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SEMI-ROTHBERGER AND RELATED SPACES

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Abstract. In this paper our focus is to study certain covering properties in topological spaces by using semi-open covers. A part of this article deals with Rothberger-type covering properties. The notions of s-Rothberger, almost s-Rothberger, star s-Rothberger, almost star s-Rothberger, strongly star s-Rothberger spaces are defined and corresponding properties are investigated.
Keywords: Semi-open set, (star) semi-compact space, semi-Lindelöf space, s-Rothberger space, star s-Rothberger space.
MSC(2010): Primary: 54D20; Secondary: 54C08.

1. Introduction

Our main focus in this paper is to study various covering properties, in particular selection principles, by using semi-open covers. We will deal with variations of the following classical selection principle:

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X. Then $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$, and $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$ (see [25]).

If $\mathcal{O}$ denotes the family of all open covers of a space X, the property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ is called the *Rothberger* (*covering*) *property*.

In 1963, N. Levine [19] defined semi-open sets in topological spaces. Since then, many mathematicians generalized different concepts and explored their properties in new setting. A set A in a topological space X is *semi-open* if and only if there exists an open set $O \subset X$ such that $O \subset A \subset \mathrm{cl}(O)$, where $\mathrm{cl}(O)$ denotes the closure of the set O. Equivalently, A is semi-open if and only if $A \subset \mathrm{cl}(\mathrm{int}(A))$ ($\mathrm{int}(A)$ is the interior of A). If A is semi-open, then its

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complement is called *semi-closed* [6]. Every open set is semi-open, whereas a semi-open set may not be open. The union of any number of semi-open sets is semi-open, but the intersection of two semi-open sets may not be semi-open. The intersection of an open set and a semi-open set is always semi-open. The collection of all semi-open subsets of X is denoted by $\mathsf{SO}(X)$. According to [6], the semi-closure and semi-interior were defined analogously to the closure and interior: the *semi-interior* $\mathrm{sint}(A)$ of a set $A \subset X$ is the union of all semi-open subsets of A); the *semi-closure* $\mathrm{scl}(A)$ of $A \subset X$ is the intersection of all semi-closed sets containing A. A set A is semi-open if and only if $\mathrm{sint}(A) = A$, and A is semi-closed if and only if $\mathrm{scl}(A) = A$. Note that for any subset A of X

$$\mathrm{int}(A) \subset \mathrm{sint}(A) \subset A \subset \mathrm{scl}(A) \subset \mathrm{cl}(A).$$

The semi θ-closure [4] of a set A is the set of all points $x \in X$ for which $A \cap \mathrm{scl}(V) \neq \phi$ for every semi-open set V containing x. It is denoted as $\mathrm{scl}_\theta(A)$.

A subset A of a topological space X is called a *semi-regular set* if it is semi-open as well as semi-closed or equivalently, $A = \mathrm{scl}(\mathrm{sint}(A))$ or $A = \mathrm{sint}(\mathrm{scl}(A))$. The collection of all semi-regular subsets of X is denoted as $\mathsf{SR}(X)$.

A mapping $f : (X, \tau_X) \to (Y, \tau_Y)$ is called:

(1) *semi-continuous* if the preimage of every open set in Y is semi open;
(2) *s-continuous* if preimage of every semi-open set in Y is open in X;
(3) *irresolute* [7] if $f^{-1}(O)$ is semi-open in X for every semi-open O in Y;
(4) *semi-homeomorphism* if f is a bijection and images and preimages of semi-open sets are semi-open. Or f is irresolute and pre-semiopen;
(5) a *quasi-irresolute* if for every semi-regular set A in Y the set $f^{-1}(A)$ is semi-regular in X [8].

For more details on semi-open sets and semi-continuity, we refer to [3,5–7,19].

2. Preliminaries

Throughout this paper a space X is an infinite topological space (X, τ) on which no separation axioms are assumed unless otherwise stated. We use the standard topological notation and terminology as in [12].

A semi-open cover $\mathcal{U}$ of a space X is called;

- an s$\omega-$cover if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$.
- an s$\gamma-$cover if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$.
- s-large if each $x \in X$ belongs to infinitely many elements of $\mathcal{U}$.
- s-groupable if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n, n \in \mathbb{N}$, such that each $x \in X$ belongs to $\cup\, \mathcal{U}_n$ for all but finitely many n.

For a topological space X we denote:

- s$\mathcal{O}-$the family of covers of X by semi-open sets in X.
- $\mathcal{D}-$the family of dense subsets of X.
- s$\Omega-$the family of s$\omega-$covers of X.
- s$\Gamma-$the family of s$\gamma-$covers of X.
- s$\Lambda-$the family of s-large covers of X.
- s$\mathcal{O}^{gp}-$the family of s-groupable covers of X.

Definition 2.1. A space X is called:

- *semi-compact* [10] if every cover of X by semi open sets has a finite subcover;
- *semi-Lindelöf* [13] if every cover of X by semi-open sets has a countable subcover.

Definition 2.2 ([11])**.** A space X is *semi-regular* if for each semi-closed set A and $x \notin A$ there exist disjoint semi-open sets U and V such that $x \in U$ and $A \subset V$.

Lemma 2.3 ([11])**.** *The following are equivalent in a space X:*

(i) *X is a semi-regular space;*
(ii) *For each $x \in X$ and $U \in \mathsf{SO}(X)$ such that $x \in U$, there exists $V \in \mathsf{SO}(X)$ such that $x \in V \subset \mathrm{scl}(V) \subset U$;*
(iii) *For each $x \in X$ and each $U \in \mathsf{SO}(X)$ with $x \in U$, there is a semi-regular set $V \subset X$ such that $x \in V \subset U$.*

3. Semi-Rothberger spaces

Definition 3.1. A space X is said to have the *semi-Rothberger property* (or *s-Rothberger property)* if it satisfies $\mathsf{S}_1(s\mathcal{O}, s\mathcal{O})$.

Clearly, we have the following diagram:

$$\begin{array}{ccc} s\text{–Rothberger} & \Rightarrow & \text{semi–}\mathsf{Lindelof} \\ \Downarrow & & \Downarrow \\ \text{Rothberger} & \Rightarrow & \text{Lindelof} \end{array}$$

Example 3.2. (1) A compact space may not be an s-Rothberger space. We know that the closed unit interval [0,1] is compact (as well as Eberlein compact), but it is not a Rothberger space as pointed out in [24]. This implies that [0,1] is not s-Rothberger.

(2) Let the real line R be endowed with the topology $\tau = \{\emptyset, R, (-\infty, x) : x \in R\}$. Then, as it is easy to see, (R, τ) is a semi-Rothberger space, but it is not semi-compact.

(3) Every semi-Rothberger space is a Rothberger space, but the converse is not true in general. The Lusin space is a Rothberger space but not s-Rothberger space. It follows from the fact that Sierpiński pointed out in [26] that Lusin sets have Rothberger property, but in [22], it is observed that Lusin space space is not s-Menger and hence it is not an s-Rothberger space.

(4) The Sorgenfrey line S is a (hereditarily) Lindelöf space which is not semi-Menger [22] hence not semi-Rothberger. The space of irrationals with the usual metric topology also is not semi-Menger [22] hence not semi-Rothberger.

