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WHEN IS THE RING OF REAL MEASURABLE FUNCTIONS A HEREDITARY RING?

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ABSTRACT. Let $M(X, \mathcal{A}, \mu)$ be the ring of real-valued measurable functions on a measure space $(X, \mathcal{A}, \mu)$. In this paper, we characterize the maximal ideals in the rings of real measurable functions and as a consequence, we determine when $M(X, \mathcal{A}, \mu)$ is a hereditary ring.
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MSC(2010): Primary: 13C10; Secondary: 28A99.

1. Introduction

A collection $\mathcal{A}$ of subsets of a set X is said to be a $\sigma - algebra$ in X if $\mathcal{A}$ has the following properties:

(a) $\emptyset \in \mathcal{A}$,
(b) if $B \in \mathcal{A}$, then its complement B^c is also in $\mathcal{A}$,
(c) if $B_1, B_2, \ldots$ is a countable collection of sets in $\mathcal{A}$, then $\bigcup_{i=1}^{\infty} B_i \in \mathcal{A}$.

If $\mathcal{A}$ is a σ-algebra in X, then $(X, \mathcal{A})$ is called a measurable space and the members of $\mathcal{A}$ are called the measurable sets in X.

A *positive measure* is a function μ, defined on a σ-algebra $\mathcal{A}$, whose range is in $[0, \infty]$ and which is countably additive. This means that if $\{A_i\}_{i=1}^{\infty}$ is a disjoint countable collection of members of $\mathcal{A}$, then $\mu(\sum_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathcal{A}$. Henceforth, all measures in this paper are positive.

A *measure space* is a triple $(X, \mathcal{A}, \mu)$, where X is a set, $\mathcal{A}$ a σ-algebra in X, and μ a (positive) measure on $\mathcal{A}$. If Y is a topological space and $f : X \longrightarrow Y$ is a function, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y. For notational convenience, we assume that $M(X, \mathcal{A}, \mu)$ is the space of measurable functions from X to $\mathbb{R}$ with arbitrary

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σ-algebra $\mathcal{A}$ in X and arbitrary measure μ on $\mathcal{A}$. The space $M(X, \mathcal{A}, \mu)$ with pointwise addition and multiplication is a ring. It is easy to check that the rings of real valued measurable functions are commutative with identity. For more information about this ring, see for example [1, 10, 15].

The statement "P holds almost everywhere on $(X, \mathcal{A}, \mu)$" (abbreviated to "P holds a.e. on $(X, \mathcal{A}, \mu)$") means that

$$\mu(\{x \in X : \text{P does not hold on } x\}) = 0.$$

A ring R is called *hereditary* if every ideal in R is projective. Equivalently, a ring R is *hereditary* if all submodules of projective modules over R are again projective, see [16, Theorem 4.23].

The following standard lemma ([16, Lemma 3.15]) is very important in this paper.

Lemma 1.1 (Projective basis lemma)**.** *Let R be a ring. An R module M is projective if and only if there are elements $\{a_\alpha : \alpha \in K\}$ of M and homomorphisms $\varphi_\alpha : M \longrightarrow R$ such that*

(a) *If $x \in M$, $\varphi_\alpha(x) = 0$ for all but finitely many $\alpha \in K$,*
(b) *If $x \in M$, $x = \Sigma_\alpha(\varphi_\alpha(x))a_\alpha$.*

For every $f \in M(X, \mathcal{A}, \mu)$, the zero set and the cozero set of f are $Z(f) := \{x \in X : f(x) = 0\}$ and $\text{coz}(f) := X \setminus Z(f)$, respectively. A covering of a space is called *star-finite* if every element of it intersects at most finite elements of it. The reader is referred to [7, 9, 11, 17] for undefined terms and concepts.

W.V. Vasconcelos in [18] shows that every projective prime ideal in a self-injective commutative ring is generated by an idempotent. Later, O.A.S. Karamzadeh in [13, 14] and [12] extended this result to non-commutative rings by replacing the self-injectivity by right self-injectivity and prime ideal by maximal right ideal. In [6] J.G. Brookshear shows the above result of Vasconcelos is true in C(X) without the assumption of the self-injectivity. In this article, using the method of Brookshear, we prove the counterpart of this result for the ring of real measurable functions and answer to our question that "when is the rings of real measurable functions, $M(X, \mathcal{A}, \mu)$, a hereditary ring?".

2. **Main Results**

To enter the discussion, first we need some features of projective ideals in the rings of real measurable functions. In the following two lemmas, we study the relationships between the projective basis of the projective ideals and the cozero sets of the functions in $M(X, \mathcal{A}, \mu)$.

