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UPPER BOUNDS FOR NOETHERIAN DIMENSION OF ALL INJECTIVE MODULES WITH KRULL DIMENSION

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Abstract. In this paper we give an upper bound for Noetherian dimension of all injective modules with Krull dimension on arbitrary rings. In particular, we also give an upper bound for Noetherian dimension of all Artinian modules on Noetherian duo rings.

Keywords: Noetherian dimension, Artinian modules, injective modules, comonoform right ideals, duo rings.
MSC(2010): Primary: 16P60; Secondary: 16P20, 16P40.

1. **Introduction**

Sharp [18] and Matlis in [15] have offered a method which shows that, for some purposes, to study Artinian modules over an arbitrary ring one may reduce the ring to be quasi-local, i.e., rings with the unique maximal ideal (and in some cases even over Noetherian quasi-local rings, i.e., local rings). In [7] this method is extended to the study of Artinian modules over duo rings (i.e., each one-sided ideal is a two-sided ideal). All rings in this article are associative with identity and all modules are unital right modules. For an R-module A, $E(A_R)$, or sometimes just $E(A)$, denotes the injective hull of A. The Noetherian dimension of an R-module A (the dual of Krull dimension $k\text{-}dim(A)$) denoted by $n\text{-}dim(A)$, is defined as follows: if $A = 0$, then $n\text{-}dim(A) = -1$, if α is an ordinal and $n\text{-}dim(A) \not< \alpha$, then $n\text{-}dim(A) = \alpha$, provided there is no infinite ascending chain $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of submodules of A such that, for $i = 1, 2 \dots$ $n\text{-}dim(\frac{A_{i+1}}{A_i}) \not< \alpha$.

This note consists of three sections. Section one is the introduction. In Section 2, we give a proof to second fact mentioned in the abstract. Finally, in Section 3, we provide an upper bound for Noetherian dimension of all injective modules with Krull dimension over any ring (not necessarily duo ring).

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In [5, Theorem 3.7], it is shown that the Noetherian dimension of an Artinian module over a valuation duo ring (i.e., all ideals are linearly ordered by inclusion) is less than or equal to 1. See also [2, Propositions 4.8, 4.9 and Corollary 4.11], where it is proved that the Artinian serial modules (i.e., a module that is a direct sum of uniserial modules, equivalently, its submodules are linearly ordered by inclusion) over commutative (or right Noetherian, or right duo, i.e., each right ideal is two-sided) rings have Noetherian dimension at most 1. In Section 2, as in [1,14] for commutative Noetherian rings, we show how one can find an upper bound for Noetherian dimension of all Artinian modules over Noetherian duo rings.

It is well-known that a nonzero module A is called α-critical if $k\text{-}dim(A) = \alpha$ and $k\text{-}dim(\frac{A}{B}) < \alpha$ for each nonzero submodule B of A. For an R-module A, it is defined in [9, Definition 2.2], $S_\alpha = \sum_{i\in I} \oplus C_i$, for each ordinal α, where $\{C_i\}_{i\in I}$ is a maximal independent set of α-critical submodules of A. A critical socle of A is defined to be a submodule S of A with $S = \sum_{\alpha\leq\lambda} S_\alpha$, where λ is the least ordinal such that each critical submodule is α-critical for some $\alpha \leq \lambda$ ($S_\alpha = 0$ if there is no α-critical submodule, for some ordinal α). Also, an R-module A is called λ-finitely embedded (λ-f.e.) if λ is the least ordinal such that each critical submodule of A is α-critical for some $\alpha \leq \lambda$ and A contains an essential critical socle with Krull dimension λ. If $k\text{-}dim(A) = \alpha$, then A is λ-f.e. module for some $\alpha \leq \lambda$, see [9, Definition 2.5] (note that, 0-f.e. modules are precisely f.e. modules in [20]). An R-module A is λ-f.e. if and only if $E(A) \cong E(C_1) \oplus \cdots \oplus E(C_n)$, where each C_i is an α_i-critical module and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n = \lambda$, see [9, Proposition 2.6]. Hence, if A is Artinian, then $E(A) \cong E(\frac{R}{M_1}) \oplus \cdots \oplus E(\frac{R}{M_n})$, in which M_i's are maximal right ideals. In addition, if A is an injective Artinian R-module then $n\text{-}dim(A) = \sup\{n\text{-}dim(E(\frac{R}{M_i}))|1 \leq i \leq n\}$. But, in relation to modules (not necessarily Artinian) with Krull dimension in [1, Proposition 11], it is shown that if A_R has Noetherian dimension and if $E(S)$ has Noetherian dimension for any simple module S, then

$$n\text{-}dim(A) \leq \sup\{n\text{-}dim(E(\tfrac{R}{M}))|M \leq R_R \text{ is maximal}\}.$$

