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Author(s):

Gh. Moghaddasi and M. Mahmoudi

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SUBDIRECTLY IRREDUCIBLE ACTS OVER SOME SEMIGROUPS

GH. MOGHADDASI* AND M. MAHMOUDI

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Abstract. In this paper, we characterize and find the number of subdirectly irreducible acts over some classes of semigroups, such as zero semigroups, right zero semigroups and strong chain of left zero semigroups.
Keywords: Subdirectly irreducible, chain of semigroups.
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1. Introduction and preliminaries

The theory of actions of semigroups on sets is applied in many branches of mathematical sciences, such as algebra, dynamical systems, computer science, automata, and computational mathematics. The book [3] is a good reference for most of we know about acts from an abstract point of view.

Recall that by Birkhoff's Representation Theorem (see [3, Theorem II.2.36]) for a semigroup S, any nontrivial S-act is a subdirect product of subdirectly irreducible S-acts. Therefore, if we characterize subdirectly irreducible S-acts then we get a description of S-acts in general. In [6], and [5], a characterization of subdirectly irreducible acts, respectively over the monoid $(\mathbb{N}\cup\{\infty\}, \min, \infty)$ and over left zero semigroups is presented. In this paper, we consider subdirectly irreducible acts over some other classes of semigroups, namely the classes of zero semigroups, right zero semigroups, and strong chain of left zero semigroups, hoping to help the research on subdirectly irreducible acts over other semigroups and over arbitrary ones.

In the following we go through some preliminaries which will be used in the sequel.

Recall that, for a semigroup S, a (right) *S-act* (or *S-system*) A is a set A together with a function $\lambda : A\times S \to A$, called the *action* of S (or the S-action)

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*Corresponding author.

on A, such that for $a \in A$ and $s,t \in S$ (denoting $\lambda(a,s)$ by as), $a(st) = (as)t$ and $a1 = a$ if S is a monoid with 1 as its identity.

If there are more than one semigroup in a discussion, to prevent any confusion, we denote an S-act A by A_S.

An S-act A is called *separated* if for $a \neq b$ in A there exists $s \in S \setminus \{1\}$ with $as \neq bs$. Also, we call a subset B of an S-act A *separated* if for $a \neq b$ in B there exists $s \in S \setminus \{1\}$ with $as \neq bs$.

A *homomorphism* $f : A \to B$ between S-acts is a function such that for each $a \in A$ and $s \in S$ we have $f(as) = f(a)s$.

An element a of an S-act A is called a *fixed* or *zero* element if $as = a$ for all $s \in S$. We denote the set of all fixed elements of an S-act A by $FixA$, which is in fact a sub-act of A. This set plays an important role in our investigation.

An equivalence relation ρ on an S-act A is called a *congruence* on A, if $a\rho a'$ implies $(as)\rho(a's)$ for $a, a' \in A$ and $s \in S$. We denote the set of all congruences on A by $ConA$. For $a, b \in A$, the symbol $\rho_{a,b}$ denotes the smallest congruence on A containing (a,b). It is in fact, the equivalence relation on A which is generated by the set $\{(as, bs) : s \in S \cup \{1\}\}$, and its elements are given by:

$$x\rho_{a,b}y \;\Leftrightarrow\; \exists s_1, s_2, \ldots, s_n \in S \cup \{1\}, \exists p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n \in A :$$

$$x = p_1 s_1 \quad q_2 s_2 = p_3 s_3 \quad \cdots \quad q_n s_n = y \quad q_1 s_1 = p_2 s_2 \quad q_3 s_3 = p_4 s_4 \quad \cdots$$

where $(p_i, q_i) = (a,b)$ or $(p_i, q_i) = (b,a)$.

Recall from [3] that, a right S-act A is called *subdirectly irreducible* if $\bigcap \rho \neq \Delta$, where ρ runs over $ConA \setminus \{\Delta\}$.

Note that for each semigroup S, every two element S-act A has exactly two congruences Δ and $\bigtriangledown$, and so it is subdirectly irreducible. Also, we apply the following remark and theorem about subdirectly irreducible and subdirectly reducible acts.

Remark 1.1. Note that every S-act with at least three fixed elements is subdirectly reducible. This is because, if a, b, c are distinct fixed elements of an S-act A, then $\rho_{a,b} \cap \rho_{a,c} = \triangle$.

The above remark was also proved in [7].

Theorem 1.2. *Let S be a semigroup. Then every subdirectly irreducible S-act A with $|A| > 2$ and two fixed elements is separated.*

Proof. Let $FixA = \{a_0, b_0\}$. On the contrary, assume that there exist $x \neq y \in A$ such that $xs = ys$ for all (non identity) element $s \in S$. Then, $\rho_{x,y} = \triangle \cup \{(x,y),(y,x)\}$. Now, $(a_0, b_0) \notin \rho_{x,y}$, and so $\rho_{x,y} \cap \rho_{a_0,b_0} = \triangle$ which is a contradiction. □

Note that the converse of Theorem 1.2 is not generally true. For example, every S-act A with identity actions, and $|A| \geq 3$ is separated but it is not subdirectly irreducible by Remark 1.1.

2. **Subdirectly irreducible acts over a strong chain of left zero semigroups**

A variety of semigroups can be decomposed into simpler types of semigroups which simplifies their structural analysis. A couple of these are:
Archimedean semigroups, completely simple semigroups, Clifford semigroups, and periodic semigroups.