Lemma 2.1. *Let I be a projective ideal in $M(X, \mathcal{A}, \mu)$ with projective basis $\{f_\alpha, \varphi_\alpha\}_{\alpha \in K}$. Then the following hold:*

(a) *$coz(\varphi_\alpha(f)) \subseteq coz(f)$ for each $\alpha \in K$ and $f \in I$,*
(b) *$\{coz(\varphi_\alpha(f_\alpha))\}_{\alpha \in K}$ is a cover of $\bigcup_{f \in I} coz(f)$,*

(c) $\{coz(\varphi_\alpha(f_\alpha))\}_{\alpha\in K}$ *is a star-finite cover of* $\bigcup_{f\in I} coz(f)$,

(d) $\bigcup_{\alpha\in K} coz(\varphi_\alpha(f_\alpha)) = \bigcup_{f\in I} coz(f)$.

Proof. (a) Let $f \in I$ and $x \notin \mathrm{coz}(f)$. Then $\chi_{Z(f)}$ is a measurable function and $\chi_{Z(f)}f = 0$. For all $\alpha \in K$, $\chi_{Z(f)}\varphi_\alpha(f) = \phi_\alpha(\chi_{Z(f)}f) = 0$ and so $\phi_\alpha(f)(x) = 0$. This implies that $x \notin \mathrm{coz}(\varphi_\alpha(f))$, for all $\alpha \in K$.

(b) Suppose that $f \in I$. By Lemma 1.1, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in K$ such that $f = \Sigma_{i=1}^n \varphi_{\alpha_i}(f) f_{\alpha_i} = \Sigma_{i=1}^n \varphi_{\alpha_i}(f_{\alpha_i}) f$. This means that $\Sigma_{i=1}^n \varphi_{\alpha_i}(f_{\alpha_i}) = 1$ on $\mathrm{coz}(f)$ and so $\{\mathrm{coz}(\varphi_\alpha(f_\alpha))\}_{\alpha\in K}$ is a cover of $\bigcup_{f\in I}\mathrm{coz}(f)$.

(c) Let $\alpha \in K$. Then by Lemma 1.1, there are at most finitely many $\beta \in K$ such that $\phi_\alpha(f_\alpha)\varphi_\beta(f_\beta) = \varphi_\alpha(\varphi_\beta(f_\alpha)f_\beta) \neq 0$ and hence $\{\mathrm{coz}(\varphi_\alpha(f_\alpha))\}_{\alpha\in K}$ is a star-finite cover of $\bigcup_{f\in I}\mathrm{coz}(f)$.

(d) The proof follows easily from the parts (a), (b) and (c). □

In the next lemma, a special projective basis is presented for every projective ideal in the rings of real measurable functions.

Lemma 2.2. *Every projective ideal* I *in* $M(X, \mathcal{A}, \mu)$ *has a projective basis* $\{\varphi_\alpha, f_\alpha\}_{\alpha\in K}$ *such that*

(a) $coz(f_\alpha) = coz(\varphi_\alpha(f_\alpha))$ *for all* $\alpha \in K$,

(b) $g \in I$ *implies* $gf_\alpha = 0$ *for all but finitely many* $\alpha \in K$.

Proof. Suppose that I is a projective ideal and $\{\psi_\alpha, g_\alpha\}_{\alpha\in K}$ is a projective basis for I. For every $h \in I$ and every $\alpha \in K$, we define $f_\alpha := \psi_\alpha(g_\alpha)g_\alpha$ and

$$h_\alpha(x) := \begin{cases} \frac{\psi_\alpha(h)(x)}{\sum_{\beta\in K}(\psi_\beta(g_\beta(x)))^2} & x \in \bigcup_{\beta\in K}\mathrm{coz}(g_\beta), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that for every $\alpha \in K$, f_α and h_α are measurable functions. Hence for every $\alpha \in K$, the module homomorphism $\phi_\alpha : I \longrightarrow M(X, \mathcal{A}, \mu)$ defined by $\phi_\alpha(h) := h_\alpha$ has the property that $\phi_\alpha(h) = 0$ for all but finitely many $\alpha \in K$. If $r \in I$, then

$$\begin{aligned} \Sigma_{\alpha\in K}\phi_\alpha(r)f_\alpha &= \Sigma_{\alpha\in K} r_\alpha \psi_\alpha(g_\alpha)g_\alpha \\ &= \frac{\Sigma_{\alpha\in K}\psi_\alpha(r)\psi_\alpha(g_\alpha)g_\alpha}{\sum_{\beta\in K}(\psi_\beta(g_\beta))^2} \\ &= r\frac{\sum_{\alpha\in K}(\psi_\alpha(g_\alpha))^2}{\sum_{\alpha\in K}(\psi_\beta(g_\beta))^2} \\ &= r. \end{aligned}$$

Therefore $\{\phi_\alpha, f_\alpha\}_{\alpha\in K}$ is a projective basis for I.