Notice that, in [6] a ring R is called right co-Noetherian if for each simple right R-module S, $E(S)$ is Artinian. In [3], a Noetherian ring R with $k\text{-}dim_r(R) = k\text{-}dim_l(R) = 1$ is constructed which possesses a simple module S such that $E(S)$ is non-Artinian, hence R is not co-Noetherian.

A proper right ideal P of R is called completely prime right ideal if for any $a, b \in R$ such that $aP \subseteq P$, $ab \in P$ implies that either $a \in P$ or $b \in P$, see [17, Definition 2.1]. Notice that every maximal ideal is manifestly a completely prime right ideal, see [17] and [8]. In [17] it is claimed that completely prime right ideals, as a one-sided generalization of the concept of a prime ideal in a commutative ring, control the one-sided structure of a ring (recall that prime ideals control the structure of commutative rings, e.g. Cohen's theorem and

Kaplansky's theorem). We observe, in Section 3, that the set comonoform right ideals (note, a proper right ideal P of R is comonoform if any nonzero $f \in Hom_R(\frac{R}{P}, E(\frac{R}{P}))$ is monomorphism, see [17, Definition 6.2]), which is a subset of completely prime right ideals, controls Noetherian dimension of injective modules with Krull dimension.

2. Artinian modules over duo rings

Let S be a finite R-algebra (i.e., $S = R + Re_1 + Re_2 + \cdots + Re_n$, where $Re_i = e_iR$ for all i). It is well-known that $n\text{-}dim(A_S) = n\text{-}dim(A_R)$ and $k\text{-}dim(A_S) = k\text{-}dim(A_R)$, where A is an S module with Krull dimension, see the comment following [2, Lemma 2.16].

Example 2.1. Let S be a finite R-algebra, where R is commutative ring (e.g., $S = M_n(R)$, the ring of all $n \times n$ matrices over R), then ω is an upper bound for the Noetherian dimension of all Artinian S-modules.

Proof. Let A be an Artinian S-module, then $n\text{-}dim(A_S) = n\text{-}dim(A_R)$, but $n\text{-}dim(A_R)$ is finite by [11], hence we are done. □

Example 2.2. Let S be a ring which is finite direct product of some rings (i.e., $S = \prod_{i \in I} R_i$, where I is a finite set), where each R_i belongs to a class of rings consisting of, commutative rings, finite R-algebras, where R is a commutative ring, valuation duo rings, and Amen rings (i.e., Noetherian and Artinian modules coincide, see [5]), then ω is an upper bound for Noetherian dimension of all Artinian S-modules.

Proof. Take A to be an Artinian S-module. Now A can uniquely be written as a direct product $A = \prod_{i \in I} A_i$, where each A_i is an R_i-module and the action is componentwise, see also [7, Remark 4.1]. Clearly, each A_i is Artinian as R_i-module, hence by what we have already observed each A_i has a finite Noethrian dimension which is naturally its Noetherian dimension as an R-module, too. Hence we are done. □

Without further ado, in this section we first observe that the study of Noetherian dimension of Artinian modules over arbitrary duo rings can be reduced to over quasi-local rings.

The following proposition, which is a combination of two results [7, Proposition 2.6 and Proposition 2.7], is needed. Notice that, $T_I(A) = \cup_{n \geq 1} Ann_A(I^n) = \sum_{n \geq 1} Ann_A(I^n)$ for any ideal I of R.