In this section, we specify subdirectly irreducible acts over a strong countable chain of left zero semigroups. It is clear that each S-act A with $|A| = 2$ has only two congruences, implying that such S-acts are subdirectly irreducible. Therefore, in this section, we assume that all S-acts have at least three elements.

Recalling the notion of a strong semilattice of completely simple semigroups from [2], we define *a strong chain of left zero semigroups* to be a semigroup $S = \bigcup_{\alpha\in Y} S_\alpha$, where $\{S_\alpha\}_{\alpha\in Y}$ is a family of disjoint left zero semigroups and Y is a chain with bottom element α_1 such that for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ there exists a semigroup homomorphism $\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$, and for $\alpha \geq \beta \geq \gamma$, $\varphi_{\beta,\gamma}\varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}$. Also, it is assumed that $\varphi_{\alpha,\alpha} = id_{S_\alpha}$ for all $\alpha \in Y$. The multiplication on S, for $s \in S_\alpha, t \in S_\beta$, with $\alpha \geq \beta$ is then given by

$$st = \varphi_{\alpha,\alpha\wedge\beta}(s)\varphi_{\beta,\alpha\wedge\beta}(t) = \varphi_{\alpha,\beta}(s)\varphi_{\beta,\beta}(t) = \varphi_{\alpha,\beta}(s)t = \varphi_{\alpha,\beta}(s)$$
$$ts = \varphi_{\beta,\alpha\wedge\beta}(t)\varphi_{\alpha,\alpha\wedge\beta}(s) = \varphi_{\beta,\beta}(t)\varphi_{\alpha,\beta}(s) = t\varphi_{\alpha,\beta}(s) = t,$$

where the last equalities are because of the assumption that S_β is a left zero semigroup.

Notation: For a strong countable chain $S = \bigcup_{\alpha\in Y} S_\alpha$ of left zero semigroups, let $T \subseteq \bigcup_{\alpha\in Y} S_\alpha$ and A be an S-act. We denote the set of all elements of A which are fixed under the action of all members of T by $FixA_T$.

Remark 2.1. Note that for every S-act A on a strong chain of left zero semigroups $S = \bigcup_{\alpha\in Y} S_\alpha$, we have:

(1) $FixA \subseteq FixA_{S_\alpha}$;

(2) if for some $\alpha \in Y$, A as an S_α-act, which we denote it by A_{S_α}, is separated then A as an S-act is separated;

(3) if α_1 is the bottom element of Y and $x \in FixA_{\bigcup_{\alpha\neq\alpha_1} S_\alpha}$ then $xS_{\alpha_1} = \{*\}$, where $* \in FixA$. This is because, for $s_1, s_2 \in S_{\alpha_1}$, and $t \in S_\alpha$ with $\alpha \neq \alpha_1$, we have

$$xs_1 = (xt)s_1 = x(ts_1) = x\varphi_{\alpha,\alpha_1}(t) = x(ts_2) = (xt)s_2 = xs_2.$$

Also for $s_1 \in S_{\alpha_1},\ r \in S$, we have $s_1r = s_1$, and hence $(xs_1)r = x(s_1r) = xs_1$.

Lemma 2.2. *Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a strong chain of left zero semigroups and A be an S-act. Then for each $x \in A$ and $t \in S_\beta$, we have $xt \in FixA_{\bigcup_{\alpha\geq\beta} S_\alpha}$.*

Proof. Let $x \in A$, $t \in S_\beta$, and $s \in S_\alpha$ for some $\alpha \geq \beta$. Then by the definition of the multiplication on S, we have $ts = t$ and hence $(xt)s = x(ts) = xt$, as required. □

Lemma 2.3. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a strong chain of left zero semigroups and $\alpha_1 = minY$ and $\alpha_2 = min(Y \setminus \{\alpha_1\})$. Then each subdirectly irreducible S-act A with $|A| \geq 3$ is separated.*

Proof. Let A be subdirectly irreducible. By Remark 1.1, $|FixA| \leq 2$. If $|FixA| = 2$ then by Theorem 1.2, A is separated. If $FixA = \{a_0\}$, then we consider three cases:

Case 1. If $FixA_{\bigcup_{\alpha \geq \alpha_2} S_\alpha} = 1$, then for each $x \in A$ and $s \in S$, $xs = a_0$. Therefore, for all $a, b, c \in A$, $\rho_{a,b} \cap \rho_{b,c} = \triangle$ which is a contradiction.

Case 2. If $|FixA_{\bigcup_{\alpha \geq \alpha_2}} S_\alpha| \geq 3$, then there exist $b, c \in A$ such that a_0, b, c are different and for all $s \in \bigcup_{\alpha \geq \alpha_2} S_\alpha$, $a_0 s = a_0$, $bs = b$, $cs = c$. But, by Lemma 2.2, for $s \in S_{\alpha_1}$, $a_0 s = bs = cs = a_0$. Thus

$$\rho_{a_0,b} \cap \rho_{a_0,c} = (\triangle \cup \{(a_0, b), (b, a_0)\}) \cap (\triangle \cup \{(a_0, c), (c, a_0)\}) = \triangle$$

which is a contradiction.