Let $\alpha \in K$ and $x \in \mathrm{coz}(f_\alpha)$. Then $f_\alpha(x) = \psi_\alpha(g_\alpha(x))g_\alpha(x) \neq 0$. This means that $\psi_\alpha(g_\alpha(x)) \neq 0$ and so $\varphi_\alpha(f_\alpha(x)) \neq 0$. Now by Lemma 2.1(a), $\{\phi_\alpha, f_\alpha\}_{\alpha\in K}$ satisfies condition (a). If $q \in I$, then $q = \Sigma_{\alpha\in K}(\varphi_\alpha(q))f_\alpha = \Sigma_{\alpha\in K}(\varphi_\alpha(f_\alpha))q$.

Thus $\{f_\alpha\}_{\alpha\in K}$ generates I. On the other hand $\{f_\alpha\}_{\alpha\in K}$ is star finite, by Lemma 2.1. This implies condition (b). □

In the following theorem, we characterize the projective prime ideals in the rings of real measurable functions. In this theorem, without invoking the fact that the ring of real measurable functions is a regular ring, we have proved that every projective prime ideal in the ring of real measurable functions is a maximal ideal generated by an idempotent. This theorem is a counterpart of similar results in [6] and [3, Proposition 1.6]. First we need the next lemma:

Lemma 2.3. *Let f be a measurable function. Then the following function is measurable:*

$$g(x) := \begin{cases} 1/f & x \in \operatorname{coz}(f), \\ 0 & x \in Z(f). \end{cases}$$

Proof. For every $x \in \operatorname{coz}(f)$, We define $\lambda(x) := (1, f(x))$. Then λ maps $\operatorname{coz}(f)$ into the plane. If U is an open set in the plane, then U is a countable union of such rectangles $I_i \times J_i$, $i \in \mathbb{N}$. For every open set W in $\mathbb{R}$, we put:

$$\tau(W) := \begin{cases} \operatorname{coz}(f) & 1 \in W, \\ \emptyset & 1 \notin W. \end{cases}$$

Since $\operatorname{coz}(f)$ and empty set are measurable, $\tau(W)$ is a measurable set. Now

$$\lambda^{-1}(U) = \lambda^{-1}(\bigcup_{i=1}^{\infty}(I_i \times J_i)) = \bigcup_{i=1}^{\infty}\lambda^{-1}(I_i \times J_i) = \bigcup_{i=1}^{\infty}(\tau(I_i) \cap f^{-1}(J_i)).$$

This implies that $\lambda^{-1}(U)$ is a measurable set and so λ is a measurable function. We define $\rho : \mathbb{R} \times \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ by $\rho(x, y) = x/y$. The function ρ is continuous and so $h := \lambda o \rho$ is a measurable function in its domain $\operatorname{coz}(f)$.

Suppose that V be an open set in $\mathbb{R}$. We consider three cases:

Case 1: If $f^{-1}(V) \subseteq \operatorname{coz}(f)$. Then $g^{-1}(V) = h^{-1}(V)$ is a measurable set.

Case 2: If $f^{-1}(V) \subseteq Z(f)$. Then $g^{-1}(V) = Z(f)$ is measurable.

Case 3: If $f^{-1}(V) \cap \operatorname{coz}(f)$ and $f^{-1}(V) \cap Z(f)$ are not empty sets. Then

$$g^{-1}(V) = g^{-1}((V \cap \mathbb{R}\backslash\{0\}) \cup \{0\}) = h^{-1}((V \cap \mathbb{R}\backslash\{0\}))) \cup Z(f)$$

and so $g^{-1}(V)$ is a measurable set. □

Theorem 2.4. *Every projective prime ideal in the rings of real measurable functions, $M(X, \mathcal{A}, \mu)$, is generated by an idempotent.*