Proposition 2.3. *Let A be an non-zero Artinian module over a duo ring R. Then $A = T_{M_1}(A) \oplus \cdots \oplus T_{M_n}(A)$ for some finite set $\{M_1, \ldots, M_n\}$ of distinct maximal ideals of R and any $T_{M_i}(A)$ can be considered as an R_{M_i}-module whose lattice of R-submodules coincides with its lattice of R_{M_i}-submodules.*

Remark 2.4. Let A be a non-zero Artinian module over duo ring R. Then by Proposition 2.3, we have $A = T_{M_1}(A) \oplus \cdots \oplus T_{M_n}(A)$, and therefore from [10, Proposition 1.6], $n\text{-}dim(A) = \sup\{n\text{-}dim(T_{M_i}(A)) | 1 \leq i \leq n\}$. By Proposition 2.3, $T_{M_i}(A)$ can be considered as an R_{M_i}-module and lattice of R-submodules coincides with its lattice of R_{M_i}-submodules for all i, $1 \leq i \leq n$. Thus for any $1 \leq i \leq n$; $n\text{-}dim((T_{M_i}(A))_R) = n\text{-}dim((T_{M_i}(A))_{R_{M_i}})$. Therefore

$$n\text{-}dim(A_R) = \sup\{n\text{-}dim((T_{M_i}(A))_{R_{M_i}}) | 1 \leq i \leq n\},$$

and so we can study the Noetherian dimension of Artinian modules over quasi-local duo rings instead of study of Noetherian dimension of Artinian modules over duo rings.

Matlis has shown that, over a commutative Noetherian ring, finitely embedded modules and Artinian modules coincide. Later, in [20, Theorem 2], he has extended this important result over locally Noetherian commutative rings (note, a commutative ring R is called locally Noetherian if R_M is Noetherian for all maximal ideals M in R). In [5] the latter result of Matlis is proved over Noetherian duo rings.

Theorem 2.5. *Let (R, M) be a local (i.e., Noetherian quasi-local) duo ring. Then*

$$\sup\{n\text{-}dim(A) | A \textit{ is an Artinian } R\textit{-module}\} = n\text{-}dim(E(\tfrac{R}{M})).$$

Proof. If A is an Artinian R-module, then A is finitely embedded and so with to be quasi-locality of R there are simple submodules $S_i \cong (\frac{R}{M})_R$ $(1 \leq i \leq n)$ such that $E(A) = E(\frac{R}{M}) \oplus \cdots \oplus E(\frac{R}{M})$. But $E(A)$ is Artinian and so has Noetherian dimension and thence $E(\frac{R}{M})$ is so. Now, by [10, Proposition 1.6], we have $n\text{-}dim(A) \leq n\text{-}dim(E(A)) = n\text{-}dim(E(\frac{R}{M}))$. Therefore

$$\sup\{n\text{-}dim(A) | A \text{ is an Artinian } R\text{-module}\} = n\text{-}dim(E(\tfrac{R}{M})),$$

as desired. □

We conclude this section with some familiar examples. Let us recall that for prime ideal P of a commutative ring R, $rank(P) = n$ (also called the height of P by various authors) if there exists a chain of prime ideals of length n descending from P, but no longer chain.

Example 2.6. Let R be a commutative Noetherian ring, and suppose that there exists a $P \in Spec(R)$ which has rank n and can be generated by n elements. Then n is an upper bound for the Noetherian dimension of all Artinian R-modules.

Proof. Without loss of generality, we may assume that (R, M) is a local ring and $rank(M) = n$, by Remark 2.4 and [19, Example 15.28]. By Theorem 2.5,

$$\sup\{n\text{-}dim(A) | A \text{ is an Artinian } R\text{-module}\} = n\text{-}dim(E(\tfrac{R}{M})).$$

Now $E(\frac{R}{M})$ has Noetherian dimension equal to n, by the comment following [2, Corollary 2.17], hence we are done. □

Example 2.7. (1) Take $R = K[x_1, \ldots, x_n]$, where K is a field, then n is an upper bound for the Noetherian dimension of all Artinian R-modules.
(2) If K is a field, then n is an upper bound for the Noetherian dimension of all Artinian R-modules which R is the ring $K[[x_1, \ldots, x_n]]$ of formal power series over K in the n indeterminates $x_1, \ldots, x_n$.
(3) Let R be a PID which is not a field (e.g., $\mathbb{Z}$). Then 1 is an upper bound for the Noetherian dimension of all Artinian R-modules.

Remark 2.8. Notice that by what we have already observed in the introduction, if R is an Amen ring then the upper bound for the Noetherian dimension of Artinian modules is zero. Also, if R is a valuation ring then the upper bound for the Noetherian dimension of Artinian modules is one. But, as in above Example 2.6 and 2.7, we are not able to express that there exists an constant positive integer n such that it is an upper bound for the Noetherian dimension of Artinian R-modules, where R is a local ring.

