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COMPOSITION OF RESOLVENTS AND QUASI-NONEXPANSIVE MULTIVALUED MAPPINGS IN HADAMARD SPACES

G. ZAMANI ESKANDANI*, S. AZARMI AND M. ESLAMIAN

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Abstract. The proximal point algorithm, which is a well-known tool for finding minima of convex functions, is generalized from the classical Hilbert space framework into a nonlinear setting, namely, geodesic metric spaces of nonpositive curvature. In this paper we propose an iterative algorithm for finding a common element of the minimizers of a finite family of convex functions and common fixed points of a finite family of quasi-nonexpansive multivalued mappings in Hadamard spaces.
Keywords: Proximal point algorithm, $CAT(0)$ spaces, nonexpansive multivalued mappings.
MSC(2010): Primary: 47H09; Secondary: 47H10.

1. Introduction

Let (X, d) be a metric space and $f : X \to (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y).$$

We denote by $argmin_{y\in X} f(y)$ the set of minimizers of f. A successful and powerful tool for solving this problem is the well-known proximal point algorithm (PPA). The proximal point algorithm is a method for finding a minimizer of a convex lower semicontinuous function defined on a Hilbert space. Its origin goes back to Martinet and Rockafellar [20, 21]. Indeed, let f be a proper convex and lower semi-continuous function on a Hilbert space H which attains its minimum. The proximal point algorithm seeks a minimizer of f by successive

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approximations

$$x_{n+1} = argmin_{y\in H}\Big(f(y) + \frac{1}{2r_n} \parallel y - x_n \parallel^2 \Big), n \in \mathbb{N},$$

where $r_n > 0$ for all $n \in \mathbb{N}$. It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of f provided $\sum_{n=1}^{\infty} r_n = \infty$. A natural question, posed by Rockafellar in [21], as to whether this convergence can be improved to strong one was answered in the negative by Guler [14]. In 2000, Kamimura and Takahashi [17] combined the PPA with Halpern's algorithm [15] so that the strong convergence is guaranteed [5]. Recently, Bačak [4, Theorem 6.3.1] investigated the convergence of the proximal point algorithm for convex functions in Hadamard spaces, which are also known as complete $CAT(0)$ space (X, d) as follows: $x_1 \in X$ and

$$x_{n+1} = argmin_{y\in X}\Big(f(y) + \frac{1}{2r_n} d(y, x_n)^2\Big), \ n \in \mathbb{N},$$

where $r_n > 0$ for all $n \in \mathbb{N}$. Based on the concept of the Fejer monotonicity, it was shown that, if f has a minimizer and $\sum_{n=1}^{\infty} r_n = \infty$, then $\{x_n\}$ $\triangle$-converges to the minimizer of f. Recently, Cholamjiak et al. [9] introduced the following modified proximal point algorithm using the S-type iteration process for two nonexpansive mappings in $CAT(0)$ spaces,

$$\begin{aligned} z_n &= argmin_{y\in X}\big(f(y) + \frac{1}{2r_n} d(y, x_n)^2\big),\\ y_n &= (1-\beta_n)x_n \oplus \beta_n T_1 z_n,\\ x_{n+1} &= (1-\alpha_n)T_1 x_n \oplus \alpha_n T_2 y_n, \end{aligned}$$

and it was shown that, $\{x_n\}$ $\triangle$-converges to a common element of

$$F(T_1) \cap F(T_2) \cap argmin_{y\in X} f(y),$$

under some mild conditions. Abkar and Eslamian in [2], introduced an iterative process for a finite family of generalized nonexpansive multivalued mappings, and proved $\triangle$-convergence and strong convergence theorems for the proposed iterative process in $CAT(0)$ spaces. In this paper, motivated and inspired by [2], we propose an iterative method for finding a common element of the minimizers of a finite family of convex functions and common fixed points of a finite family of quasi-nonexpansive multivalued mappings in Hadamard spaces.

2. **Preliminaries**

Let (X, d) be a metric space. A geodesic path joining x to y in X is a mapping γ from a closed interval $[0, l] \subseteq \mathbb{R}$ to X such that $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, l]$. In particular, the mapping γ is an isometry and $d(x, y) = l$. The image of γ is called a geodesic segment joining

x and y which when is unique denoted by $[x, y]$. We denote the unique point $z \in [x, y]$ such that

$$d(x, z) = \alpha d(x, y) \quad \text{and} \quad d(y, z) = (1 - \alpha)d(x, y), \tag{2.1}$$

by $(1 - \alpha)x \oplus \alpha y$, where $0 \le \alpha \le 1$.

The metric space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic segment joining x and y for each $x, y \in X$. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of points (the edges of Δ). A comparison triangle for $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$.
A geodesic space X is called a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom:
Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle in $\mathbb{R}^2$. Then the triangle Δ is said to satisfy the $CAT(0)$ inequality if

$$d(x, y) \le d_{\mathbb{R}^2}(\overline{x}, \overline{y}),$$

for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$. A subset C of a $CAT(0)$ space is convex if $[x, y] \subseteq C$ for all $x, y \in C$. It is well known that any complete simply connected Riemannian manifold of nonpositive sectional curvature is a $CAT(0)$ space. Other examples include Pre- Hilbert spaces, $\mathbb{R}$-trees [6], Euclidean buildings, the complex Hilbert ball with a hyperbolic metric [13] and many others. If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the geodesic segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies the so-called (CN) inequality, i.e.,

$$d(x, y_0)^2 \le \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

It is known that a uniquely geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality [6].
Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x_n, x).$$

The asymptotic radius $r(x_n)$ of $\{x_n\}$ is given by:

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point [11].
A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be $\triangle$-convergent to $x \in X$ if

x is the unique asymptotic center of every subsequence of $\{x_n\}$.
We will use the following lemmas.