Case 3. If $|FixA_{\bigcup_{\alpha \geq \alpha_2}} S_\alpha| = 2$, then $FixA_{\bigcup_{\alpha \geq \alpha_2}} S_\alpha = \{a_0, a\}$ where $a_0 s = a_0$ for all $s \in S$, $at = a_0$ for all $t \in \bigcup_{\alpha \geq \alpha_2} S_\alpha$, $ar = a$ for all $r \in \bigcup_{\alpha \geq \alpha_2} S_\alpha$. Now, on the contrary, assume that there exist $x \neq y \in A$ such that $xs = ys$ for all $s \in S$. Since $x, y \notin \{a_0, a\}$ and $xs, ys \in \{a_0, a\}$ for all $s \in S$, we get $\rho_{x,y} \cap \rho_{a_0,a} = (\Delta \cup \{(x, y), (y, x)\}) \cap (\Delta \cup \{(a_0, a), (a, a_0)\}) = \Delta$ which is a contradiction. Therefore A is separated. □

In the following, we characterize subdirectly irreducible acts over the two element strong chain of left zero semigroups, and then over an arbitrary strong chain of left zero semigroups.

Theorem 2.4. *Let $S = S_{\alpha_1} \cup S_{\alpha_2}, \alpha_1 < \alpha_2$, be a strong chain of left zero semigroups. Then an S-act A is subdirectly irreducible if and only if A is separated and $|FixA_{S_{\alpha_2}}| = 2$.*

Proof. Let A be subdirectly irreducible. Then $|FixA| \leq 2$. Two cases may occur:

Case 1. $FixA = \{a_0\}$. Then by Lemma 2.2, for each $s \in S_{\alpha_1}$ and each $x \in A$, $xs = a_0$. Now if $|FixA_{S_{\alpha_2}}| = 1$ then for each $x \in A$ and each $s \in S_{\alpha_2}$, $xs = a_0$. Therefore, for all $a, b, c \in A$, $\rho_{a,b} \cap \rho_{b,c} = \triangle$ which is a contradiction. If $|FixA_{S_{\alpha_2}}| \geq 3$, then there exist elements $b, c \in A$ such that a_0, b, c are different and for all $s \in S_{\alpha_2}$, $a_0 s = a_0$, $bs = b$, $cs = c$. But, by Lemma 2.2, for $s \in S_{\alpha_1}$, $a_0 s = bs = cs = a_0$. Thus

$$\rho_{a_0,b} \cap \rho_{a_0,c} = (\triangle \cup \{(a_0, b), (b, a_0)\}) \cap (\triangle \cup \{(a_0, c), (c, a_0)\}) = \triangle$$

which is a contradiction. Therefore $|FixA_{S_{\alpha_2}}| = 2$. Also, A is separated by Lemma 2.3.

Case 2. $FixA = \{a_0, b_0\}$. Then $|FixA_{S_{\alpha_2}}| \geq 2$. Let $FixA_{S_{\alpha_2}} = \{a_0, b_0, c, \cdots\}$. Then by Remark 2.1(3), for all $s \in S_{\alpha_1}$, $cs = a_0$ or for all $s \in S_{\alpha_1}$, $cs = b_0$. Let for all $s \in S_{\alpha_1}$, $cs = a_0$. Then for $s \in S_{\alpha_1}$, $(a_0 s, cs) = (a_0, a_0)$, and for $t \in S_{\alpha_2}$, $(a_0t, ct) = (a_0, c)$. Therefore, $(a_0, b_0) \notin \rho_{c,a_0}$, and hence $\rho_{a_0,b_0} \cap \rho_{c,a_0} = \triangle$, which is a contradiction. If for all $s \in S$, $cs = b_0$, it is proved similarly that $\rho_{a_0,b_0} \cap \rho_{c,b_0} = \triangle$ which is again a contradiction. Thus $|FixA_{S_{\alpha_2}}| = 2$. Also, by Lemma 2.3, A is separated.

Conversely, let A be separated and $FixA_{S_{\alpha_2}} = \{a_0, b_0\}$. We claim that each non trivial congruence θ on A contains (a_0, b_0). Let θ be a non trivial congruence on A. Then there exist $x \neq y \in A$ such that $(x, y) \in \theta$. Thus for all $s \in S_{\alpha_2}$, $(xs, ys) \in \theta$. But, by Lemma 2.2, for all $s \in S_{\alpha_1}$, $xs, ys \in FixA$ and for $s \in S_{\alpha_2}$, $xs, ys \in FixA_{S_{\alpha_2}} = \{a_0, b_0\}$. On the other hand, since A is separated, there exists $s \in S$ such that $xs \neq ys$. Now, since $FixA \subseteq FixA_{S_{\alpha_2}}$, it is concluded that $(a_0, b_0) \in \theta$, and so A is subdirectly irreducible. □

Corollary 2.5. *Let $S = S_{\alpha_1} \cup S_{\alpha_2}$ with $\alpha_1 < \alpha_2$, be a strong chain of left zero semigroups. An S-act A with exactly two fixed elements and $|FixA_{S_{\alpha_2}}| = 2$ is subdirectly irreducible if and only if A is separated.*

Proof. Let $FixA = \{a_0, b_0\}$ and A be subdirectly irreducible. By Theorem 1.2, A is separated. The converse is true by the above theorem. □

Now, we recall a theorem of I.B. Kozhukhov and A.R. Haliullina from [4].

Theorem 2.6 ([4]). *Let A be an S-act with two fixed elements θ_1, θ_2. Then A is subdirectly irreducible if and only if for any $a \neq b$ of A, there exists $s \in S$ such that $\{as, bs\} = \{\theta_1, \theta_2\}$.*

Here, we prove a similar result for the case of a strong countable chain of left zero semigroups.