Proposition 3.3. *The following statements are true:*

(1) *An irresolute image of a semi-Rothberger space is semi-Rothberger; in particular, continuous open images of semi-Rothberger spaces are semi-Rothberger;*

(2) *An s-continuous image of a Rothberger space is semi-Rothberger;*

(3) *A semi-continuous* (*in particular, continuous*) *image of a semi-Rothberger space is Rothberger;*

(4) *A semi-regular subspace of a semi-Rothberger space is also semi-Rothberger.*

Proof. By applying definitions, we can easily prove the statements (1) -(3). We prove only (4). Let S be a semi-regular subspace of X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of S. As $S \in SO(X)$, thus each $\mathcal{U}_n$ is a collection of semi-open sets in X. On the other hand, since S is also semi-closed, we conclude that each $\mathcal{U}_n \cup \{X \setminus S\} = \mathcal{G}_n$ is a semi-open cover of X. Semi-Rothbergerness of X implies the existence of sets $G_n \in \mathcal{G}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}}\{G_n\}$ is a semi-open cover of X. Let $\mathcal{W} = \{G_n \in \mathcal{G}_n,\ n \in \mathbb{N}\}$. It follows that the set $\mathcal{V} = \mathcal{W} \setminus \{X \setminus S\}$ witness for $(\mathcal{U}_n : n \in \mathbb{N})$ that S is semi-Rothberger. □

A property which is preserved by semi-homeomorphisms is called a *semi-topological property* [6].

Remark 3.4. From the previous proposition we see that the semi-Rothbergerness is a semi-topological property.

Let $X = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ be the upper half-plane. Endow X with the following two topologies: τ_1 is the subspace topology of the usual metric topology on $\mathbb{R}^2$, and τ_2 is the Niemytzki tangent disc topology (called also the Niemytzki plane) [12]. (X, τ_2) is not s-Rothberger because it is not Lindelöf. On the other hand, $\mathsf{SO}(X, \tau_1) = \mathsf{SO}(X, \tau_2)$ [7]. Therefore, the mapping $id_X : (X, \tau_1) \to (X, \tau_2)$ is semi-homeomorphism. Hence, by Proposition 3.3, we conclude that (X, τ_1) is not s-Rothberger.

A mapping $f : X \to Y$ is called *s-perfect* if for each semi-closed set $A \subset X$ the set $f(A)$ is semi-closed in Y and for each $y \in Y$ its preimage $f^{-1}(y)$ is semi-compact relative to X.

Theorem 3.5. *If f is an s-perfect mapping from a space X onto a semi-Rothberger space Y, then X is also semi-Rothberger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. For each n and each $y \in Y$ there is a finite subcollection $\mathcal{G}_n^y$ of $\mathcal{U}_n$ covering $f^{-1}(y)$. Set $G_n^y = \cup \mathcal{G}_n^y$ and $W_n^y = Y \setminus f(X \setminus G_n^y)$. Then $y \in W_n^y$, and $\mathcal{W}_n = \{W_n^y : y \in Y\}$ is a semi-open cover of Y for each $n \in \mathbb{N}$. Since Y is semi-Rothberger for each n, there is a set $H_n \in \mathcal{W}_n$ such that $Y = \bigcup_{n\in\mathbb{N}} H_n$. To each H_n associate the set $\mathcal{G}_n^y$ which occur in the representation of G_n^y for which $H = Y \setminus f(X \setminus G_n^y)$. In this way for each n we have chosen a set U_n of $\mathcal{U}_n$. Evidently, $X = \bigcup_{n\in\mathbb{N}} U_n$, so that X is semi-Rothberger. □

Theorem 3.6. *For a space X the following are equivalent:*

(1) *X is s-Rothberger;*
(2) *X satisfies $\mathsf{S}_1(s\Omega, s\mathcal{O})$.*

Proof. (1) $\Rightarrow$ (2): It follows from the fact that every semi-ω-cover of X is a semi-open cover for X.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Partition $\mathbb{N}$ into pairwise disjoint infinite subsets N_i: $\mathbb{N} = N_1 \cup N_2 \cup \cdots \cup N_m \cup \cdots$. For each n let $\mathcal{V}_n$ be the set of all elements of the form

$$U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_k},\ n_1 \le \cdots \le n_k,\ n_i \in N_n, U_{n_i} \in \mathcal{U}_n, i \le k, k \in \mathbb{N}$$

which are not equal to X. Then every $\mathcal{V}_n$ is a semi-ω-cover of X. Applying (2) to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ we can choose a sequence $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ of sets such that for each n, $W_n \in \mathcal{V}_n$ and $\bigcup_{n\in\mathbb{N}} \bigcup_{W\in\mathcal{W}} W = X$. By the construction, each $W_n^i = U_n^{n_{i_1}} \cup \cdots \cup U_n^{n_{i_k}}$, so that in this way we get a member of $\mathcal{U}_p$ for some $p \in \mathbb{N}$ and these elements cover X. If there are no elements from some $\mathcal{U}_q$ chosen in this way we put $\mathcal{W}_q = \emptyset$. This gives that X is semi-Rothberger. □

It is known that Rothberger's covering property can be characterized game-theoretically and Ramsey-theoretically [25]. We do not know if it is the case for the semi-Rothberger property.

Problem 3.7. *Can semi-Rothbergerness be characterized game-theoretically or Ramsey-theoretically?*

In [16], the notion of almost Rothberger spaces was introduced, and in [14] this class of spaces has been studied. We make use of this concept and define analogously spaces with the help of semi-open covers.

Definition 3.8. A space X is *almost semi-Rothberger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of X there exists a sequence $(U_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{\mathrm{scl}(U_n) : U_n \in \mathcal{U}_{\mathrm{n}}\} = X$.

Proposition 3.9. *If a space X contains a dense subset which is semi-Rothberger in X, then X is almost semi-Rothberger.*

Proof. Let D be a dense subset of X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Since D is semi-Rothberger in X, there are sets $V_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$ such that $D \subset \bigcup_{n\in\mathbb{N}} V_n \subset \bigcup_{n\in\mathbb{N}} \mathrm{scl}(V_n)$. Since D is dense in X and $\mathrm{scl}(D) = \mathrm{cl}(D)$, we have $X = \bigcup_{n\in\mathbb{N}} \mathrm{scl}(V_n)$. □

The following two theorems show when an almost s-Rothberger space becomes s-Rothberger.

Theorem 3.10. *Let X be a semi-regular space. If X is an almost s-Rothberger space, then X is an s-Rothberger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Since X is a semi-regular space, using the equivalence condition there exists for each n a semi-open cover $\mathcal{V}_n$ of X such that $\mathcal{V}'_n = \{\mathrm{scl}(V) : V \in \mathcal{V}_n\}$ forms a refinement of $\mathcal{U}_n$. By assumption, there exists $\mathcal{W} =\{W_n : n \in \mathbb{N}\}$ such that for each n, W_n is a member of $\mathcal{V}_n$ and $\cup(\mathcal{W}' : n \in \mathbb{N})$ is a cover of X, where $\mathcal{W}' = \{\mathrm{scl}(W) : W \in \mathcal{W}\}$. For every $n \in \mathbb{N}$ and every $W \in \mathcal{W}$ we can choose $U_W \in \mathcal{U}_n$ such that $\mathrm{scl}(W) \subset U_W$. Let $\mathcal{U}' = \{U_W : W \in \mathcal{W}\}$. We shall prove that $\cup(\mathcal{U}' : n \in \mathbb{N})$ is a semi-open cover of X. Let $x \in X$. There exists $n \in \mathbb{N}$ and $\mathrm{scl}(W) \in \mathcal{W}'$ such that $x \in \mathrm{scl}(W)$. By construction, there exists $U_W \in \mathcal{U}'$ such that $\mathrm{scl}(W) \subset U_W$. Hence, $x \in U_W$. □

Theorem 3.11. *A space X is almost s-Rothberger if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-regular sets, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a member of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X.*