Lemma 2.1 ([19]). *Every bounded sequence in an Hadamard space has a $\triangle$-convergent subsequence.*

Lemma 2.2 ([10]). *If D is a closed convex subset of an Hadamard space and $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 2.3 ([12]). *If $\{x_n\}$ is a bounded sequence in an Hadamard space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.4 ([12]). *Let X be a $CAT(0)$ space. Then for all $x, y, z \in X$ and all $t \in [0, 1]$, we have*

(i) $d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z)$,
(ii) $d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$.

Let D be a nonempty subset of $CAT(0)$ space X. Then a mapping T of D into itself is called nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in D$. A point $x \in D$ is called a fixed point of T if $Tx = x$. We denote by $F(T)$ the set of all fixed points of T. $W.A.$ Kirk showed that the fixed point set of a nonexpansive mapping T is closed and convex [18].

Lemma 2.5 ([12]). *Let D be a closed and convex subset of an Hadamard space X and $T : D \to D$ be a nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in D such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\triangle-\lim_{n\to\infty} x_n = x$. Then $Tx = x$.*

Firmly nonexpansive mappings were first introduced by Browder [7], under the name of firmly contractive, in the setting of Hilbert spaces, and later by Bruck [8] in the context of Banach spaces. Recently Bruck's definition was extended to a nonlinear setting in [3].

Definition 2.6. Let D be a nonempty subset of a $CAT(0)$ space (X, d). We say that a mapping $T : D \longrightarrow X$ is firmly nonexpansive if

$$d(Tx, Ty) \leq d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty),$$

for all $x, y \in D$ and $\lambda \in (0, 1)$.

Remark 2.7. Any firmly nonexpansive mapping is nonexpansive.

For $\lambda > 0$, define the Moreau-Yosida resolvent of f in $CAT(0)$ space (X, d) as

$$Prox_\lambda^f(x) := argmin_{y\in X}\big(f(y) + \frac{1}{2\lambda}d(x, y)^2\big),$$

for all $x \in X$. The mapping $Prox_\lambda^f$ is well defined for all $\lambda > 0$ [16]. Recall that a function $f : X \longrightarrow (-\infty, +\infty]$ defined on a convex subset D of a $CAT(0)$ space is convex if, for any geodesic $\gamma : [a, b] \longrightarrow D$, the function $fo\gamma$ is convex. The following lemmas play an important role in this paper.

Lemma 2.8 ([3]). *Let (X,d) be an Hadamard space and $f : X \longrightarrow (-\infty,+\infty]$ be a proper convex and lower semicontinuous function. Then for every $\lambda > 0$,*

(i) *$Prox^f_\lambda$ is a firmly nonexpansive mapping.*
(ii) *$F(Prox^f_\lambda) = argmin_{y\in X} f(y)$.*

Lemma 2.9 ([16]). *Let (X,d) be an Hadamard space and $f : X \to (-\infty,\infty]$ be a proper convex and lower semi-continuous function. Then the following identity holds:*

$$Prox^f_\lambda x = Prox^f_\mu\Big(\frac{\lambda-\mu}{\lambda}Prox^f_\lambda x \oplus \frac{\mu}{\lambda}x\Big),$$

for all $x \in X$ and $\lambda > \mu > 0$.

Let D be a subset of a $CAT(0)$ space X. We denote by $CB(D)$, $K(D)$, $KC(D)$ and $P(D)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, nonempty convex compact subsets and proximinal bounded subsets of D, respectively. The Hausdorff metric H on $CB(X)$ is defined by:

$$H(A,B) := \max\{\sup_{x\in A}\operatorname{dist}(x,B), \sup_{y\in B}\operatorname{dist}(y,A)\},$$

for all $A, B \in CB(X)$, where $\operatorname{dist}(x,B) = \inf\{d(x,z) : z \in B\}$.

Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denoted by $F(T)$.

Definition 2.10. A multivalued mapping $T : X \to CB(X)$ is called

(i) Nonexpansive if $H(Tx,Ty) \le d(x,y)$ for all $x,y \in X$;
(ii) Quasi-nonexpansive if $F(T) \ne \emptyset$ and $H(Tx,Tp) \le d(x,p)$ for all $x \in X$ and all $p \in F(T)$.

We state the multivalued analogs of the conditions (E) in the following way (see [1]).

Definition 2.11. A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition (E_μ) provided that

$$\operatorname{dist}(x,Ty) \le \mu \operatorname{dist}(x,Tx) + d(x,y), \qquad x,y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \ge 1$.

We will use the following lemma.

Lemma 2.12 ([2]). *Let D be a nonempty closed convex subset of an Hadamard space X and $T : D \to K(D)$ satisfies the condition (E). If $\{x_n\}$ is a sequence in D such that $\lim_{n\to\infty}\operatorname{dist}(Tx_n,x_n) = 0$ and $\triangle - \lim_n x_n = v$, then $v \in Tv$.*

3. Main results

Let D be a nonempty convex subset of an Hadamard space X and $f_i : D \to (-\infty, \infty]$ $(i = 1, 2, \dots, r)$ be r proper convex and lower semi-continuous functions. Let $T_j : D \to CB(D)$ $(j = 1, \dots, m)$ be m given mappings. Then for $x_0 \in D$, $a_{n,j} \in [0, 1]$ $(j = 0, 1, \dots, m)$ and $\lambda_n^i > 0$ $(i = 1, \dots, r)$, we consider the following iterative process:

$$\begin{cases} w_n = Prox^{f_r}_{\lambda_n^r} \circ \cdots \circ Prox^{f_1}_{\lambda_n^1} x_n, \\ y_{n,0} = (1 - a_{n,0})x_n \oplus a_{n,0} w_n, \\ y_{n,1} = (1 - a_{n,1})x_n \oplus a_{n,1} z_{n,1}, \\ y_{n,2} = (1 - a_{n,2})x_n \oplus a_{n,2} z_{n,2}, \\ . \\ . \\ . \\ y_{n,m-1} = (1 - a_{n,m-1})x_n \oplus a_{n,m-1} z_{n,m-1}, \\ x_{n+1} = (1 - a_{n,m})x_n \oplus a_{n,m} z_{n,m}, \end{cases} \tag{3.1}$$

where $z_{n,j} \in T_j(y_{n,j-1})$ for $j = 1, \dots, m$.