Lemma 2.7. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a strong chain of left zero semigroups. If A is a subdirectly irreducible S-act with exactly two fixed elements a_0, b_0 then $FixA_{\bigcup_{\alpha \neq \alpha_1} S_\alpha} = \{a_0, b_0\}$, and for every different $x, y \in A$ there exists $s \in S$ such that $xs \neq ys$ and $xs, ys \in FixA_{\bigcup_{\alpha \neq \alpha_1} S_\alpha}$.*

Proof. Let $FixA = \{a_0, b_0\}$ and $(a_0, b_0 \neq) c_0 \in FixA_{\bigcup_{\alpha \neq \alpha_1} S_\alpha}$. Then by Remark 2.1, for all $s \in S_{\alpha_1}$, $c_0s = a_0$ or for all $s \in S_{\alpha_1}$, $c_0s = b_0$. Now if for all $s \in S_{\alpha_1}$, $c_0s = a_0$ then $\rho_{a_0,b_0} \cap \rho_{a_0,c_0} = \triangle$, and if for all $s \in S_{\alpha_1}$, $c_0s = b_0$ then $\rho_{a_0,b_0} \cap \rho_{b_0,c_0} = \triangle$, both of which are contradictions. Hence $|FixA_{\bigcup_{\alpha \neq \alpha_1} S_\alpha}| = 2$. To prove the other part, on the contrary, let there exist different $x, y \in A$ such that for all $s \in S$, $xs = ys$ for $xs, ys \in FixA_{\bigcup_{\alpha \neq \alpha_1} S_\alpha}$. Then $\rho_{x,y} = \triangle \cup \{(x, y), (y, x)\}$. Thus $\{x, y\} \neq \{a_0, b_0\}$ which implies $(x, y) \notin \rho_{a_0,b_0}$, and so $(a_0, b_0) \notin \rho_{x,y}$. Thus $\rho_{x,y} \cap \rho_{a_0,b_0} = \triangle$ which is a contradiction. □

Theorem 2.8. *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a strong finite chain of left zero semigroups. An S-act A is subdirectly irreducible if and only if there exists $\beta \in Y$*

such that $|FixA_{\bigcup_{\alpha>\beta} S_\alpha}| = 2$*, and for every different* x, y *of* A *there exists* $s \in S$ *such that* $xs \neq ys$ *and* $xs, ys \in FixA_{\bigcup_{\alpha>\beta} S_\alpha}$.

Proof. Let A be subdirectly irreducible. Then, by Remark 1.1, $|FixA| \leq 2$. If $|FixA| = 2$ then by Lemma 2.7 the result is true (take $\beta = \alpha_1$). If $|FixA| = 1$, we prove the result by induction on $|Y|$. If $|Y| = 2$, then by Theorem 2.4, A is separated, so for $x \neq y \in A$ there exists $s \in S$ such that $xs \neq ys$. Now $s \in S_{\alpha_1}$ or $s \in S_{\alpha_2}$. If $s \in S_{\alpha_1}$ then $xs, ys \in FixA$, but $|FixA| = 1$ hence $xs = ys$ which is a contradiction therefore $s \in S_{\alpha_2}$, hence $xs, ys \in FixA_{S_{\alpha_2}}$. Then by the proof of Theorem 2.4 (case 1), we have the result (take $\beta = \alpha_1$). Now, by induction we assume that the result is true for each subdirectly irreducible T-act B, where $T = \bigcup_{\alpha\in Y} S_\alpha$ with $|Y| < n$. Let A be a subdirectly irreducible S-act, where $S = \bigcup_{\alpha\in Y} S_\alpha$, and $|Y| = n$. Then, by assumption $1 = |FixA| = |FixA_{\bigcup_{\alpha\geq\alpha_1} S_\alpha}|$, we get $|FixA_{\bigcup_{\alpha\neq\alpha_1} S_\alpha}| \leq 2$. This is because, if on the contrary, $|FixA_{\bigcup_{\alpha\neq\alpha_1} S_\alpha}| \geq 3$ then we have at least three different elements $a_0, b_0, c_0 \in A$ such that for every $s \in \bigcup_{\alpha\neq\alpha_1} S_\alpha, a_0 s = a_0, b_0 s = b_0, c_0 s = c_0$. Also, since $|FixA_{\bigcup_{\alpha\geq\alpha_1} S_\alpha}| = 1$, for every $s \in S_{\alpha_1}$ we have $a_0 s = b_0 s = c_0 s$. Therefore, $\rho_{a_0,b_0} \cap \rho_{a_0,c_0} = \Delta$, which contradicts the hypothesis that A is subdirectly irreducible. Therefore, $|FixA_{\bigcup_{\alpha\neq\alpha_1} S_\alpha}| \leq 2$.

Let $|FixA_{\bigcup_{\alpha\neq\alpha_1} S_\alpha}| = 1$, and consider $T = \bigcup_{\alpha\neq\alpha_1} S_\alpha$. Then T is a strong chain of left zero semigroups (by the hypothesis that $Y \setminus \{\alpha_1\}$ has a least element), and A is subdirectly irreducible as a T-act, too. The latter is because $Con(A_S) = Con(A_T)$. To see this, let ρ be a congruence on A_T. Then for each $(a, b) \in \rho$, and $t \in S_{\alpha_1}$, by Lemma 2.2, $at, bt \in FixA_{\bigcup_{\alpha\geq\alpha_1} S_\alpha} = FixA$. But, by hypothesis $|FixA| = 1$, and hence $(at, bt) \in \Delta \subseteq \rho$. Therefore, $Con(A_T) \subseteq Con(A_S)$, the converse is clear. Now, by applying induction hypothesis for A_T, we get the result for A_S.