Proof. Let X be an almost s-Rothberger space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-regular sets. Since every semi-regular set is semi-open (as well as semi-closed), $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of semi-open covers of X. By assumption, there exists a sequence $(V_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, V_n is a member of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} V_n$ is a cover of X, where $\mathrm{scl}(V_n) = V_n$ for all $n \in \mathbb{N}$.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Let $(\mathcal{U}'_n : n \in \mathbb{N})$ be a sequence defined by $\mathcal{U}'_n = \{\mathrm{scl}(U) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}'_n$ is a cover of X by semi-regular sets. Thus there exists a sequence $(V_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, V_n is a member of $\mathcal{U}'_n$ and $\bigcup_{n\in\mathbb{N}} V_n$ is a cover of X. Let $\mathcal{V} =\{V_n : n \in \mathbb{N}\}$. By construction, for each $n \in \mathbb{N}$ and $V \in \mathcal{V}$, there exists $U_V \in \mathcal{U}_n$ such that $V = \mathrm{scl}(U_V)$. Hence, $\bigcup_{n\in\mathbb{N}}\{\mathrm{scl}(U_V) : V \in \mathcal{V}\} = X$. So, X is an almost s-Rothberger space. □

Theorem 3.12. *Let X be an almost s-Rothberger space, and Y be a topological space. If $f : X \to Y$ is a quasi-irresolute surjection, then Y is an almost s-Rothberger space.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of Y by semi-regular sets. Let $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ for each $n \in \mathbb{N}$. Then $(\mathcal{U}'_n : n \in \mathbb{N})$ is a sequence of semi-regular covers of X, since f is a quasi-irresolute surjection. Since X is an almost s-Rothberger space, there exists a sequence $(V_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $V_n \in \mathcal{U}'_n$ and $\bigcup_{n\in\mathbb{N}} V_n$ is a cover of X. Let $\mathcal{V} =\{V_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ and $V \in \mathcal{V}$ we can choose $U_V \in \mathcal{U}_n$ such that $V = f^{-1}(U_V)$. Let $\mathcal{W} = \{\mathrm{scl}(U_V) = U_V : V \in \mathcal{V}\}$. We will show that $\bigcup \mathcal{W}$ is a cover of Y.

If $y = f(x) \in Y$, then there exists $n \in \mathbb{N}$ and $V \in \mathcal{V}$ such that $x \in V$. Since $V = f^{-1}(U_V)$, $y = f(x) \in U_V \in \mathcal{W}$. □

A subset A of space X is called semi G_δ subset of X if it is a countable intersection of semi-open sets of X. A topological space X is called semi P-space if every intersection of countably many semi-open subsets of X is semi-open. Equivalently X is semi P-space if every semi G_δ subset of X is semi-open.

A space X is called weakly s-Rothberger space if it satisfies $\mathsf{S}_1(s\mathcal{O}, \mathcal{D})$. A space X is called almost semi-Lindelöf [22] if there is for each semi-open cover $\mathcal{U}$ of X a countable subset $\mathcal{V}$ such that $\{\mathrm{scl}(V) : V \in \mathcal{V}\}$ is a cover of X.

The following diagram shows relation between Rothberger type spaces and Lindelöf type spaces.

$$\begin{array}{ccccc}
s\text{–Rothberger} & \Rightarrow & \text{almost s} - \text{Rothberger} & \Rightarrow & \text{weakly s} - \text{Rothberger} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{semi} - \text{Lindelöf} & \Rightarrow & \text{almost semi} - \text{Lindelöf} & \Rightarrow & \text{weakly semi} - \text{Lindelöf} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{Lindelöf} & \Rightarrow & \text{almost Lindelöf} & \Rightarrow & \text{weakly Lindelöf}
\end{array}$$

Theorem 3.13. *If a topological space (X, τ) is weakly s-Rothberger semi P-space, then (X, τ) is almost s-Rothberger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Since X is weakly s-Rothberger, there exists a sequence $(U_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, U_n is an element of $\mathcal{U}_n$ and $\cup_{n\in\mathbb{N}} U_n$ is dense in X. Let $x \in X$. By the condition X is semi P-space, the intersection of every countable family of semi-open subsets of X is semi-open and hence, every countable union of semi-closed sets is semi-closed. So, $\mathrm{scl}(\cup_{n\in\mathbb{N}} U_n) = \cup_{n\in\mathbb{N}}\mathrm{scl}(U_n) = X$, which shows that X is an almost s-Rothberger space. □

A space X is called almost semi-Lindelöf if there is for each semi-open cover $\mathcal{U}$ of X a countable subset $\mathcal{V}$ such that $\{\mathrm{scl}(V) : V \in \mathcal{V}\}$ is a cover of X.

Theorem 3.14. *Every almost semi-Lindelöf semi P-space is almost s-Rothberger.*

Proof. Let X be an almost semi-Lindelöf semi P-space and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Assume that for each $n \in \mathbb{N}, \mathcal{U}_n$ is closed under finite unions. Put $\mathcal{U} = \{\cap_{n\in\mathbb{N}} U_n : U_n \in \mathcal{U}_n\}$. Then $\mathcal{U}$ is semi-open cover for X, since X is a semi P-space. As X is almost semi-Lindelöf, there exists a countable subset $\{V_n : n \in \mathbb{N}\}$ of $\mathcal{U}$ such that $\cup_{n\in\mathbb{N}}\mathrm{scl}(V_n) = X$. For all $n \in \mathbb{N}$, we can write $V_n = \cap_{k\in\mathbb{N}} U_{n_k}$, where $U_{n_k} \in \mathcal{U}_k$. But $\cup_{n\in\mathbb{N}}\mathrm{scl}(U_{n_n}) = X$, since $V_n \subset U_{n_n}$, for every $n \in \mathbb{N}$. Hence, X is almost s-Rothberger. □

Corollary 3.15. *Let X be a semi-regular semi P-space, then the following are equivalent:*

(1) X is s-Rothberger.
(2) X is almost s-Rothberger.
(3) X is weakly s-Rothberger.
(4) X is semi-Lindelöf.
(5) X is almost semi-Lindelöf.
(6) X is weakly semi-Lindelöf.

A topological space X is called d-paracompact space if every family of dense subsets of X has a locally finite refinement.

Theorem 3.16. *If X is weakly s-Rothberger and d-paracompact space, then X is almost Rothberger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Since X is weakly s-Rothberger, there exists a sequence $(V_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, V_n is a member of $\mathcal{U}_n$ and $\cup_{n\in\mathbb{N}} V_n$ is dense in X. Let $x \in X$. By the assumption $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ has a locally finite refinement $\mathcal{W}$ and therefore $\mathrm{cl}(\cup\mathcal{W}) = \mathrm{cl}(\cup_{n\in\mathbb{N}} V_n)$. As $\mathcal{W}$ is a locally finite family, $\mathrm{cl}(\cup\mathcal{W}) = \cup_{W\in\mathcal{W}}\mathrm{cl}(W)$. Since for every $W \in \mathcal{W}$ there exists $n \in \mathbb{N}$ and $V_W \in \mathcal{V}$, so that $W \subset V_W$, we have that $\cup_{n\in\mathbb{N}}\{\mathrm{cl}(V) : V \in \mathcal{V}\} = X$. Hence, it is shown that X is almost Rothberger. □

Theorem 3.17. *Every semi-regular subset of an almost s-Rothberger space is almost s-Rothberger.*