We shall make use of the following lemma.

Lemma 3.1. *Let D be a nonempty closed convex subset of an Hadamard space X and $f_i : D \to (-\infty, \infty]$ $(i = 1, 2, \dots, r)$ be r proper convex and lower semi-continuous functions. Let $T_j : D \to CB(D)$ $(j = 1, \dots, m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E) such that*

$$\Omega = \cap_{j=1}^m F(T_j) \bigcap \cap_{i=1}^r argmin_{y \in D} f_i(y)$$

is nonempty and $T_j(p) = \{p\}$ for each $p \in \Omega$. Let $\{x_n\}$ be the iterative process defined by (3.1), $a_{n,j} \in [a, b] \subset (0, 1)(j = 0, 1, \dots, m)$ and $\{\lambda_n^i\}$ is a sequence such that $\lambda_n^i \geq \lambda_0 > 0$ for all $n \in \mathbb{N}$ $(i = 1, \dots, r)$ and for some λ_0. Then

(i) $\lim_{n\to\infty} d(x_n, p)$ *exists for all* $p \in \Omega$,

(ii) $\lim_{n\to\infty} d(x_n, Prox^{f_i}_{\lambda_n^i} \circ \cdots \circ Prox^{f_1}_{\lambda_n^1} x_n) = 0 \quad (i = 1, \dots, r)$,

(iii) $\lim_{n\to\infty} \operatorname{dist}(T_j x_n, x_n) = 0 \quad (j = 1, \dots, m)$.

Proof. Let $p \in \Omega$. By Lemma 2.8, $p = Prox^{f_i}_{\lambda_n^i} p$ for any $i = 1, \dots, r$ and $n \in \mathbb{N}$.
(i) We show that $\lim_{n\to\infty} d(x_n, p)$ exists.
We denote by S_n^i the composition $Prox^{f_i}_{\lambda_n^i} \circ \cdots \circ Prox^{f_1}_{\lambda_n^1}$ for any $i = 1, 2, \dots, r$ and $n \in \mathbb{N}$. Therefore $w_n = S_n^r x_n$. We also assume that $S_n^0 = I$ where I is the identity operator. By Lemma 2.8, we have

$$\begin{aligned} d(S_n^i x_n, p) &= d(Prox^{f_i}_{\lambda_n^i} \circ \cdots \circ Prox^{f_1}_{\lambda_n^1} x_n, Prox^{f_i}_{\lambda_n^i} p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.2}$$

By employing Lemma 2.4, we obtain

$$\begin{aligned} d(y_{n,0},p) &= d((1-a_{n,o})x_n \oplus a_{n,o}w_n,p) \\ &\leq (1-a_{n,o})d(x_n,p) + a_{n,o}d(w_n,p) \\ &\leq d(x_n,p). \end{aligned} \tag{3.3}$$

So by (3.3), we obtain

$$\begin{aligned} d(y_{n,1},p) &= d((1-a_{n,1})x_n \oplus a_{n,1}z_{n,1},p) \\ &\leq (1-a_{n,1})d(x_n,p) + a_{n,1}d(z_{n,1},p) \\ &= (1-a_{n,1})d(x_n,p) + a_{n,1}\operatorname{dist}(z_{n,1},T_1(p)) \\ &\leq (1-a_{n,1})d(x_n,p) + a_{n,1}H(T_1(y_{n,0}),T_1(p)) \\ &\leq (1-a_{n,1})d(x_n,p) + a_{n,1}d(y_{n,0},p) \\ &\leq (1-a_{n,1})d(x_n,p) + a_{n,1}d(x_n,p) \\ &= d(x_n,p), \end{aligned}$$

and

$$\begin{aligned} d(y_{n,2},p) &= d((1-a_{n,2})x_n \oplus a_{n,2}z_{n,2},p) \\ &\leq (1-a_{n,2})d(x_n,p) + a_{n,2}d(z_{n,2},p) \\ &= (1-a_{n,2})d(x_n,p) + a_{n,2}\operatorname{dist}(z_{n,2},T_2(p)) \\ &\leq (1-a_{n,2})d(x_n,p) + a_{n,2}H(T_2(y_{n,1}),T_2(p)) \\ &\leq (1-a_{n,2})d(x_n,p) + a_{n,2}d(y_{n,1},p) \\ &\leq d(x_n,p). \end{aligned}$$

By induction, we have

$$\begin{aligned} d(y_{n,m-1},p) &= d((1-a_{n,m-1})x_n \oplus a_{n,m-1}z_{n,m-1},p) \\ &\leq (1-a_{n,m-1})d(x_n,p) + a_{n,m-1}d(z_{n,m-1},p) \\ &= (1-a_{n,m-1})d(x_n,p) + a_{n,m-1}\operatorname{dist}(z_{n,m-1},T_{m-1}(p)) \\ &\leq (1-a_{n,m-1})d(x_n,p) + a_{n,m-1}H(T_{m-1}(y_{n,m-2}),T_{m-1}(p)) \\ &\leq (1-a_{n,m-1})d(x_n,p) + a_{n,m-1}d(y_{n,m-2},p) \\ &\leq d(x_n,p), \end{aligned}$$

and also

$$\begin{aligned} d(x_{n+1},p) &= d((1-a_{n,m})x_n \oplus a_{n,m}z_{n,m},p) \\ &\leq (1-a_{n,m})d(x_n,p) + a_{n,m}d(z_{n,m},p) \\ &= (1-a_{n,m})d(x_n,p) + a_{n,m}\operatorname{dist}(z_{n,m},T_m(p)) \\ &\leq (1-a_{n,m})d(x_n,p) + a_{n,m}H(T_m(y_{n,m-1}),T_m(p)) \\ &\leq (1-a_{n,m})d(x_n,p) + a_{n,m}d(y_{n,m-1},p) \\ &\leq d(x_n,p). \end{aligned}$$