Also, if $|FixA_{\bigcup_{\alpha\neq\alpha_1} S_\alpha}| = 2$ then by applying Lemma 2.7 for A as a T-act, where $T = \bigcup_{\alpha\neq\alpha_1} S_\alpha$, we get the result.

Conversely, let β be as in the statement. By Lemma 2.2, for each $x \in A$, and $s \in S_\beta$, $xs \in FixA_{\bigcup_{\alpha\geq\beta} S_\alpha}$. Let $FixA_{\bigcup_{\alpha>\beta} S_\alpha} = \{a_0, b_0\}$. We claim that each non trivial congruence ρ on A contains (a_0, b_0). Let ρ be a nontrivial congruence on A, and $(x, y) \in \rho$, where $x \neq y$. By hypothesis, for some $s \in S$, $xs \neq ys$ and $xs, ys \in FixA_{\bigcup_{\alpha>\beta} S_\alpha}$. Thus $(a_0, b_0) = (xs, ys) \in \rho$ or $(a_0, b_0) = (ys, xs) \in \rho$, and hence A is subdirectly irreducible. □

3. Subdirectly irreducible acts over a zero semigroup

Another class of semigroups over which we characterize the subdirectly irreducible acts is the class of zero semigroups. Recall that a semigroup S is a zero semigroup, if $st = 0$ for all $s, t \in S$, where 0 is the zero element of S.

Lemma 3.1. *Let S be a zero semigroup. Then every S-act A with $|A| \geq 3$ and $|FixA| = 2$ is subdirectly reducible.*

Proof. Let $FixA = \{a_0, b_0\}$ and $c \in A \setminus FixA$. We know that $c0 \in FixA$. Let $c0 = a_0$. Then for each $s \in S$, $cs \neq b_0$. This is because, if $cs = b_0$ for some $s \in S$, then $c0 = c(ss) = (cs)s = b_0 s = b_0$ which is a contradiction. Let $s \in S$ and $cs = x$. Then for each $t \in S$, $xt = (cs)t = c(st) = c0 = a_0$. Now, if $x = a_0$ then $(a_0, b_0) \notin \rho_{a_0,c}$, that is $\rho_{a_0,b_0} \cap \rho_{a_0,c} = \Delta$, and if $x \neq a_0$ then $\rho_{a_0,b_0} \cap \rho_{a_0,x} = \Delta$. Therefore, A is subdirectly reducible. □

By the above lemma and Remark 1.1, we get the following fact.

Corollary 3.2. *Let S be a zero semigroup. Then every S-act A with $|A| \geq 3$ and $|FixA| \geq 2$ is subdirectly reducible.*

Lemma 3.3. *Let S be a zero semigroup. If A is a separated S-act with $|FixA| = 1$, then $|A| = 1$ and is subdirectly irreducible.*

Proof. Let $FixA = \{a_0\}$. If $|A| \geq 2$ then there exists a non fixed element $b \in A$ such that $b0 = a_0$. Since A is separated, there exists $t \in S$ such that $bt \neq a_0 t \ (= a_0)$, let $bt = c$. Now, for every $s \in S$, $cs = (bt)s = b(ts) = b0 = a_0 = a_0 s$, which is a contradiction, and so $|A| = 1$. The fact that such an A is subdirectly irreducible is then clear. □

The following results describe subdirectly irreducible S-acts over a zero semigroup S.

Theorem 3.4. *Let S be a zero semigroup. An S-act A with $|A| \geq 3$ is subdirectly irreducible if and only if $|FixA| = 1$ and $A \setminus FixA$ is separated.*

Proof. Let $FixA = \{a_0\}$ and $A \setminus FixA$ be separated. Then for all $x \in A$, $x0 = a_0$. Now, for every $(0 \neq)s \in S$ and $(a_0 \neq)x \in A$, $xs \neq x$, since otherwise $x = xss = x0 = a_0$, which is a contradiction. Thus, for $x \in A \setminus FixA$, two cases may occur:

Case 1. For all $s \in S$, $xs = a_0$. Note that since $A \setminus FixA$ is separated, such an x (if exists) is unique.

Case 2. There exists $s \in S$ such that $xs = c \neq a_0$. Then for all $t \in S$, $ct = (xs)t = x(st) = x0 = a_0$, and such a c is unique, as we explained in case (1).

Therefore, there exists a unique element $c \in A$ such that $cs = a_0$ for all $s \in S$ and $xS \subseteq \{a_0, c\}$, for all $x \in A$. Note that if $xs = d \neq a_0, c$ for some $s \in S$, then similar to the discussion of case (2), $dt = a_0 = ct$ for all $t \in S$, which contradicts the fact that $A \setminus FixA$ is separated.

Now, we claim that each $(\Delta \neq)\theta \in ConA$ contains $\{(a_0, c), (c, a_0)\}$. Let $(\Delta \neq)\theta \in ConA$. Then, there exist $x \neq y \in A$ such that $(x, y) \in \theta$. Now, two cases may occur:

Case 1. One of x or y is equal to a_0, say $x = a_0$. Then $(a_0, y) \in \theta$. Now, if for each $s \in S$, $ys = a_0$ then $y = c$, and hence $(a_0, c) = (ys, a_0 s) \in \theta$. If $ys \neq a_0$ for some $s \in S$, then $ys = c$, and hence $(a_0, c) = (a_0 s, ys) \in \theta$.

Case 2. $x, y \neq a_0$. Then $x0 = y0 = a_0$. But, since $A \setminus FixA$ is separated, there exists $t \in S$ such that $xt \neq yt$. Let $xt = c$. Then $yt = a_0$, and hence $(a_0, c) = (yt, xt) \in \theta$. Therefore A is subdirectly irreducible.