Proof. Let F be a semi-regular subset of an almost s-Rothberger space and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of F. Then $\mathcal{V}_n = \mathcal{U}_n \cup \{X - F\}$ is a semi-open cover for X for every $n \in \mathbb{N}$. Since X is an almost s-Rothberger space, there exists $\mathcal{V} = \{V_n : V_n \in \mathcal{V}_n; n \in \mathbb{N}\}$ for which

$$\cup_{n\in\mathbb{N}}\{\mathrm{scl}(V) : V \in \mathcal{V}\} = X$$

for each $n \in \mathbb{N}$. By semi-regularity of $X - F, \mathrm{scl}(X - F) = X - F$ and $\cup_{n\in\mathbb{N}}\{\mathrm{scl}(V) : V \in \mathcal{V}, V \neq X - F\}$ covers F. □

Theorem 3.18. *Every semi-regular subset of a weakly s-Rothberger space is weakly s-Rothberger.*

Proof. Let F be a semi-regular subset of a weakly s-Rothberger space and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of F. Then $\mathcal{V}_n = \mathcal{U}_n \cup \{X - F\}$ is a semi-open cover for X for every $n \in \mathbb{N}$. Since X is a weakly s-Rothberger space, there exists a set $\mathcal{V} = \{V_n : V_n \in \mathcal{V}_n; n \in \mathbb{N}\}$ such that $\cup_{n\in\mathbb{N}} V_n$ is dense in X. Put $\mathcal{W} = \cup_{n\in\mathbb{N}}\{V : V \in \mathcal{V}, V \neq X - F\}$. Then $\mathrm{scl}(\mathcal{W}) \cup (X - F) = X$. Since $F = \mathrm{scl}(\mathrm{sint}(F))$ we have $\mathrm{sint}(F) \cap \mathrm{scl}(X - F) = \phi$. So, $\mathrm{sint}(F) \subset \mathrm{scl}(\mathcal{W})$ and $F = \mathrm{scl}(\mathrm{sint}(F)) \subset \mathrm{scl}(\mathcal{W})$. □

4. Star covering properties

In this section we consider some properties defined in terms of stars with respect to semi-open covers.

The method of stars has been used quite extensively in recent years for definition and investigations of several important classical topological notions. For star covering properties we refer the reader to see [2, 21, 31]. Kočinac [15], gave motivation for the study of star selection principles and consequently, a substantial work on star covering properties appeared in [16–18, 20, 23, 27–30].

Let A be a subset of X and $\mathcal{U}$ be a collection of subsets of X. Then

$$\mathrm{st}(A,\mathcal{U}) = \mathrm{st}^1(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\},$$

and for $n = 1, 2, 3, \ldots$

$$\mathrm{st}^{n+1}(A,\mathcal{U}) = \mathrm{st}(\mathrm{st}^n(A,\mathcal{U}),\mathcal{U}).$$

We usually write $\mathrm{st}(x, U)$ for $\mathrm{st}(\{x\},\mathcal{U})$.

4.1. Star semi-Rothberger spaces.

Definition 4.1 ([15])**.** $\mathsf{S}_1{}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$, and $\{\mathrm{st}(U_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

Definition 4.2 ([15])**.** $\mathsf{SS}_1{}^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(x_n : n \in \mathbb{N})$ of elemens of X such that $\{\mathrm{st}(x_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

The symbols $\mathsf{S}_1{}^*(\mathcal{O},\mathcal{O})$ and $\mathsf{SS}_1{}^*(\mathcal{O},\mathcal{O})$ denotes the *star-Rothberger property* and *strongly star-Rothberger property*, respectively.

In a similar way we introduce the following definition.

Definition 4.3. (1) A space X is said to have the *star s-Rothberger property* if it satisfies $\mathsf{S}_1{}^*(s\mathcal{O}, s\mathcal{O})$.

(2) X is *strongly star s-Rothberger space* if it satisfies $\mathsf{SS}_1{}^*(s\mathcal{O}, s\mathcal{O})$.

Definition 4.4. A space X is called star semi-Lindelöf if every cover $\mathcal{U}$ of semi-open subsets of X has a countable subset $\mathcal{V}$ such that $\mathrm{st}(\cup\mathcal{V},\mathcal{U})$ covers X.

Definition 4.5. A space X is called strongly star semi-Lindelöf if for every cover $\mathcal{U}$ of semi-open subsets of X, there is a countable subset A of X such that $\mathrm{st}(A,\mathcal{U})$ covers X.

It is understood that every star s-Rothberger space is star semi-Lindelöf, and every strongly star s-Rothberger space is strongly star semi-Lindelöf. Every semi-Rothberger space is strongly star s-Rothberger.

Example 4.6. There is a strongly star s-Rothberger space which is not semi-Rothberger. Endow the real line $\mathbb{R}$ with the topology $\tau = \{\mathbb{R}, \emptyset, \{p\}\}$, where p is a point in $\mathbb{R}$. Each subset of $\mathbb{R}$ containing p is semi-open. Let $\mathcal{U} = \{\{p, x\} : x \in \mathbb{R}\}$ be the semi-open cover of $\mathbb{R}$. This cover does not contain a countable subcover, so that this space is not semi-Lindelöf and thus cannot be semi-Rothberger. On the other hand, if $\mathcal{U}$ is any semi-open cover, then for the finite set $F = \{p\}$ we have $\mathrm{st}(F,\mathcal{U}) = \mathbb{R}$, i.e., $(\mathbb{R}, \tau)$ is strongly star compact, hence strongly star s-Rothberger.

Definition 4.7. A space X is called *meta semi-compact* if every semi-open cover $\mathcal{U}$ of X has a point-finite semi-open refinement $\mathcal{V}$ (that is, every point of X belongs to at most finitely many members of $\mathcal{V}$).

Theorem 4.8. *A topological space X is star s-Rothberger if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-open sets, there exist $O_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that* $\mathrm{st}(\{x\},\mathcal{U}_n) \cap O_n \neq \phi$.

Proof. Let X be star s-Rothberger space, then X satisfies $\mathcal{S}_1^*(s\mathcal{O}, s\mathcal{O})$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. By definition there exists $O_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\mathrm{st}(O_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is a semi-open cover of X. That is, $\cup_{n\in\mathbb{N}}\mathrm{st}(O_n,\mathcal{U}_n) = X$. Let $x \in X$, then $x \in \mathrm{st}(O_k,\mathcal{U}_k)$ for some $k \in \mathbb{N}$. That is, there exists $O_k \in \mathcal{U}_k$ containing x such that $O_k \cap \mathcal{U}_k \neq \phi$. Also $\mathrm{st}(\{x\},\mathcal{U}_k) \cap O_k \neq \phi$.

Conversely, Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Then, there exist $O_n \in \mathcal{U}_n, n \in \mathbb{N}$ such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\mathrm{st}(\{x\},\mathcal{U}_n) \cap O_n \neq \phi$. $\mathrm{st}(\{x\},\mathcal{U}_n)$ contains the elements of $\mathcal{U}_n$ which contains x. This implies $x \in \mathrm{st}(O_n,\mathcal{U}_n)$. That is for every $x \in X$ there exists $n \in \mathbb{N}$ such that $x \in \mathrm{st}(O_n,\mathcal{U}_n)$. This implies that $\cup_{n\in\mathbb{N}}\mathrm{st}(O_n,\mathcal{U}_n) = X$. Hence, X is star s-Rothberger. □

Theorem 4.9. *A topological space X is strongly star s-Rothberger if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-open sets, there exist $(x_n : n \in \mathbb{N})$ of points of X such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $x \in st(x_n,\mathcal{U}_n)$.*

Proof. Let X be strongly star s-Rothberger space. Then X satisfies $SS_1^*(s\mathcal{O},$-$s\mathcal{O})$. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Let $\mathcal{K}$ be a collection of one-point subsets of X. Then, by definition there exists a sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that $\{\mathrm{st}(K_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is a semi-open cover of X. Let $\{x_n\} = K_n$, $n \in \mathbb{N}$, then $(x_n : n \in \mathbb{N})$ is a sequence of points of X such that $\{\mathrm{st}(\{x_n\},\mathcal{U}_n) : n \in \mathbb{N}\}$ is a semi-open cover of X. For every $x \in X$ there exists $k \in \mathbb{N}$ such that $x \in \mathrm{st}(\{x_k\},\mathcal{U}_k)$.

