi_1) \\
&\geq \frac{\sum_{j=1}^{m-2}\alpha_j}{1-\sum_{j=1}^{m-2}\theta_j}\left(\int_{\xi_j}^{1} q(s)f(s,u(s))\nabla s + \sum_{\xi_j<t_k<1}\bar{I}_k(u(t_k)) + A\right) \\
&> \frac{\sum_{j=1}^{m-2}\alpha_j}{1-\sum_{j=1}^{m-2}\theta_j}\left(\int_{\xi_{m-2}}^{1} q(s)\nabla s\right)\frac{b}{\Omega} \\
&= b.
\end{aligned}$$

So, $\beta(Tu) > b$. Hence, condition (ii) is satisfied.

Finally, we show that the condition (iii) of Lemma 3.1 is satisfied. We note that $u(t) = \dfrac{a}{2},\ t \in [0,1]_{\mathbb{T}}$ is a member of $\mathcal{P}(\alpha,a)$, and so $\mathcal{P}(\alpha,a) \neq \emptyset$.

Now, let $u \in \partial\mathcal{P}(\alpha,a)$. Then $\alpha(u) = \max_{t\in[0,\xi_{m-2}]_{\mathbb{T}}} u(t) = u(\xi_{m-2}) = a$. This implies $0 \leq u(t) \leq a,\ t \in [0,\xi_{m-2}]_{\mathbb{T}}$. Noticing that $\|u\| \leq \dfrac{1}{\xi_1}\gamma(u) \leq \dfrac{1}{\xi_1}\alpha(u) = \dfrac{a}{\xi_1}$, we get

$$0 \leq u(t) \leq \frac{a}{\xi_1}, \text{ for } t \in [0,1]_{\mathbb{T}}.$$

By assumption (C6), we have for all $(t,u) \in [0,1]_{\mathbb{T}} \times \left[0, \dfrac{a}{\xi_1}\right]$,

$$\begin{gathered}
f(t,u) < \min\left\{\frac{a}{\Lambda},\ \phi_p\left(\frac{a}{\Lambda}\right)\right\},\ I_k(u(t_k)) \leq \frac{a}{\Lambda}, \\
\bar{I}_k(u(t_k)) \leq \min\left\{\frac{a}{\Lambda},\ \phi_p\left(\frac{a}{\Lambda}\right)\right\},\ k = 1,2,\ldots,n.
\end{gathered}$$

Therefore, we get

$$\begin{aligned}
\alpha(Tu) &= \max_{t\in[0,\xi_{m-2}]_{\mathbb{T}}} (Tu)(t) = (Tu)(\xi_{m-2}) \\
&< \frac{a}{\Lambda}\left(\xi_{m-2}\phi_q(B) + \frac{\sum_{j=1}^{m-2}\theta_j}{1-\sum_{j=1}^{m-2}\theta_j}(n+\phi_q(B)) + \frac{\sum_{j=1}^{m-2}\alpha_j}{1-\sum_{j=1}^{m-2}\theta_j}B + n\right) \\
&= a.
\end{aligned}$$

So, we have $\alpha(Tu) < a$. Thus, (iii) of Lemma 3.1 is satisfied.

Therefore, by Lemma 3.1, the impulsive boundary value problem (1.1) has at least three positive solutions u_1, u_2 and u_3 which belong to $\overline{\mathcal{P}(\gamma, c)}$ such that

$$0 < \alpha(u_1) < a < \alpha(u_2),\ \beta(u_2) < b < \beta(u_3),\ \gamma(u_3) < c.$$