Conversely, by Lemma 3.1, Remark 1.1, and the fact that each S-act (S being a zero semigroup) has at least one fixed element, $|FixA| = 1$. Let $FixA = \{a_0\}$. To prove that $A \setminus FixA$ is separated let, on the contrary that there exist $b \neq c \in A \setminus FixA$ with $bs = cs$ for every $s \in S$. Then, two cases may happen:

Case 1. For some $s \in S$, $bs = cs = x \neq a_0$ (note that $x \neq b, c$ since if $bs = cs = b$ then for every $t \in S$, $bt = a_0$, which is a contradiction). Then for each $s \in S$, $xs = b0 = a_0$. Now $\rho_{a_0,x} \cap \rho_{b,c} = \Delta$, which means that A is subdirectly reducible, which is a contradiction.

Case 2. For each $s \in S$, $bs = cs = a_0$. Then $\rho_{b,c} \cap \rho_{a_0,c} = \Delta$, which is again a contradiction and so we get the result. □

Remark 3.5. For a zero semigroup $S = \{0, s, t, \ldots\}$ with 0 as the zero element, the structure of a subdirectly irreducible S-act A with at least three elements is as follow:

(1) $FixA$ is a singleton set $\{a_0\}$.
(2) There exists a unique element $c_0 \neq a_0 \in A$ such that $c_0 S = \{a_0\}$.
(3) For all $x \in A$, $x0 = a_0$ and $xS \subseteq \{a_0, c_0\}$.
(4) $A \setminus FixA$ is separated.

Now, we find the number of all subdirectly irreducible S-acts over finite zero semigroups.

Theorem 3.6. *Let S be a zero semigroup with $|S| = n$. The number of non isomorphic subdirectly irreducible S-acts is $2^{2^{n-1}-1} + 2$.*

Proof. Let $S = \{0, s_1, \ldots, s_{n-1}\}$ and A be a subdirectly irreducible S-act with at least three elements. Then, by Remark 3.5, there exist two elements a_0, c_0 in A, as described there. Also for every $x \in A$ (distinct from a_0, c_0), taking $\rho_x : S \to A$, by $\rho_x(s) = xs$, we have $\rho_x^{-1}\{a_0\} \subseteq S$ and $|\rho_x^{-1}\{a_0\}| = 1$ or 2 or $\cdots$ or $n-1$. (Note that, by the above remark the cases that $|\rho_x^{-1}\{a_0\}| = 0$ or n are impossible). Also, note that since $A \setminus FixA$ is separated, for $x, y \in A \setminus FixA$ we have

$$\rho_x^{-1}\{a_0\} = \rho_y^{-1}\{a_0\} \Rightarrow x = y.$$

This is because, for $x \neq y$ in $A \setminus FixA$ there exists $s \in S$ such that $xs \neq ys$. But $xs, ys \subseteq \{a_0, c_0\}$, and so $xs = a_0$, $ys = c_0$ or $xs = c_0$, $ys = a_0$ which contradicts $\rho_x^{-1}\{a_0\} = \rho_y^{-1}\{a_0\}$.

This shows that for each $m \in \{1, 2, \ldots, n-1\}$, the number of $x \in A$ with $|\rho_x^{-1}\{a_0\}| = m$ is the same as the number of subsets T of $S \setminus \{s_0\}$ with $|T| = m-1$; that is $\binom{n-1}{m-1}$. Therefore A has at most $2 + \sum_{k=0}^{n-2} \binom{n-1}{k} = 2 + 2^{n-1} - 1$ elements.

In fact, the largest subdirectly irreducible S-act is

$$A_1 = \{a_0, c_0, x_1, x_2, \ldots, x_{2^{n-1}-1}\},$$

where

$$x_1 0 = a_0,\ x_1 s_1 = x_1 s_2 = \cdots = x_1 s_{n-1} = c_0, \text{ and } \quad x_2, \ldots, x_n$$

satisfy

$$x0 = xs_i = a_0, \text{ for some } i = 1, \cdots, n-1, \text{ and } xs_j = c_0 \text{ for all } j \neq i,$$

$x_n, \cdots, x_{\binom{n-1}{2}}$ satisfy $x0 = xs_i = xs_j = a_0,$ for some $i, j \in \{1, \cdots, n-1\}$ and $xs_k = c_0$ for all $k \neq i, j$ and so on to $x_{(2^{n-1}-1)-\binom{n-1}{n-2}}, \cdots, x_{2^{n-1}-1}$ which satisfy

$$xs_i = c_0, \text{ for some } i = 1, \cdots, n-1, \text{ and } x0 = xs_j = a_0 \text{ for all } j \neq i.$$

Note that, since $A \setminus FixA$ is separated, the elements of A_1 are distinct.

Now, we see that every $B \leq A_1$ with $a_0, c_0 \in B$ is also subdirectly irreducible. The number of nontrivial subdirectly irreducible subacts of A_1 is $|\{B :\ B \subseteq A_1,\ a_0, c_0 \in B\}|$ which is the same as $|\{C | C \subseteq \{x_1, \ldots, x_{2^{n-1}-1}\}\}| = 2^{2^{n-1}-1}$. Finally, by a similar discussion as above, any other subdirectly irreducible S-act with at least three elements is isomorphic to A_1 or one of the subacts of A_1. Therefore, the number of all subdirectly irreducible S-acts is $2^{2^{n-1}-1} + 2$, where the added 2, comes by considering the trivial subdirectly irreducible act $\{a_0\}$, and the two elements (both fixed) subdirectly irreducible act. □

Recall that an S-act A is called *simple* if $ConA = \{\triangle, \nabla\}$. It is easy to check that every S-act A with $|A| \leq 2$ is simple, but there exists no simple S-act A with trivial action and $|A| > 2$. We close this section, by generalizing the latter fact for any S-act.