Conversely, Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Then, there exist $(x_n : n \in \mathbb{N})$ of points of X such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $x_n \in \mathrm{st}(\{x\},\mathcal{U}_n)$. Let $\{x_n\} = K_n$, $n \in \mathbb{N}$. Then we get the existence of the sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$. Since $x_n \in \mathrm{st}(\{x\},\mathcal{U}_n)$ for some n, then $x \in \mathrm{st}(\{x_n\},\mathcal{U}_n) = \mathrm{st}(K_n,\mathcal{U}_n)$. This implies that $\cup_{n\in\mathbb{N}}\mathrm{st}(K_n,\mathcal{U}_n) = X$. Hence, X is strongly star s-Rothberger. □

Theorem 4.10. *A space X is an almost star s-Rothberger space if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-regular sets, there exists a sequence $\mathcal{V} =\{V_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, V_n is a member of $\mathcal{U}_n$ and $\{\mathrm{scl}(\mathrm{st}(\cup\mathcal{V},\mathcal{U}_n)) : n \in \mathbb{N}\}$ is a cover of X.*

Proof. Since every semi-regular set is semi-open, therefore necessity follows.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be the sequence of semi-open covers of X. Let $\mathcal{U}'_n = \{\mathrm{scl}(U) : U \in \mathcal{U}_n\}$. Then $\mathcal{U}'_n$ is a cover of X by semi-regular sets. Then by assumption there exists $\mathcal{V} =\{V_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, V_n is a member of $\mathcal{U}'_n$ and $\{\mathrm{scl}(\mathrm{st}(\cup\mathcal{V},\mathcal{U}'_n) : n \in \mathbb{N}\}$ is a cover of X.

First we shall prove that $\mathrm{st}(U,\mathcal{U}_n) = \mathrm{st}(\mathrm{scl}(U),\mathcal{U}_n)$ for all $U \in \mathcal{U}_n$. It is obvious that $\mathrm{st}(U,\mathcal{U}_n) \subset \mathrm{st}(\mathrm{scl}(U),\mathcal{U}_n)$ since $U \subset \mathrm{scl}(U)$. Let $x \in \mathrm{st}(\mathrm{scl}(U),\mathcal{U}_n)$. Then there exists some $U' \in \mathcal{U}_n$ such that $x \in U'$ and $U' \cap \mathrm{scl}(U) \neq \phi$. Then $U' \cap \mathrm{scl}(U) \neq \phi$ which implies that $x \in \mathrm{st}(U,\mathcal{U}_n)$. Hence, $\mathrm{st}(\mathrm{scl}(U),\mathcal{U}_n) \subset \mathrm{st}(U,\mathcal{U}_n)$.

For each $V \in \mathcal{V}$ we can find $U_V \in \mathcal{U}_n$ such that $V = \mathrm{scl}(U_V)$. Let $\mathcal{V}' = \{U_V : V \in \mathcal{V}\}$.

Let $x \in X$. Then there exists $n \in \mathbb{N}$ such that $x \in \mathrm{scl}(\mathrm{st}(\cup\mathcal{V},\mathcal{U}'_n))$. For each semi-open set V of x, we have $V \cap \mathrm{st}(\cup\mathcal{V},\mathcal{U}'_n) \neq \phi$. Then there exists $U \in \mathcal{U}_n$ such that $(V \cap \mathrm{scl}(U) \neq \phi) \wedge (\cup\mathcal{V}_n \cap \mathrm{scl}(U) \neq \phi)$ implies that $(V \cap U \neq \phi) \wedge (\cup\mathcal{V} \cap \mathrm{scl}(U) \neq \phi)$. We have that $\cup\mathcal{V}' \cap U \neq \phi$, so $x \in \mathrm{scl}(\mathrm{st}(\cup\mathcal{V}',\mathcal{U}_n))$. Hence, $\{\mathrm{scl}(\mathrm{st}(\cup\mathcal{V}',\mathcal{U}_n) : n \in \mathbb{N}\}$ is a cover of X. □

Definition 4.11. A space X is an *almost star s-Rothberger space* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of X there exists a sequence $(U_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, U_n is an element of $\mathcal{U}_n$ and $\{\mathrm{scl}(\mathrm{st}(U_n,\mathcal{U}_n)) : n \in \mathbb{N}\}$ is a cover of X.

Theorem 4.12. *Quasi-irresolute surjective image of an almost star s-Rothberger space is an almost star s-Rothberger space.*

Proof. Let X be an almost star s-Rothberger space and Y be any topological space. Let $f : X \to Y$ be a quasi irresolute surjection Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of Y by semi-regular sets. Let $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}'_n$ is a cover of X by semi-regular sets since f is quasi-irresolute. Since X is an almost star s-Rothberger space, there exists a sequence $(V'_n : n \in \mathbb{N}) = \mathcal{V}'$ such that for every $n \in \mathbb{N}$, V'_n is a member of $\mathcal{U}'_n$ and $\{\mathrm{scl}(\mathrm{st}(\cup V'_n, \mathcal{U}'_n)) : n \in \mathbb{N}\}$ is a cover of X.

Let $\mathcal{V} = \{U \in \mathcal{U}_n : f^{-1}(U) \in \mathcal{V}'\}$ and $x \in X$. Then $f^{-1}(\cup\mathcal{V}) = \cup\mathcal{V}'$ and there is $n \in \mathbb{N}$ such that $x \in \mathrm{scl}(\mathrm{st}(f^{-1}(\cup\mathcal{V}, \mathcal{U}'_n))$. For $y = f(x) \in Y$,

$$\begin{aligned} y \in f(\mathrm{scl}(\mathrm{st}(f^{-1}(\cup\mathcal{V}, \mathcal{U}'_n)))) &\subseteq \mathrm{scl}(f(\mathrm{st}(f^{-1}(\cup\mathcal{V}, \mathcal{U}'_n)))) \\ &\subseteq \mathrm{scl}_\theta(f(\mathrm{st}(f^{-1}(\cup\mathcal{V}, \mathcal{U}'_n)))) = \mathrm{scl}(\mathrm{st}(\cup\mathcal{V}, \mathcal{U}_n)). \end{aligned}$$

Now assume that $f^{-1}(\cup\mathcal{V}) \cap f^{-1}(U) \neq \phi$. Then $f(f^{-1}(\cup\mathcal{V})) \cap f(f^{-1}(U)) \neq \phi$, hence $\cup\mathcal{V} \cap U \neq \phi$. So, it is shown that Y is an almost star s-Rothberger space. □

Definition 4.13. A space X is said to be *meta semi-Lindelöf* if every semi-open cover $\mathcal{U}$ of X has a point-countable semi-open refinement $\mathcal{V}$.

Theorem 4.14. *Every strongly star s-Rothberger meta semi-Lindelöf space is a semi-Lindelöf space.*

Proof. Let X be a strongly star s-Rothberger meta semi-Lindelöf space. Let $\mathcal{U}$ be a semi-open cover of X and let $\mathcal{V}$ be a point-countable semi-open refinement of $\mathcal{U}$. Since X is strongly star s-Rothberger, there is a sequence $(a_n : n \in \mathbb{N})$ of elements of X such that $\bigcup_{n\in\mathbb{N}} \mathrm{st}(a_n, \mathcal{V}_n) = X$.