The proof of Teorem 3.2 is complete. □

4. An example

Example 4.1. Let $\mathbb{T} = \left[0, \frac{1}{5}\right] \cup \left[\frac{2}{5}, \frac{4}{5}\right] \cup \{1\}$. Consider the following second-order multipoint p-Laplacian impulsive boundary value problem:

$$(4.1) \qquad \begin{cases} (\phi_3(u^\Delta(t)))^\nabla + tf(t, u(t)) = 0, \quad t \in [0,1]_\mathbb{T},\ t \neq \frac{1}{2}, \\ \Delta u\left(\frac{1}{2}\right) = I_1\left(u\left(\frac{1}{2}\right)\right), \\ \Delta\phi_3\left(u^\Delta\left(\frac{1}{2}\right)\right) = -\bar{I}_1\left(u\left(\frac{1}{2}\right)\right), \\ u(0) = \frac{1}{6}\phi_3\left(u^\Delta\left(\frac{1}{7}\right)\right) + \frac{1}{3}\phi_3\left(u^\Delta\left(\frac{2}{7}\right)\right) + \frac{1}{5}u\left(\frac{3}{5}\right) + \frac{2}{5}u\left(\frac{7}{10}\right), \\ \phi_3(u^\Delta(1)) = \frac{1}{4}\phi_3\left(u^\Delta\left(\frac{1}{9}\right)\right) + \frac{1}{8}\phi_3\left(u^\Delta\left(\frac{3}{4}\right)\right), \end{cases}$$

where

$$f(t,u) = \begin{cases} 8, & u \in [0, 581], \\ 33u - 19165, & u \in (581, 660], \\ 2615, & u > 660, \end{cases}$$

$$I_1(u) = \frac{u}{100},\ u \geq 0,\quad \bar{I}_1(u) = \frac{3}{400}u,\quad u \geq 0.$$

By simple calculation, we get $\Omega = \frac{11}{32}$, $B = \frac{308}{125}$, $\Lambda \approx 8.383$. Taking $a = 83$, $b = 660$ and $c = 25149$, it is easy to check that

$$a = 83 < \frac{660}{7} = \xi_1 b < \frac{\xi_1\Omega}{\Lambda}c \approx 147.32 < 25149 = c,$$

and the conditions (C1)-(C3) are satisfied. Now, we show that conditions (C4)-(C6) are satisfied:

$$f(t,u) \leq 2615 < \min\left\{\frac{c}{\Lambda},\ \phi_3\left(\frac{c}{\Lambda}\right)\right\} = 3000,$$

$$I_1\left(u\left(\frac{1}{2}\right)\right) \leq 1760.43 < \frac{c}{\Lambda} = 3000,$$

$$\bar{I}_1\left(u\left(\frac{1}{2}\right)\right) \leq 1320.3225 < \min\left\{\frac{c}{\Lambda},\ \phi_3\left(\frac{c}{\Lambda}\right)\right\} = 3000,$$

for $(t,u) \in [0,1]_\mathbb{T} \times [0, 176043]$;

$$f(t,u) = 2615 > \frac{b}{\Omega} = 1920 \text{ for } (t,u) \in \left[\frac{1}{7}, 1\right]_{\mathbb{T}} \times [660, 4620]\,;$$
$$f(t,u) = 8 < \min\left\{\frac{a}{\Lambda},\ \phi_3\left(\frac{a}{\Lambda}\right)\right\} = \frac{1000}{101},\ I_1\left(u\left(\frac{1}{2}\right)\right) \le 5.81 < \frac{a}{\Lambda},$$
$$\bar{I}_1\left(u\left(\frac{1}{2}\right)\right) \le 4.3575 < \min\left\{\frac{a}{\Lambda},\ \phi_3\left(\frac{a}{\Lambda}\right)\right\} = \frac{1000}{101}$$
$$\text{for } (t,u) \in [0,1]_{\mathbb{T}} \times [0, 581]\,.$$

So, all conditions of Theorem 3.2 hold. Thus by Theorem 3.2, the BVP (4.1) has at least three positive solutions $u_1,\ u_2$ and u_3 which belong to $\overline{\mathcal{P}(\gamma, 25149)}$ such that

$$0 \le \max_{t\in\left[0,\frac{2}{7}\right]_{\mathbb{T}}} u_1(t) < 83 < \max_{t\in[0,\frac{2}{7}]_{\mathbb{T}}} u_2(t),$$
$$\min_{t\in[\frac{1}{7},\frac{2}{7}]_{\mathbb{T}}} u_2(t) < 660 < \min_{t\in[\frac{1}{7},\frac{2}{7}]} u_3(t),$$
$$\max_{t\in\left[0,\frac{1}{7}\right]_{\mathbb{T}}} u_3(t) < 25149.$$

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