Theorem 3.7. *For a zero semigroup S, there exists no simple S-act A with $|A| > 2$.*

Proof. First note that an S-act A with $|A| > 2$ and at least two fixed elements, is not simple. Now, let $FixA = \{a_0\}$. For $b, c \in A$ we have $b0 = c0 = a_0$. Assume that for some $s \in S$, $bs = x \neq a_0$. Then for each $t \in S$, $xt = bst = b0 = a_0$. Therefore, $\rho_{a_0,x} \neq \triangle, \nabla$, and so A can not be simple. □

4. Subdirectly irreducible acts over a right zero semigroup

The last kind of semigroups over which we consider the subdirectly irreducible acts, are right zero semigroups. Recall that a semigroup S is a right zero semigroup, if $st = t$, for all $s, t \in S$.

Also Note that for a right zero semigroup S, the only separated S-acts are S-acts with trivial actions. This is because if A is an S-act, and $a, b \in A$ are distinct elements with $as = b$ for some $s \in S$, then $at = bt$ for all $t \in S$ which means that A is not separated.

Definition 4.1. Let S be a right zero semigroup and A be an S-act. Define the relation $\sim$ on A as follow:

$$a \sim b \Leftrightarrow (a = bs \text{ or } b = as \text{ for some } s \in S).$$

We call each subset $a_\sim = \{b \in A | b \sim a\}$ a $\sim$*-part* of A.

Remark 4.2. (1) Note that although $\sim$ is a symmetric relation, it is not necessarily an equivalence relation. For example, take $S = \{s, t\}$ be a two element right zero semigroup, and consider the S-act $A = \{a, b, c\}$ with the action given by $as = bs = cs = b$, $at = bt = ct = c$. Then A has two $\sim$-parts: $a_\sim = \{b, c\}$ and $b_\sim = c_\sim = \{a, b, c\}$. We further see that $a \nsim a$.

(2) Each element of A belongs to at least one $\sim$-part. In fact, for each $a \in A$ and $s \in S$, $a \in (as)_\sim$.

(3) If $a \sim b$ then $at = bt$ for all $t \in S$. This is because, if $a = bs$ or $b = as$, for some $s \in S$, then for all $t \in S$, $at = (bs)t = b(st) = bt$ or $bt = (as)t = a(st) = at$.

Lemma 4.3. *Let S be a right zero semigroup and A be an S-act. If A has a $\sim$-part with at least three elements then A is subdirectly reducible.*

Proof. Let $x_\sim$ be a $\sim$-part of A and $\{a, b, c\} \subseteq x_\sim$. By Remark 4.2(3), for all $s \in S$, $as = bs = cs$. Therefore $\rho_{a,b} \cap \rho_{a,c} = \Delta$. Thus A is subdirectly reducible. □

Lemma 4.4. *Let S be a right zero semigroup and A be an S-act. If A has two $\sim$-parts with at least two elements in each $\sim$-part then A is subdirectly reducible.*

Proof. Let $x_\sim$ and $y_\sim$ be two $\sim$-parts and $\{a, a'\} \subseteq x_\sim$, $\{b, b'\} \subseteq y_\sim$. Then by Remark 4.2(3), $as = a's$ and $bs = b's$ for every $s \in S$. Therefore $\rho_{a,a'} \cap \rho_{b,b'} = \Delta$. Thus A is subdirectly reducible. □

Lemma 4.5. *Let S be a right zero semigroup and A be an S-act with $|A| \geq 3$. If A has only two fixed elements then A is subdirectly reducible.*

Proof. Let $FixA = \{a, b\}$ and $c \in A \setminus FixA$. Two cases may occur:

$$(1)\ c \sim a \text{ or } c \sim b \qquad (2)\ c \nsim a \text{ and } c \nsim b$$

Let $c \sim a$. Then $cs = as$, for all $s \in S$. Now $\rho_{a,b} \cap \rho_{a,c} = \Delta$. Thus A is subdirectly reducible. Similarly, if $c \sim b$ then A is subdirectly reducible.

Let $c \nsim a$ and $c \nsim b$. Then there dose not exist $t \in S$ with $ct = a$ or $ct = b$. So $(a,b),(b,a) \notin \rho_{a,c}$, and hence $\rho_{a,b} \cap \rho_{a,c} = \Delta$. Thus A is subdirectly reducible. □

Theorem 4.6. *Let S be a right zero semigroup with $|S| > 3$. An S-act A with at least three elements is subdirectly irreducible if and only if $|A| = 3$, $|FixA| = 1$ and A has exactly two $\sim$-parts.*

Proof. Let A be subdirectly irreducible. Then first we see that $|A| = 3$. On the contrary, let $|A| > 3$. If A has only one $\sim$-part then it is subdirectly reducible, by Lemma 4.3, which is a contradiction. If A has two $\sim$-parts then it is subdirectly reducible, by Lemmas 4.3 and 4.4, which is again a contradiction. If A has more than two $\sim$-parts then it clearly is subdirectly reducible, which is again a contradiction. Therefore, $|A| = 3$. Now using this, by a similar discussion to the above, we get that A has exactly two $\sim$-parts. Finally $|FixA| = 1$. This is because, if $|FixA| = 2$ then by Lemma 4.5 it is subdirectly reducible. Also, if $|FixA| = 0$ then A does not have a one element $\sim$-part, and hence each $\sim$-part has at least two elements. Therefore by Lemma 4.4, A is subdirectly reducible, which is a contradiction.