For every $n \in \mathbb{N}$ denote by $\mathcal{W}_n$ the collection of all members of $\mathcal{V}$ which intersects a_n. Since $\mathcal{V}$ is point-countable, $\mathcal{W}_n$ is countable. So, the collection $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a countable subfamily of $\mathcal{V}$ and is a cover of X. For every $W \in \mathcal{W}$ pick a member $U_W \in \mathcal{U}$ such that $W \in U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$. Hence, X is a semi-Lindelöf space. □

Definition 4.15. A space X is said to be semi neighborhood star s-Rothberger if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-open sets, for each $n \in \mathbb{N}$, there exists $x_n \in X$ such that for every semi-open set O_n containing x_n, $\{\mathrm{st}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in s\mathcal{O}$.

Theorem 4.16. *A topological space X is semi neighborhood star s-Rothberger if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-open sets, there exist a sequence $(x_n : n \in \mathbb{N})$ of points of X such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $x_n \in \mathrm{scl}(\mathrm{st}(\{x\}, \mathcal{U}_n))$.*

Proof. Let X be a semi neighborhood star s-Rothberger space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. For each $n \in \mathbb{N}$, there exist $x_n \in X$ such that for every semi-open set O_n containing x_n, $n \in \mathbb{N}$, $\{\mathrm{st}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in s\mathcal{O}$. This implies $\cup_{n\in\mathbb{N}}\mathrm{st}(O_n, \mathcal{U}_n) = X$. Let $x \in X$ there exists $k \in \mathbb{N}$

such that $x \in st(O_k,\mathcal{U}_k)$. Now, it can be easily seen that $\mathrm{st}(\{x\},\mathcal{U}_k) \cap O_k \neq \phi$. This implies that $x_k \in \mathrm{scl}(\mathrm{st}(\{x\},\mathcal{U}_k))$.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Then there exists a sequence $(x_n : n \in \mathbb{N})$ of points of X such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $x_n \in \mathrm{scl}(\mathrm{st}(\{x\},\mathcal{U}_n))$. This implies that for every semi-open set O_n containing x_n, $n \in \mathbb{N}, \mathrm{st}(\{x\},\mathcal{U}_n) \cap O_n \neq \phi$. This implies that $x \in \mathrm{st}(O_n,\mathcal{U}_n)$. Hence, $\{\mathrm{st}(O_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in s\mathcal{O}$. □

Theorem 4.17. *Let a space X satisfies the following condition: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of X there is a sequence $(F_n : n \in \mathbb{N})$ of subsets of X such that for each n, $|F_n| \leq n$ and $\{\mathrm{st}(F_n,\mathcal{U}_n) : n \in \mathbb{N})\}$ is an $s\gamma-$cover of X. Then X satisfies $\mathcal{SS}_1^*(s\mathcal{O}, s\mathcal{O}^{gp})$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Let M be a clopen subset of X. For each n let

$$\mathcal{V}_n = \{M \cap U_i : \frac{(n-1)n}{2} < i \leq \frac{n(n+1)}{2}; U_i \in \mathcal{U}_i\}.$$

$(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of semi-open covers of M. Let $\mathcal{W}_n = \mathcal{V}_n \cup \{X - M\}$, then $(\mathcal{W}_n : n \in \mathbb{N})$ is a sequence of semi-open covers of X. Now applying the assumption of the theorem we find a sequence $(F_n : n \in \mathbb{N})$ of subsets of X such that for each n, $|F_n| \leq n$ and $\{\mathrm{st}(F_n, \mathcal{W}_n) : n \in \mathbb{N}\}$ is an $s\gamma-$cover of X. For each $x \in X$ there exists n_0 such that $x \in \mathrm{st}(F_n, \mathcal{W}_n)$ for all $n > n_0$. For each n we can write F_n as

$$F_n = \Big\{x_i : \frac{(n-1)n}{2} < i \leq \frac{n(n+1)}{2}\Big\}.$$

Then $\{\mathrm{st}(x_i,\mathcal{U}_i) : i \in \mathbb{N}\}$ is a semi-open groupable cover of X. Consider the sequence $n_1 < n_2 < \cdots < n_t < \cdots$ of natural numbers defined by $n_t = \frac{(t-1)t}{2}$. Then for each point $x \in X$ we have $x \in \cup_{n_t < n_{t+1}} \mathrm{st}(x_i, \mathcal{U}_i)$ for all but finitely many t. Thus X satisfies $\mathcal{SS}_1^*(s\mathcal{O}, s\mathcal{O}^{gp})$.

□

In topology, an F_σ-set is a countable union of closed sets.

Theorem 4.18. *A semi-open F_σ-subset of a strongly star s-Rothberger space is strongly star s-Rothberger.*

Proof. Let X be a strongly star s-Rothberger space and let $A = \cup\{M_n : n \in \mathbb{N}\}$ be a semi-open F_σ-subset of X, where each M_n is closed in X for each $n \in \mathbb{N}$. Without loss of generality, we can assume that $M_n \subseteq M_{n+1}$ for each $n \in \mathbb{N}$. Now we show that A is strongly star s-Rothberger space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of A. We need to find a set $K = \{a_n : n \in \mathbb{N}\}$ of elements of A such that $\{\mathrm{st}(a_n,\mathcal{U}_n) : n \in \mathbb{N}\}$ is a cover for A. For each $n \in \mathbb{N}$, let

$$\mathcal{V}_n = \mathcal{U}_n \cup \{X - M_n\}.$$

Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of semi-open covers of X. There exists a set $T = \{x_n : n \in \mathbb{N})$ of elements of X such that $\{\mathrm{st}(x_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ is a cover for X, since X is a strongly star s-Rothberger space. Let $K = T \cap A$. Thus K is a subset of A. For every $a \in A$, there exists $k \in \mathbb{N}$ such that $a \in \mathrm{st}(T, \mathcal{V}_k)$. Hence $a \in \mathrm{st}(K, \mathcal{U}_k)$, which shows that A is strongly star s-Rothberger. □

A cozero-set in a space X is a set of the form $f^{-1}(\mathbb{R} - \{0\})$ for some real valued continuous function f on X. Since a cozero-set is a semi-open F_σ-set, we have:

Corollary 4.19. *A cozero-set of a strongly star s-Rothberger space is strongly star s-Rothberger*

Theorem 4.20. *Every strongly star-s-Rothberger space is strongly star-semi Lindelöf.*

Proof. Let X be a strongly star-s-Rothberger space. This implies that X satisfies $S\mathcal{S}_1^*(s\mathcal{O}, s\mathcal{O})$. Let $\mathcal{U}$ be a semi-open cover of X. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence such that each $\mathcal{U}_n = \mathcal{U}$. Let $\mathcal{K}$ is the collection of all singleton subsets of X. Then, by definition, there is a sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that $\cup_{n\in\mathbb{N}}(\mathrm{st}(K_n, \mathcal{U}_n)) = X$. Let $A = \cup_{n\in\mathbb{N}} K_n$; then A is a countable set being countable union of finite sets. Also, $\cup_{n\in\mathbb{N}}\mathrm{st}(K_n, \mathcal{U}_n) = \cup_{n\in\mathbb{N}}(\mathrm{st}(\cup_{n\in\mathbb{N}} K_n, \mathcal{U}_n)) = \mathrm{st}(A, \mathcal{U}_n) = X$. Hence, X is strongly star-semiLindelöf space. □

Theorem 4.21. *Every strongly star s-Rothberger meta semicompact space is s-Menger space.*

Proof. Let X be a strongly star s-Rothberger meta semicompact space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be a point-finite semi-open refinement of $\mathcal{U}_n$. Since X is strongly star s-Rothberger, there is a sequence $(a_n : n \in \mathbb{N})$ of points of X such that $\cup_{n\in\mathbb{N}}(\mathrm{st}(a_n, \mathcal{V}_n)) = X$.