Proof. Suppose that P is a non finitely generated projective prime ideal in $M(X, \mathcal{A}, \mu)$. By Lemma 2.1 and Lemma 2.2, P is generated by a family $\{f_\alpha\}_{\alpha\in K}$ such that $\{\operatorname{coz}(f_\alpha)\}_{\alpha\in K}$ is star-finite. Since P is non finitely generated, there is a countably infinite subset $\{\operatorname{coz}(f_i)\}_{i=1}^{\infty} \subseteq \{\operatorname{coz}(f_\alpha)\}_{\alpha\in B}$ such that $\operatorname{coz}(f_i) \cap \operatorname{coz}(f_j) = \emptyset$, where $i \neq j$. For all $i \in \mathbb{N}$, we define

$$g_i(x) := \begin{cases} 1/2^i & x \in \mathrm{coz}(f_i), \\ 0 & x \in Z(f_i). \end{cases}$$

For every $i \in \mathbb{N}$, $\mathrm{coz}(f_i)$ and $Z(f_i)$ are measurable sets and so $g_i = 1/2^i \chi_{\mathrm{coz}(f_i)}$ is a measurable function. Since $\{\mathrm{coz}(f_\alpha)\}_{\alpha \ K}$ is star-finite, the functions $h_1 = \Sigma_{i=1}^{\infty} g_{2i}$ and $h_2 = \Sigma_{i=1}^{\infty} g_{2i+1}$ are measurable functions and so belong to $M(X, \mathcal{A}, \mu)$. Neither h_1 nor h_2 can be a finite linear combinations of elements of $\{f_\alpha\}_{\alpha \in B}$ and so $h_1, h_2 \notin P$. But $h_1 h_2 = 0 \in P$ which implies that P is not prime, a contradiction.

Now suppose that $P =< f_1, f_2, \ldots, f_n >$. We put $f := \sum_{i=1}^{n} f_i$. It is easy to check that f is a measurable function. Let $g \in P$. Then there exist $h_1, h_2, \ldots, h_n \in M(X, \mathcal{A}, \mu)$ such that $g = h_1 f_1 + h_2 f_2 + \cdots + h_n f_n$. We define

$$h(x) := \begin{cases} h_1(x) & x \in \mathrm{coz}(f_1), \\ h_2(x) & x \in \mathrm{coz}(f_2), \\ \vdots & \vdots \\ h_n(x) & x \in \mathrm{coz}(f_n), \\ 0 & \text{otherwise}. \end{cases}$$

Since $h_1, h_2, \ldots, h_n$ are measurable functions and $\mathrm{coz}(f_1), \mathrm{coz}(f_2), \ldots, \mathrm{coz}(f_n)$ are (disjoint) members in σ-algebra $\mathcal{A}$, h is a measurable function. Now by definition of h, $g = fh$ and hence P is generated by f.

We define

$$r(x) := \begin{cases} 1/f^{2/3} & x \in \mathrm{coz}(f), \\ 0 & x \in Z(f), \end{cases}$$

By Lemma 2.3, r is a measurable function and hence $f^{1/3} = rf \in P$. Thus $\chi_{\mathrm{coz}(f)} = r(f^{1/3})^2 \in P$ is an idempotent and so $P =< \chi_{\mathrm{coz}(f)} >$. □

Let $C(X)$ be the ring of all real valued continuous functions on a topological space X. The topological properties of X has played an important role in the study of $C(X)$ (see [2–6,8]). But in the study of $M(X, \mathcal{A}, \mu)$, the properties of σ-algebra $\mathcal{A}$ and its measure are essential keys. The concept of isolated points in X which are related to the existence of maximal ideals in $C(X)$, generated by idempotents, play an important role in the context of $C(X)$. Below, we define special measurable sets as generalization of these points in X.

Definition 2.5. Suppose that $E \in \mathcal{A}$ and $\mu(E) \neq 0$. The set E is *near-zero* if for every subset $A \subseteq E$ such that $\mu(A) \neq 0$, $A = E$ a.e. on $(X, \mathcal{A}, \mu)$.

Now we are in a position to present one of the main results in this paper. We record Theorem 2.6 which gives a necessary and sufficient condition for an ideal in $M(X, \mathcal{A}, \mu)$ to be a projective prime ideal.