This shows that $\lim_{n\to\infty} d(x_n,p)$ exists.
(ii) We show that $\lim_{n\to\infty} d(x_n, Prox^{f_i}_{\lambda^i_n} \circ \cdots \circ Prox^{f_1}_{\lambda^1_n} x_n) = 0\,(i=1,\ldots,r)$.
Since $d(S^i_n x_n,p) - d(x_n,p) \leq d(x_n,p) - d(x_n,p)$, we get

$$\limsup_{n\to\infty} \big(d(S^i_n x_n,p) - d(x_n,p)\big) \leq 0, \tag{3.4}$$

for all $(i=1,\ldots,r)$. By using Lemma 2.4 and nonexpansivity of $Prox^{f_i}_{\lambda^i_n}$, for all $i=1,\ldots,r$, we have

$$\begin{aligned} d(y_{n,0},p) &= d((1-a_{n,o})x_n \oplus a_{n,o}w_n,p) \\ &\leq (1-a_{n,o})d(x_n,p) + a_{n,o}d(w_n,p) \\ &\leq (1-a_{n,o})d(x_n,p) + a_{n,o}d(S^i_n x_n,p), \end{aligned}$$

and

$$\begin{aligned} d(y_{n,1},p) &= d((1-a_{n,1})x_n \oplus a_{n,1}z_{n,1},p) \\ &\leq (1-a_{n,1})d(x_n,p) + a_{n,1}d(z_{n,1},p) \\ &= (1-a_{n,1})d(x_n,p) + a_{n,1}\operatorname{dist}(z_{n,1},T_1(p)) \\ &\leq (1-a_{n,1})d(x_n,p) + a_{n,1}H(T_1(y_{n,0}),T_1(p)) \\ &\leq (1-a_{n,1})d(x_n,p) + a_{n,1}d(y_{n,0},p) \\ &\leq (1-a_{n,1})d(x_n,p) + a_{n,1}((1-a_{n,o})d(x_n,p) + a_{n,o}d(S^i_n x_n,p)) \\ &= d(x_n,p) + a_{n,1}a_{n,0}(d(S^i_n x_n,p) - d(x_n,p)). \end{aligned}$$

Similarly, we have

$$\begin{aligned} d(y_{n,m-1},p) &= d((1-a_{n,m-1})x_n \oplus a_{n,m-1}z_{n,m-1},p) \\ &\leq (1-a_{n,m-1})d(x_n,p) + a_{n,m-1}d(z_{n,m-1},p) \\ &= (1-a_{n,m-1})d(x_n,p) + a_{n,m-1}\operatorname{dist}(z_{n,m-1},T_{m-1}(p)) \\ &\leq (1-a_{n,m-1})d(x_n,p) + a_{n,m-1}H(T_{m-1}(y_{n,m-2}),T_{m-1}(p)) \\ &\leq (1-a_{n,m-1})d(x_n,p) + a_{n,m-1}d(y_{n,m-2},p) \\ &\leq d(x_n,p) + a_{n,m-1},\ldots,a_{n,0}(d(S^i_n x_n,p) - d(x_n,p)), \end{aligned}$$

and also

$$\begin{aligned} d(x_{n+1}, p) &= d((1-a_{n,m})x_n \oplus a_{n,m} z_{n,m}, p) \\ &\leq (1-a_{n,m})d(x_n, p) + a_{n,m} d(z_{n,m}, p) \\ &= (1-a_{n,m})d(x_n, p) + a_{n,m} \operatorname{dist}(z_{n,m}, T_m(p)) \\ &\leq (1-a_{n,m})d(x_n, p) + a_{n,m} H(T_m(y_{n,m-1}), T_m(p)) \\ &\leq (1-a_{n,m})d(x_n, p) + a_{n,m} d(y_{n,m-1}, p) \\ &\leq d(x_n, p) + a_{n,m}, \ldots, a_{n,0}(d(S_n^i x_n, p) - d(x_n, p)), \end{aligned}$$

and hence

$$\liminf_{n\to\infty} \big(d(S_n^i x_n, p) - d(x_n, p)\big) \geq 0.$$

From the above inequality and (3.4), we obtain that

$$\lim_{n\to\infty} \big(d(S_n^i x_n, p) - d(x_n, p)\big) = 0.$$

Using Lemma 2.4 and firmly nonexpansivity of $Prox_{\lambda_n^i}^{f_i}$ for all $i = 1, 2, \ldots, r$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(S_n^i x_n, p)^2 =& d(Prox_{\lambda_n^i}^{f_i}(S_n^{i-1} x_n), Prox_{\lambda_n^i}^{f_i} p)^2 \\ \leq& d((1-\lambda)S_n^{i-1} x_n \oplus \lambda S_n^i x_n, (1-\lambda)p \oplus \lambda p)^2 \\ \leq& (1-\lambda)d(S_n^{i-1} x_n, p)^2 + \lambda d(S_n^i x_n, p)^2 \\ &- \lambda(1-\lambda)d(S_n^{i-1} x_n, S_n^i x_n)^2 \\ \leq& (1-\lambda)d(x_n, p)^2 + \lambda d(x_n, p)^2 - \lambda(1-\lambda)d(S_n^{i-1} x_n, S_n^i x_n)^2 \\ =& d^2(x_n, p) - \lambda(1-\lambda)d(S_n^{i-1} x_n, S_n^i x_n)^2, \end{aligned}$$

for all $\lambda \in (0, 1)$, which implies that

$$d(S_n^{i-1} x_n, S_n^i x_n)^2 \leq \frac{1}{\lambda(1-\lambda)}(d(x_n, p)^2 - d(S_n^i x_n, p)^2).$$