Since $\mathcal{V}_n$ is a point-finite refinement, each a_n belongs to finite members of $\mathcal{V}_n$ say $V_{n_1}, V_{n_2}, V_{n_3}, \ldots, V_{n_k}$. Let $\mathcal{V}'_n = \{V_{n_1}, V_{n_2}, V_{n_3}, \ldots, V_{n_k}\}$. Then $\mathrm{st}(a_n, \mathcal{V}_n) = \cup\mathcal{V}'_n$ for each $n \in \mathbb{N}$. We have that $\cup_{n\in\mathbb{N}}(\cup\mathcal{V}'_n) = X$. For every $V \in \mathcal{V}'_n$ choose $U_V \in \mathcal{U}_n$ such that $V \subset U_V$. Then, for every n, $\{U_V : V \in \mathcal{V}'_n\} = \mathcal{W}_n$ is such that $\cup_{n\in\mathbb{N}}\mathcal{W}_n = X$, that is X is s-Menger space. □

4.2. **Finite Powers.** All covers that we consider are infinite and countable.

Lemma 4.22 ([9]). *X is an extremely disconnected space if, and only if, $SO(X)$ is a topology.*

Theorem 4.23. *For an extremely disconnected space X,*

$$\mathsf{S}_1(s\Gamma, s\Gamma) = \mathsf{S}_{fin}(s\Gamma, s\Gamma).$$

Proof. It is clear that $\mathsf{S}_1(s\Gamma, s\Gamma)$ is contained in the class $\mathcal{S}_{fin}(s\Gamma, s\Gamma)$. We show that these classes are infact equal.

Let us assume that X has the property $\mathsf{S}_{fin}(s\Gamma, s\Gamma)$, and for each n let $\mathcal{U}_n$ be an $s\gamma-$cover of X, enumerated bijectively as $(U_1^n, U_2^n, U_3^n, \ldots)$. For each n define $\mathcal{V}_n$ to be $\{V_1^n, V_2^n, V_3^n, \ldots\}$, where $V_k^n = U_k^1 \cap U_k^2 \cap \ldots \cap U_k^n$. For each n, $\mathcal{V}_n$ is a $s\gamma-$cover. For each x, and for each $i \in \{1, \ldots, n\}$ there exists an N_i such that x is in U_m^i for all $m > N_i$. And x is in V_m^n for all $m > \max\{N_i : i = 1, 2, \ldots\}$. Now since X is $\mathsf{S}_{fin}(s\Gamma, s\Gamma)$, for $(\mathcal{V}_n : n = 1, 2, \ldots)$, we get a sequence $(\mathcal{W}_n : n \in \omega)$ such that $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ for each n, such that $\bigcup_{n=1}^{\infty} \mathcal{W}_n$ is an $s\gamma-$cover of X. Choose an increasing sequence $n_1 < n_n < \cdots$ such that for each j, $\mathcal{W}_{n_j} \setminus \cup_{i<j} \mathcal{W}_{n_i}$. Then $\{V_{m_k}^{n_k} : k = 1, 2, \ldots\}$ is an $s\gamma-$cover of X. For each n in $(n_k, n_{k+1}]$ we define $U_n = U_{n_{k+1}}^n$. Then $\{U_n : n = 1, 2, \ldots\}$ is an $s\gamma-$cover of X. □

Theorem 4.24. *If for each $n \in \mathbb{N}$, X^n is an almost s-Rothberger space for a topological space X, then X satisfies the selection hypothesis:*

- *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $s\omega-$covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a singleton subset of $\mathcal{U}_n$ and for every $F \subset X$ there exist $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ such that $F \subset \mathrm{scl}(V)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $s\omega$-covers of X. Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots \cup N_n \cup \cdots$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite subsets. For every $i \in \mathbb{N}$ and every $j \in N_i$ let $\mathcal{V}_j = \{U^i : U \in \mathcal{U}_j\}$. The sequence $\{\mathcal{V}_j : j \in N_i\}$ is a sequence of semi-open covers of X^i. Since X^i is an almost semi-Rotherger space, for every $i \in \mathbb{N}$, we can choose a sequence $(\mathcal{W}_j : j \in N_i)$ so that for each j, $\mathcal{W}_j = \{U_{j_1}^i, U_{j_2}^i, \ldots, U_{j_{k(j)}}^i\}$ is a finite subset of $\mathcal{V}_j$ and $\bigcup_{j \in N_i} \{\mathrm{scl}(W) : W \in \mathcal{W}_j\}$ is a cover of X^i. We shall show that $\{\mathrm{scl}(U_{j_p}) : 1 \leq p \leq k(j), j \in \mathbb{N}\}$ is an $s\omega$-cover of X. Let $F = \{x_1, x_2, \ldots, x_t\}$ be a finite subset of X. Then $(x_1, x_2, \ldots, x_t) \in X^t$, so there is some $l \in N_t$ such that $(x_1, x_2, \ldots, x_t) \in \mathcal{W}_l$. So, we can find $1 \leq r \leq k(l)$ such that $(x_1, x_2, \ldots, x_t) \in \mathrm{scl}(U_{l_{k(l)}}^t) = (\mathrm{scl}(U_{l_{k(l)}}))^t$. It is clear that $F \subset \mathrm{scl}(U_{l_{k(l)}})$. □

Lemma 4.25 ([1])**.** *Let $A_1, A_2, \ldots, A_m$ be subsets of X, then* $\mathrm{scl}(\prod_{n=1}^{n=m}(A_n)) = \prod_{n=1}^{n=m}(\mathrm{scl}(A_n))$.

Lemma 4.26. *Let $\mathcal{U}$ be a cover of X and let U be any subset of X, then* $\mathrm{scl}(\mathrm{st}(U, \mathcal{U})) = \mathrm{st}(\mathrm{scl}(U), \mathcal{U})$.

Lemma 4.27. *Let A be a semi-open set, then* $\mathrm{scl}(A)$ *is semi-open.*

Lemma 4.28. $\mathrm{cl}(A)$ *is semi-open if A is semi-open.*

Theorem 4.29. *If X^n is an almost star s-Rothberger space for the topological space X, then X satisfies the following selection hypothesis:*

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $s\omega$-covers of X there exists a set $\mathcal{V} =\{U_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$ and for every finite $F \subset X$, there exists $U \in \mathcal{V}$ such that $F \subset \mathrm{scl}(U)$.

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $s\omega$-covers of X where $\mathcal{U}_1 = \{U_{11}, U_{12}, U_{13}, \ldots\}, \mathcal{U}_2 = \{U_{21}, U_{22}, U_{23}, \ldots\}, \mathcal{U}_3 = \{U_{31}, U_{32}, U_{33}, \ldots\}, \ldots, \mathcal{U}_n = \{U_{nk} : k \in K_n\}, \ldots$, where K_n is an infinite countable index set. Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots \cup N_n \cup \cdots$ be a partition of $\mathbb{N}$ into countably many pairwise disjoint infinite substes.

For every $i \in \mathbb{N}$ and every $j \in N_i$ let $\mathcal{V}_j = \{U^i : U \in \mathcal{U}_j\}$. We have U^i is semi open for each i because finite product of semi-open sets is semi-open. Hence, the sequence $\{\mathcal{V}_j : j \in N_i\}$ is a sequence of semi-open covers of X^i.