Notation. Let E be a subset of X. We set:

$$M_E := \{f \in M(X, \mathcal{A}, \mu) : f(E) = 0\}.$$

Theorem 2.6. *A proper ideal in $M(X, \mathcal{A}, \mu)$ is a projective prime ideal if and only if it has the form M_E where E is a near-zero set in $\mathcal{A}$.*

Proof. Let $E \in \mathcal{A}$ be a near-zero set. It is easy to see that M_E is an ideal. Suppose that I is an ideal in $M(X, \mathcal{A}, \mu)$ such that $M_E \subseteq I \subseteq M(X, \mathcal{A}, \mu)$ and $M_E \neq I$. Let $f \in I \setminus M_E$. We claim that

$$E \subseteq \{x \in X : f(x) \neq 0 \text{ a.e. on } (X, \mathcal{A}, \mu)\}.$$

Otherwise, $\mu(E \cap Z(f)) \neq 0$. Since E is a near-zero set, $E \cap Z(f) = E$ a.e. on $(X, \mathcal{A}, \mu)$. Therefore $\mu(E \backslash Z(f)) = 0$, which is a contradiction.

We define

$$g(x) := \begin{cases} f(x) + 1 & x \in E^c, \\ 0 & x \in E. \end{cases}$$

Since E is a measurable set and f is a measurable function, g is measurable and belongs to $M_E \subseteq I$. Hence $g - f \in I$ is a unit element of $M(X, \mathcal{A}, \mu)$. This means that M_E is a maximal ideal in $M(X, \mathcal{A}, \mu)$ and so it is a prime ideal. Now suppose that $f \in M(X, \mathcal{A}, \mu)$. For $f_1 := \chi_{\text{coz}(f)} f$ and $f_2 := \chi_{Z(f)} f$, f_1 and f_2 are measurable functions and $f = f_1 + f_2$. This means that M_E is a summand of $M(X, \mathcal{A}, \mu)$ and so it is a projective ideal.

Conversely, suppose that P is a proper projective prime ideal in $M(X, \mathcal{A}, \mu)$. Since $M(X, \mathcal{A}, \mu)$ is a commutative regular ring, P is a maximal ideal. By Theorem 2.4, there exists $E \in \mathcal{A}$ such that $P =< \chi_E >$. It is easy to check that $P = M_{E^c}$. If E^c is not near-zero, then there exists a measurable set $A \subseteq E^c$ such that $\mu(A) \neq 0$ and $E^c \neq A$ a.e. on $(X\mathcal{A}, \mu)$. This implies that $P = M_{E^c}$ is a proper subset of M_A a.e. on $(X, \mathcal{A}, \mu)$, which is a contradiction. $\square$

In the next corollary, we characterize hereditary rings in the rings of real measurable functions, $M(X, \mathcal{A}, \mu)$, by the structure of maximal ideals in this rings.

Corollary 2.7. *The ring of real measurable functions, $M(X, \mathcal{A}, \mu)$, is a hereditary ring if and only if every maximal ideal in $M(X, \mathcal{A}, \mu)$ has the form M_E where E is a near-zero set in $\mathcal{A}$.*

Proof. Let $M(X, \mathcal{A}, \mu)$ is a hereditary ring and M is a maximal ideal in $M(X, \mathcal{A},$
$\mu)$. Then by Theorem 2.6, there exists a near-zero set $E \in \mathcal{A}$ such that $M = M_E$.

Conversely, suppose that every maximal ideal in $M(X, \mathcal{A}, \mu)$ has the form $M_E = \{f \in M(X, \mathcal{A}, \mu) : f(E) = 0\}$ such that E is a near-zero set. If I is an ideal in $M(X, \mathcal{A}, \mu)$, there exists a maximal ideal M and a near-zero set E such that $M = M_E$ and $I \subseteq M$. For $f \in I$, we define

$$g(x) = \begin{cases} 1/f(x) & x \in \mathrm{coz}(f), \\ 0 & x \in Z(f). \end{cases}$$

By Lemma 2.3, g is a measurable function and so belongs to $M(X, \mathcal{A}, \mu)$. Hence $fg = \chi_E \in I$ and so $I = M = M_E$. Now by Theorem 2.6, I is a projective ideal and $M(X, \mathcal{A}, \mu)$ is a hereditary ring. □

Remark 2.8. It is well-known that if every maximal right ideal of an associative ring is a direct summand, then the ring is Artin semisimple, see [12, Lemma 1]. This immediately shows that X is finite if and only if $C(X)$ is hereditary, in which case $C(X)$ is Artin semisimple and it is, in fact, a finite direct product of fields, each of which is isomorphic to $\mathbb{R}$. In [14] and [13], O.A.S. Karamzadeh has shown that a right self-injective ring R is right hereditary (in fact, Artin semisimple) if and only if its right maximal ideals are projective. In Corollary 2.7, which is similar to the result of Karamzadeh [13, Corollary 2.8], we have shown that the ring of measurable functions is hereditary if and only if every maximal ideal is projective, in which case, this ring is Artin semisimple and hence a finite direct product of fields.

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