Conversely, let $|A| = 3$, $|FixA| = 1$, and A have exactly two $\sim$-parts. Then one $\sim$-part has one element which is a fixed element, and another $\sim$-part has two elements which are not fixed elements. Let $A = \{a, b, c\}$, where a is a fixed element and $b \sim c$. Then, using the hypothesis that $|FixA| = 1$, there exist $t, s \in S$ such that $bs = cs = b$ and $ct = bt = c$. Now, the only nontrivial congruences on A are: $\rho_{a,b} = \rho_{a,c} = \nabla$ and $\rho_{b,c}$. Also, $(b,c) \in \rho_{a,b} \cap \rho_{a,c} \cap \rho_{b,c}$. Therefore, A is subdirectly irreducible. □

Since every S-act with one or two elements is subdirectly irreducible, the above theorem gives a characterization of all subdirectly irreducible S-acts, for a right zero semigroup S. Theorem 4.6 can be also obtained from [4], where a characterization of subdirectly irreducible acts over a rectangular band is given.

Remark 4.7. For a right zero semigroup S, the structure of each subdirectly irreducible S-act A with $|A| \geq 3$ is as follow:

(1) $|A| = 3$, let $A = \{a_0, b, c\}$
(2) $FixA = \{a_0\}$
(3) There exist $s, t \in S$ such that $bs = cs = c$ and $bt = ct = b$, and for all $r \in S$, $br = cr \neq a_0$.

Now we find the number of subdirectly irreducible S-acts over a right zero semigroup S.

Theorem 4.8. *Let S be a right zero semigroup with $|S| = n$. Then the number of non-isomorphic subdirectly irreducible S-acts is $n + 2$.*

Proof. Let $S = \{s_1, s_2, \ldots, s_n\}$. Consider the set $\mathcal{X} = \{A_i | \ 1 \leq i \leq n-1\}$ in which for all i, $1 \leq i \leq n-1$, $A_i = \{a_0, b, c\}$ is the S-act given as $FixA_i = \{a_0\}$, for all j with $1 \leq j \leq i$, $bs_j = cs_j = b$, and for all $l > i$, $bs_l = cs_l = c$. By Remark 4.7, all A_i in $\mathcal{X}$ are subdirectly irreducible. On the other hand, if an S-act A is subdirectly irreducible with at least three elements, then again by the above remark, A is isomorphic to one of A_i in $\mathcal{X}$. Therefore the number of all subdirectly irreducible S-acts with at least three elements is $|\mathcal{X}| = n-1$. Also, computing the number of subdirectly irreducible S-acts A with $|A| \leq 2$, we get that there are (up to isomorphism) only three such acts: the singleton S-act $\{a\}$, the two fixed element act $\mathbf{2}$, and the two element act $\{a, b\}$ with only one fixed element. In all, the number of subdirectly irreducible S-acts is $(n-1)+3 = n+2$. □

Theorem 4.9. *For a right zero semigroup S, there exists no simple S-act with at least three elements.*

Proof. Let A be an S-act with $|A| \geq 3$. Two cases may happen:

Case 1. A has exactly one $\sim$-part. Then we can take $a, b, c \in A$ with $as = bs = cs$, for all $s \in S$. Thereafter, $\rho_{a,b} = \triangle \cup \{(a,b),(b,a)\} \neq \nabla, \triangle$.

Case 2. A has more than one $\sim$-part. Let $a_\sim$, $b_\sim$ be two $\sim$-parts of A such that $c \in a_\sim$. Then $\rho_{a,c} = \triangle \cup \{(a,c),(c,a)\} \neq \nabla, \triangle$. Therefore, A is not simple. □

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References

[1] M.M. Ebrahimi, M. Mahmoudi and Gh. Moghaddasi Angizan, Injective hulls of acts over left zero semigroups, *Semigroup Forum* **75** (2007) 212–220.
[2] J.M. Howie, Fundamentals of Semigroup Theory, Clarendon Press, 1995.
[3] M. Kilp, U. Knauer and A.V. Mikhalev, Monoids, Acts and Categories, Walter De Gruyter, Berlin-New York, 2000.
[4] I.B. Kozhukhov and A.R. Haliullina, A characterization of subdirectly irreducible acts (Russian), Appl. Discrete Math. (Prikl. Diskr. Mat.) **27** (2015) 5–16.
[5] Gh. Moghaddasi, A note on acts over left zero semigroups, *Turkish. J. Math.* **36** (2012) 359–365.
[6] M. Mahmoudi, Separation axioms on $\mathbb{N}^\infty$-systems, *Semigroup Forum* **70** (2005) 97–106.
[7] E.N. Roiz, On subdirectly irreducible monars, in: Ordered Sets and Lattices, No. 2 (Russian), pp. 80–84. Izdat. Saratov. Univ. Saratov, 1974.

(Gholamreza Moghaddasi) Department of Mathematics, Hakim Sabzevari University, Sabzevar, Iran.
E-mail address: r.moghadasi@hsu.ac.ir

(Mojgan Mahmoudi) Department of Mathematics, Shahid Beheshti University, Tehran, Iran.
E-mail address: m-mahmoudi@sbu.ac.ir