Therefore

$$\lim_{n\to\infty} d(S_n^{i-1} x_n, S_n^i x_n) = 0,$$

and hence for all $i = 1, 2, \ldots, r$,

$$d(x_n, S_n^i x_n) \leq d(x_n, S_n^1 x_n) + \cdots + d(S_n^{i-1} x_n, S_n^i x_n) \to 0.$$

(iii) We show that $\lim_{n\to\infty} \operatorname{dist}(x_n, T_j x_n) = 0$ $(j = 1, \ldots, m)$. By using Lemma 2.4 and (3.3), we get

$$\begin{aligned}
&d(y_{n,1}, p)^2 = d((1-a_{n,1})x_n \oplus a_{n,1}z_{n,1}, p)^2\\
&\le (1-a_{n,1})d(x_n,p)^2 + a_{n,1}d(z_{n,1},p)^2 - a_{n,1}(1-a_{n,1})d(x_n,z_{n,1})^2\\
&= (1-a_{n,1})d(x_n,p)^2 + a_{n,1}\operatorname{dist}(z_{n,1},T_1(p))^2 - a_{n,1}(1-a_{n,1})d(x_n,z_{n,1})^2\\
&\le (1-a_{n,1})d(x_n,p)^2 + a_{n,1}H(T_1(y_{n,0}),T_1(p))^2 - a_{n,1}(1-a_{n,1})d(x_n,z_{n,1})^2\\
&\le (1-a_{n,1})d(x_n,p)^2 + a_{n,1}d(y_{n,0},p)^2 - a_{n,1}(1-a_{n,1})d(x_n,z_{n,1})^2\\
&\le (1-a_{n,1})d(x_n,p)^2 + a_{n,1}d(x_n,p)^2 - a_{n,1}(1-a_{n,1})d(x_n,z_{n,1})^2\\
&= d(x_n,p)^2 - a_{n,1}(1-a_{n,1})d(x_n,z_{n,1})^2.
\end{aligned}$$

By induction, we have

$$\begin{aligned}
&d(x_{n+1}, p)^2 = d((1-a_{n,m})x_n \oplus a_{n,m}z_{n,m}, p)^2\\
&\le (1-a_{n,m})d(x_n,p)^2 + a_{n,m}d(z_{n,m},p)^2 - a_{n,m}(1-a_{n,m})d(x_n,z_{n,m})^2\\
&= (1-a_{n,m})d(x_n,p)^2 + a_{n,m}\operatorname{dist}(z_{n,m},T_m(p))^2\\
&\quad - a_{n,m}(1-a_{n,m})d(x_n,z_{n,m})^2\\
&\le (1-a_{n,m})d(x_n,p)^2 + a_{n,m}H(T_m(y_{n,m-1}),T_m(p))^2\\
&\quad - a_{n,m}(1-a_{n,m})d(x_n,z_{n,m})^2\\
&\le (1-a_{n,m})d(x_n,p)^2 + a_{n,m}d(y_{n,m-1},p)^2 - a_{n,m}(1-a_{n,m})d(x_n,z_{n,m})^2\\
&\le d(x_n,p)^2 - a_{n,m}(1-a_{n,m})d(x_n,z_{n,m})^2\\
&\quad - a_{n,m}a_{n,m-1}(1-a_{n,m-1})d(x_n,z_{n,m-1})^2\\
&\quad - \cdots - a_{n,m}a_{n,m-1}a_{n,m-2}\cdots a_{n,1}(1-a_{n,1})d(x_n,z_{n,1})^2.
\end{aligned}$$

So we obtain

$$\begin{aligned}
a^m(1-b)d(x_n,z_{n,1})^2 &\le a_{n,m}a_{n,m-1}\cdots a_{n,1}(1-a_{n,1})d(x_n,z_{n,1})^2\\
&\le d(x_n,p)^2 - d(x_{n+1},p)^2.
\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} a^m(1-b)d(x_n,z_{n,1})^2 \le d(x_1,p)^2 < \infty,$$

thus $\lim_{n\to\infty} d(x_n, z_{n,1}) = 0$. Similarly for $k = 2, \ldots, m$, we obtain

$$\lim_{n\to\infty} d(x_n, z_{n,k}) = 0.$$

Hence $\operatorname{dist}(x_n, T_k y_{n,k-1}) \le d(x_n, z_{n,k}) \to 0$ as $n \to \infty$ for $k = 1, \ldots, m$. We have

$$\lim_{n\to\infty} d(x_n, y_{n,k-1}) = \lim_{n\to\infty} a_{n,k-1}d(x_n, z_{n,k-1}) = 0.$$

By the condition (E), we get for some $\mu \geq 1$,

$$\begin{aligned}
&\operatorname{dist}(x_n, T_k x_n) \leq d(x_n, y_{n,k-1}) + \operatorname{dist}(y_{n,k-1}, T_k x_n)\\
&\leq d(x_n, y_{n,k-1}) + \mu \operatorname{dist}(T_k y_{n,k-1}, y_{n,k-1}) + d(x_n, y_{n,k-1})\\
&\leq d(x_n, y_{n,k-1}) + \mu \operatorname{dist}(x_n, T_k y_{n,k-1}) + \mu d(x_n, y_{n,k-1}) + d(x_n, y_{n,k-1})\\
&= (\mu+2) d(x_n, y_{n,k-1}) + \mu \operatorname{dist}(x_n, T_k y_{n,k-1}).
\end{aligned}$$

Hence for $k = 1, \ldots, m$, we have

$$\lim_{n\to\infty} \operatorname{dist}(x_n, T_k x_n) = 0.$$

□

Now we are ready to state and prove our main results.