Since X^i is an almost star s-Rothberger space, by the hypothesis, for every $i \in \mathbb{N}$, we can choose a sequence $(W_j : j \in N_i)$ so that for each j there exists $U^i_{jk} \in \mathcal{V}_j$ such that $W_j = \mathrm{scl}(U^i_{jk})$ and $\mathcal{W} = \{\mathrm{scl}(\mathrm{st}(U^i_{jk}, \mathcal{U}_j)) : j \in N_i\} = \{\mathrm{st}(\mathrm{scl}(U^i_{jk}), \mathcal{U}_j) : j \in N_i\} = \{\mathrm{st}(W_j, \mathcal{U}_j) : j \in N_i\}$ is a cover of X^i.

We show that $\mathcal{W}$ is an $s\omega$-cover of X. Let $F = \{x_1, x_2, \ldots, x_r\}$ be a finite subset of X. Then $(x_1, x_2, \ldots, x_r) \in X^r$, so there is some $l \in N_r$ such that $(x_1, x_2, \ldots, x_r) \in \mathrm{st}(W_l, \mathcal{U}_l) \in \mathcal{W}$ for some k such that $(x_1, x_2, \ldots, x_r) \in V^r_{lk} \subseteq \mathrm{scl}(V_{lk})^r = (\mathrm{scl}(V_{lk}))^r$, where $\mathrm{scl}(V_{lk}) \cap W_l \neq \phi; V^r_{lk} \in \mathcal{U}_l$. It is clear that $F \subset \mathrm{scl}(V_{lk})$. □

Corollary 4.30. *If X^n is an almost star s-Rothberger space for the topological space X, then X satisfies the following selection hypothesis:*

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $s\omega$-covers of X there exists a set $\mathcal{V} =\{U_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$ and for every finite $F \subset X$ there exists $U \in \mathcal{V}$ such that $F \subset \mathrm{cl}(U)$.

Proof. Since, $\mathrm{scl}(U) \subset \mathrm{cl}(U)$, for any subset U of X, the statement is true. □

Corollary 4.31. *If X^n is an s-Rothberger space for the topological space X, then X satisfies the following selection hypothesis:*

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $s\omega$-covers of X there exists a set $\mathcal{V} =\{U_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$ and for every finite $F \subset X$ there exists $U \in \mathcal{V}$ such that $F \subset \mathrm{scl}(U)$.

Proof. The proof is similar to the previous result. □

Corollary 4.32. *If X^n is an s-Rothberger space for a topological space X, then X satisfies the following selection hypothesis:*

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $s\omega$-covers of X there exists a set $\mathcal{V} = \{U_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$ and for every finite $F \subset X$ there exists $U \in \mathcal{V}$ such that $F \subset \mathrm{cl}(U)$.

Corollary 4.33. *If X^n is an s-Rothberger space for a topological space X, then X satisfies the following selection hypothesis:*

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $s\omega$-covers of X there exists a set $\mathcal{V} = \{U_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$ and for every finite $F \subset X$ there exists $U \in \mathcal{V}$ such that $F \subset U$.

Corollary 4.34. *If X^n is a star s-Rothberger space for the topological space X, then X satisfies the following selection hypothesis:*

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $s\omega$-covers of X there exists a set $\mathcal{V} = \{U_n : n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$, U_n is a member of $\mathcal{U}_n$ and for every $F \subset X$ there exists $n \in \mathbb{N}$ and $U \in \mathcal{V}$ such that $F \subset \mathrm{scl}(U)$.

Proof. Since, every s-Rothberger space is star s-Rothberger, the result holds. □

Theorem 4.35. *If each finite power of a space X is star s-Rothberger, then X satisfies $\mathsf{S}_1{}^*(s\mathcal{O}, s\Omega)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a partition of $\mathbb{N}$ into infinitely many infinite pairwise disjoint sets. For every $k \in \mathbb{N}$ and every $t \in N_k$ let $\mathcal{W}_t = \{U_1 \times U_2 \times \cdots \times U_k : U_1, \dots U_k \in \mathcal{U}_t\} = \mathcal{U}_t^k$. Then $(\mathcal{W}_t : t \in N_k)$ is a sequence of semi-open covers of X^k, and since X^k is a star s-Rothberger space, we can choose a sequence $(\mathcal{H}_t : t \in N_k)$ such that for each t, $\mathcal{H}_t$ is a sigleton subset of $\mathcal{W}_t$ and $\bigcup_{t \in N_k} \{\mathrm{st}(H, \mathcal{W}_t) : H \in \mathcal{H}_t\}$ is a semi-open cover of X^k. For every $t \in N_k$ and $H \in \mathcal{H}_t$ we have $H = U_1(H) \times U_2(H) \times \cdots \times U_k(H)$, where $U_i(H) \in \mathcal{U}_t$ for every $i \leq k$. Set $\mathcal{V}_t = \{U_i(H) : i \leq k, H \in \mathcal{H}_t\}$. Then for each $t \in N_k$ $\mathcal{V}_t$ is a finite subset of $\mathcal{U}_t$.

We claim that $\{\mathrm{st}(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an $s\omega$-cover of X. Let $F = \{x_1, \dots, x_p\}$ be a finite subset of X. Then $y = (x_1, \dots, x_p) \in X^p$ so that there is an $n \in N_p$ such that $y \in \mathrm{st}(H, \mathcal{W}_n)$ for $H \in \mathcal{H}_n$. But $H = U_1(H) \times U_2(H) \times \cdots \times U_p(H)$, where $U_1(H), U_2(H), \dots, U_p(H) \in \mathcal{V}_n$. The point y belongs to some $W \in \mathcal{W}_n$ of the form $V_1 \times V_2 \times \cdots \times V_p$, $V_i \in \mathcal{U}_n$ for each $i \leq p$, which meets $U_1(H) \times U_2(H) \times \cdots \times U_p(H)$. This implies that for each $i \leq p$, we have $x_i \in \mathrm{st}(U_i(H), \mathcal{U}_n) \subset \mathrm{st}(\cup\mathcal{V}_n, \mathcal{U}_n)$, that is, $F \subset \mathrm{st}(\cup\mathcal{V}_n, \mathcal{U}_n)$. Hence, X satisfies $\mathsf{S}_1{}^*(s\mathcal{O}, s\Omega)$. □

Theorem 4.36. *If all finite powers of a space X are strongly star s-Rothberger, then X satisfies $SS_1^*(s\mathcal{O}, s\Omega)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi open sets. Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots$ be a partition of $\mathbb{N}$ into infinite pairwise disjoint sets. For every $k \in \mathbb{N}$ and every $t \in N_k$ let $\mathcal{W}_t = \mathcal{U}_t^k$. Then $(\mathcal{W}_t : t \in N_k)$ is a sequence of semi open covers of X^k. Applying strongly star s-Rothberger property of X^k we can get a sequence $(x_t : t \in N_k)$ of elements of X^k such that $\{\mathrm{st}(x_t, \mathcal{W}_t) : t \in N_k\}$ is a semi open cover of X^k. For each t consider A_t a finite subset of X such that $x_t \in A_t^k$.

We show that $\{\mathrm{st}(x_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an $s\omega$-cover of X. Let $F = \{x_1, \ldots, x_p\}$ be a finite subset of X. Then $(x_1, \ldots, x_p) \in X^p$ such that there is $n \in \mathbb{N}_p$ and $(x_1, \ldots, x_p) \in \mathrm{st}(\{x_1, \ldots, x_p\}, \mathcal{W}_n) \subset \mathrm{st}(A_n^p, \mathcal{W}_n)$. Consequently, $F \subset \cup \mathrm{st}(A_n, \mathcal{U}_n)$. □

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