Theorem 3.2. *Let D be a nonempty closed convex subset of an Hadamard space X and $f_i : D \to (-\infty, \infty]$ $(i = 1, 2, \ldots, r)$ be r proper convex and lower semi-continuous functions. Let $T_j : D \to K(D)$ $(j = 1, \ldots, m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E) such that*

$$\Omega = \cap_{j=1}^{m} F(T_j) \bigcap \cap_{i=1}^{r} argmin_{y\in D} f_i(y)$$

is nonempty and $T_j(p) = \{p\}$ for each $p \in \Omega$. Let $\{x_n\}$ be the iterative process defined by (3.1), $a_{n,j} \in [a, b] \subset (0, 1)$ $(j = 0, 1, \ldots, m)$ and $\{\lambda_n^i\}$ is a sequence such that $\lambda_n^i \geq \lambda_0 > 0$ for all $n \in \mathbb{N}$ $(i = 1, \ldots, r)$ and for some λ_0. Then $\{x_n\}$ is $\triangle$-convergent to an element of Ω.

Proof. Since $\lambda_n^i \geq \lambda_0 > 0$, by Lemmas 2.9 and 3.1, we have

$$\begin{aligned}
&d\big(Prox_{\lambda_0}^{f_i}(S_n^{i-1}x_n), S_n^i x_n\big) = d\big(Prox_{\lambda_0}^{f_i}(S_n^{i-1}x_n), Prox_{\lambda_n^i}^{f_i}(S_n^{i-1}x_n)\big)\\
&= d\big(Prox_{\lambda_0}^{f_i}(S_n^{i-1}x_n), Prox_{\lambda_0}^{f_i}(\frac{\lambda_n^i - \lambda_0}{\lambda_n^i} Prox_{\lambda_n^i}^{f_i}(S_n^{i-1}x_n) \oplus \frac{\lambda_0}{\lambda_n^i} S_n^{i-1}x_n)\big)\\
&\leq d\big(S_n^{i-1}x_n, (1-\frac{\lambda_0}{\lambda_n^i}) Prox_{\lambda_n^i}^{f_i}(S_n^{i-1}x_n) \oplus \frac{\lambda_0}{\lambda_n^i} S_n^{i-1}x_n\big)\\
&\leq (1-\frac{\lambda_0}{\lambda_n^i}) d\big(S_n^{i-1}x_n, Prox_{\lambda_n^i}^{f_i}(S_n^{i-1}x_n)\big)\\
&= (1-\frac{\lambda_0}{\lambda_n^i}) d(S_n^{i-1}x_n, S_n^i x_n) \to 0,
\end{aligned}$$

as $n \to \infty$. So, by Lemma 3.1, we obtain

$$\begin{aligned}
d(x_n, Prox_{\lambda_0}^{f_i} x_n) &\leq d\big(Prox_{\lambda_0}^{f_i} x_n, Prox_{\lambda_0}^{f_i}(S_n^{i-1}x_n)\big)\\
&\quad + d\big(Prox_{\lambda_0}^{f_i}(S_n^{i-1}x_n), S^i x_n\big) + d(S_n^i x_n, x_n)\\
&\leq d(x_n, S_n^{i-1}x_n)\\
&\quad + d\big(Prox_{\lambda_0}^{f_i}(S_n^{i-1}x_n), S_n^i x_n\big) + d(S_n^i x_n, x_n) \to 0,
\end{aligned}$$

as $n \to \infty$. Lemma 3.1(i) shows that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \Omega$ and Lemma 3.1(iii) also implies that $\lim_{n\to\infty} \text{dist}(x_n, T_j x_n) = 0$ for all $j = 1, \ldots, m$. Now we let $W_\triangle(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_\triangle(x_n) \subset \Omega$. Let $u \in W_\triangle(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.1 and 2.2 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim_{n\to\infty} v_n = v \in D$. Since $\lim_{n\to\infty} \text{dist}(v_n, T_j v_n) = 0$, by Lemma 2.12, we have $v \in \cap_{j=1}^m F(T_j)$ for $j = 1, \ldots, m$. Also by Lemma 2.5, nonexpansivity of $Prox_{\lambda_0}^{f_i}$ and

$$\lim_{n\to\infty} d(x_n, Prox_{\lambda_0}^{f_i} x_n) = 0,$$

we get $Prox_{\lambda_0}^{f_i}(v) = v$ for all $i = 1, 2, \ldots, r$. By using Lemma 2.8, we get

$$v \in \cap_{i=1}^r argmin f_i.$$

So, $u = v$ by Lemma 2.3. This shows that $W_\triangle(x_n) \subset \Omega$.
Finally, we show that the sequence $\{x_n\}$ $\triangle$-converges to a point in Ω. To this end, it suffices to show that $W_\triangle(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $\{d(x_n, u)\}$ converges, by Lemma 2.3, we have $x = u$. Hence $W_\triangle(x_n) = \{x\}$. This completes the proof. □

Theorem 3.3. *Let D be a nonempty compact convex subset of an Hadamard space X and $f_i : D \to (-\infty, \infty]$ $(i = 1, 2, \ldots, r)$ be r proper convex and lower semi-continuous functions. Let $T_j : D \to CB(D)$ $(j = 1, \ldots, m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E) such that*

$$\Omega = \cap_{j=1}^m F(T_j) \bigcap \cap_{i=1}^r argmin_{y\in D} f_i(y)$$

is nonempty and $T_j(p) = \{p\}$ for each $p \in \Omega$. Let $\{x_n\}$ be the iterative process defined by (3.1), $a_{n,j} \in [a, b] \subset (0, 1)$ $(j = 0, 1, \ldots, m)$ and $\{\lambda_n^i\}$ is a sequence such that $\lambda_n^i \geq \lambda_0 > 0$ for all $n \in \mathbb{N}$ $(i = 1, \ldots, r)$ and for some λ_0. Then $\{x_n\}$ converges strongly to an element of Ω.

Proof. By Lemma 3.1 (iii), we have $\lim_{n\to\infty} \text{dist}(T_j x_n, x_n) = 0$ for $j = 1, \ldots, m$. Since D is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = w$ for some $w \in D$. As in the proof of Theorem 3.2,

$$\lim_{n\to\infty} d(x_{n_k}, Prox_{\lambda_0}^{f_i} x_{n_k}) = 0,$$

for all $i = 1, \ldots, r$. From nonexpansivity of $Prox_{\lambda_0}^{f_i}$ and Lemma 2.12, we have $w \in \cap_{i=1}^r argmin f_i$. On the other hand by the condition (E), we have for some $\mu \geq 1$

$$\begin{aligned} \text{dist}(w, T_j w) &\leq d(w, x_{n_k}) + \text{dist}(x_{n_k}, T_j w) \\ &\leq \mu \, \text{dist}(x_{n_k}, T_j x_{n_k}) + 2d(w, x_{n_k}) \to 0, \end{aligned}$$

as $k \to \infty$. Therefore, we get $w \in \Omega$. Since $\{x_{n_k}\}$ converges strongly to w and $\lim_{n\to\infty} d(x_n, w)$ exists (by Lemma 3.1), it follows that $\{x_n\}$ converges strongly to w. □

Let $\{C_j\}_{j=1}^r$ be a family of nonempty closed convex subsets of an Hadamard space (X, d) such that $\cap_{j=1}^r C_j \neq \emptyset$. The convex feasibility problem (CFP) is to find x in $\bigcap_{j=1}^r C_j$.

For a nonempty closed convex subset C of an Hadamard space (X, d) the indicator function

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \in X \backslash C, \end{cases}$$

is a proper convex and lower semi-continuous and $Prox_\lambda^{i_C} = P_C$. Therefore by letting $f_j = i_{C_j}$ $(j = 1, 2, \ldots, r)$ in Theorems 3.2 and 3.3, we get the following corollary.

Corollary 3.4. *Let D be a nonempty closed convex subset of an Hadamard space X and $\{C_i\}_{i=1}^r$ be a family of nonempty closed convex subsets of X. Let $T_j : D \to K(D)$ $(j = 1, \ldots, m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E) such that $\Omega = \cap_{j=1}^m F(T_j) \bigcap \cap_{i=1}^r C_i$ is nonempty and $T_j(p) = \{p\}$ for each $p \in \Omega$. Let for $x_0 \in D$, $\{x_n\}$ be the iterative process defined by:*

$$\begin{cases} w_n = P_{C_r} \circ \cdots \circ P_{C_1} x_n \\ y_{n,0} = (1 - a_{n,0})x_n \oplus a_{n,0} w_n, \\ y_{n,1} = (1 - a_{n,1})x_n \oplus a_{n,1} z_{n,1}, \\ y_{n,2} = (1 - a_{n,2})x_n \oplus a_{n,2} z_{n,2}, \\ . \\ . \\ . \\ y_{n,m-1} = (1 - a_{n,m-1})x_n \oplus a_{n,m-1} z_{n,m-1}, \\ x_{n+1} = (1 - a_{n,m})x_n \oplus a_{n,m} z_{n,m}, \end{cases}$$

where $z_{n,j} \in T_j(y_{n,j-1})$ for $j = 1, \ldots, m$ and $a_{n,j} \in [a, b] \subset (0, 1)(j = 0, 1, \ldots, m)$ then $\{x_n\}$ is $\triangle$-convergent to an element of Ω. Moreover, if D is a nonempty compact convex subset of X, then $\{x_n\}$ converges strongly to an element of Ω.

Since every Hilbert space is an Hadamard space, we obtain the following corollary.

Corollary 3.5. *Let D be a nonempty closed convex subset of a Hilbert space X and $f_i : D \to (-\infty, \infty]$ $(i = 1, 2, \ldots, r)$ be r proper convex and lower semi-continuous functions. Let $T_j : D \to K(D)$ $(j = 1, \ldots, m)$ be a finite family* *of quasi-nonexpansive multivalued mappings satisfying the condition* (E) *such that*

$$\Omega = \cap_{j=1}^{m} F(T_j) \bigcap \cap_{i=1}^{r} argmin_{y\in D} f_i(y)$$

is nonempty and $T_j(p) = \{p\}$ *for each* $p \in \Omega$*. Let for* $x_0 \in D$*,* $\{x_n\}$ *be the iterative process defined by:*

$$\begin{cases} w_n = Prox_{\lambda_n^r}^{f_r} \circ \cdots \circ Prox_{\lambda_n^1}^{f_1} x_n, \\ y_{n,0} = (1 - a_{n,0})x_n + a_{n,0}w_n, \\ y_{n,1} = (1 - a_{n,1})x_n + a_{n,1}z_{n,1}, \\ y_{n,2} = (1 - a_{n,2})x_n + a_{n,2}z_{n,2}, \\ . \\ . \\ . \\ y_{n,m-1} = (1 - a_{n,m-1})x_n + a_{n,m-1}z_{n,m-1}, \\ x_{n+1} = (1 - a_{n,m})x_n + a_{n,m}z_{n,m}, \end{cases}$$

where $z_{n,j} \in T_j(y_{n,j-1})$ *for* $j = 1, \ldots, m$ *and* $a_{n,j} \in [a, b] \subset (0, 1)$ $(j = 0, 1, \ldots, m)$ *also* $\lambda_n^i \geq \lambda_0 > 0$ *for all* $n \in \mathbb{N}$ $(i = 1, \ldots, r)$ *and for some* λ_0*. Then* $\{x_n\}$ *is* $\triangle$*-convergent to an element of* Ω*. Moreover, if* D *is a nonempty compact convex subset of* X*, then* $\{x_n\}$ *converges strongly to an element of* Ω*.*

Now, by using some idea from [22], we remove the restriction $T_j(p) = \{p\}$ $(j = 1, \ldots, m)$ for each $p \in \Omega$ and define the following iteration process.
Let D be a nonempty convex subset of an Hadamard space X and $f_i : D \to (-\infty, \infty]$ $(i = 1, 2, \ldots, r)$ be r proper convex and lower semi-continuous functions. Let $T_j : D \to P(D)$ $(j = 1, \ldots, m)$ be m given mappings and

$$P_{T_j}(x) = \{y \in T_j(x) : d(x, y) = \mathrm{dist}(x, T_j(x))\}.$$

Then for $x_0 \in D$, $a_{n,j} \in [0, 1]$ $(j = 0, 1, \ldots, m)$ and $\lambda_n^i > 0$ $(i = 1, \ldots, r)$, we consider the following iterative process:

$$\begin{cases} w_n = Prox_{\lambda_n^r}^{f_r} \circ \cdots \circ Prox_{\lambda_n^1}^{f_1} x_n, \\ y_{n,0} = (1 - a_{n,0})x_n \oplus a_{n,0}w_n, \\ y_{n,1} = (1 - a_{n,1})x_n \oplus a_{n,1}z_{n,1}, \\ y_{n,2} = (1 - a_{n,2})x_n \oplus a_{n,2}z_{n,2}, \\ . \\ . \\ . \\ y_{n,m-1} = (1 - a_{n,m-1})x_n \oplus a_{n,m-1}z_{n,m-1}, \\ x_{n+1} = (1 - a_{n,m})x_n \oplus a_{n,m}z_{n,m}, \end{cases} \tag{3.5}$$

where $z_{n,j} \in P_{T_j}(y_{n,j-1})$ for $j = 1, \ldots, m$.

Theorem 3.6. *Let D be a nonempty closed convex subset of an Hadamard space X and $f_i : D \to (-\infty, \infty]$ $(i = 1, \ldots, r)$ be r proper convex and lower semi-continuous functions. Let $T_j : D \to P(D)$ $(j = 1, \ldots, m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E) such that*

$$\Omega = \cap_{j=1}^{m} F(T_j) \bigcap \cap_{i=1}^{r} argmin_{y \in D} f_i(y)$$

is nonempty and P_{T_j} is nonexpansive. Let $\{x_n\}$ be the iterative process defined by (3.4), $a_{n,j} \in [a, b] \subset (0, 1)$ $(j = 0, 1, \ldots, m)$ and $\{\lambda_n^i\}$ is a sequence such that $\lambda_n^i \geq \lambda_0 > 0$ for all $n \in \mathbb{N}$ $(i = 1, \ldots, r)$ and for some λ_0. Assume that there exists an increasing function $g : [0, \infty) \to [0, \infty)$ with $g(r) > 0$ for all $r > 0$ such that for some $j = 1, \ldots, m$

$$\operatorname{dist}(x_n, T_j(x_n)) \geq g(\operatorname{dist}(x_n, \Omega)).$$

Then $\{x_n\}$ converges strongly to an element of Ω.

Proof. Let $p \in \Omega$. Then $p \in P_{T_j}(p) = \{p\}$ for $j = 1, \ldots, m$. As in the proof of Lemma 3.1, $\lim_{n\to\infty} \operatorname{dist}(T_j x_n, x_n) = 0$ and $d(x_{n+1}, p) \leq d(x_n, p)$. Hence by assumption $\lim_{n\to\infty} \operatorname{dist}(x_n, \Omega) = 0$. Therefore, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\}$ in Ω such that for all $k \in \mathbb{N}$

$$d(x_{n_k}, p_k) < \frac{1}{2^k},$$

then, we have

$$d(x_{n_{k+1}}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Hence

$$d(p_{k+1}, p_k) \leq d(x_{n_{k+1}}, p_{k+1}) + d(x_{n_{k+1}}, p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

Consequently, $\{p_k\}$ is a Cauchy sequence in D and hence converges to some $q \in D$. Since for $j = 1, \ldots, m$,

$$\operatorname{dist}(p_k, T_j(q)) \leq \operatorname{dist}(p_k, P_{T_j}(q)) \leq H(P_{T_j}(p_k), P_{T_j}(q)) \leq d(p_k, q),$$

and $p_k \to q$ as $k \to \infty$, we get $\operatorname{dist}(q, T_j(q)) = 0$ for $j = 1, \ldots, m$, so $q \in \cap_{j=1}^{m} F(T_j)$. Also since

$$\begin{aligned} d(p_k, Prox_{\lambda_0}^{f_i} q) &= d(Prox_{\lambda_0}^{f_i} p_k, Prox_{\lambda_0}^{f_i} q) \\ &\leq d(p_k, q), \end{aligned}$$

and $p_k \to q$ as $k \to \infty$, we get $Prox_{\lambda_0}^{f_i} q = q$ for $i = 1, \ldots, r$, so

$$q \in \cap_{i=1}^{r} argmin f_i.$$

Therefor $q \in \Omega$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} d(x_n, q)$ exists, we conclude that $\{x_n\}$ converges strongly to q. □

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