3. Upper bounds for Noetherian dimension of injective modules with Krull dimension

We first record some well-known definitions, see [4, 12].

Definition 3.1. Let A be a module and C a submodule.

- C is a complement in A (written $C \subseteq_c A$) if there exists a submodule $S \subseteq A$ such that C is maximal with respect to the property that $C \cap S = 0$.
- C is essentially closed in A if and only if $C \leq_e B \leq A$ always implies $B = C$.

Definition 3.2 ([13]). A right ideal $I \subsetneqq R$ is called right irreducible (note, it is also meet-irreducible in the literature, see [12]) if the cyclic module $(\frac{R}{I})_R$ is uniform.

Without further ado we present our main result.

Theorem 3.3. *Let R be a ring (not necessarily commutative) and A be an injective module with Krull dimension. Then*

$$n\text{-}dim(A) \leq \sup\{n\text{-}dim(E(\tfrac{R}{I}))|I \leq R_R \text{ is irreducible, } E(\tfrac{R}{I}) \text{ has } n\text{-}dim\}.$$

Proof. Let $G\text{-}dim(A) = n$. From [12, Exercise 6.4], there exist essentially closed submodules $A_i \subseteq_c A$ $(1 \leq i \leq n)$ such that any $\frac{A}{A_i}$ is uniform and $A = E(A) \cong \sum_{i=1}^{n} \oplus E(\frac{A}{A_i})$. By [12, Theorem 3.52], for any $1 \leq i \leq n$,

$$E(\tfrac{A}{A_i}) \cong E((\tfrac{R}{J_i})_R),$$

where J_i is an irreducible right ideal of R. Hence we have $A \cong \sum_{i=1}^{n} \oplus E(\frac{R}{J_i})$ and so

$$n\text{-}dim(A) = \sup\{n\text{-}dim(E(\frac{R}{J_1})), \dots, n\text{-}dim(E(\frac{R}{J_n}))\},$$

therefore we are through. $\square$

In particular, we show that if in Theorem 3.3, R is right Noetherian, then the set comonoform right ideals exactly describes the Noetherian dimension of all injective modules with Krull dimension.
We cite the following well-known results, all of which, with our Theorem 3.3, are needed for the proofs of our main results in Corollary 3.12 and Corollary 3.14.

Remark 3.4. Following [13], two irreducible right ideals I and J of R are called related if $E(\frac{R}{I}) \cong E(\frac{R}{J})$ (or equivalently, if there exist elements $s \notin I$ and $t \notin J$ of R such that $s^{-1}I = t^{-1}J$. Here $s^{-1}I = \{r \in R | sr \in I\}$). If I is an irreducible right ideal of R, then $E(\frac{R}{I}) \cong E(\frac{R}{s^{-1}I})$ for every element $s \notin I$. Moreover, $s^{-1}I$ is also irreducible.

Hence relatedness is an equivalence relation on the set of all irreducible right ideals of the ring R. In a commutative Noetherian ring every equivalence class of related irreducible ideals contains a unique maximal element, which is a prime ideal. As such, in noncommutative rings, the maximal elements of equivalence classes of irreducible right ideals enjoy many properties of prime ideals in commutative rings.

An R-module A is called indecomposable if $A \neq 0$ and it is not a direct sum of two nonzero submodules.

Proposition 3.5 ([13, Proposition 2.7])**.** *The following properties of a right ideal I of the ring R are equivalent.*

(1) *I is comonoform.*
(2) *I is irreducible and is a maximal element in its equivalence class of related irreducible right ideals.*
(3) *There is an indecomposable injective module A such that I is maximal among the right annihilators of nonzero element of A.*

Proposition 3.6. *Let R be a ring that satisfies the ascending chain condition on every equivalence class of related irreducible right ideals. Then every indecomposable injective R-modules is isomorphic to $E(\frac{R}{P})$ for some comonoform right ideal P of R, and P is uniquely determind up to relatedness.*

Proof. Let A be an indecomposable injective R-module. We have $A \cong E(\frac{R}{I})$, where $I \subsetneq R$ is a irreducible right ideal of R, by [12, Theorem 3.52]. Now, the equivalence class of I has maximal element, say P. Hence $E(\frac{R}{I}) \cong E(\frac{R}{P})$ and P is comonoform, by Proposition 3.5. $\square$

Notice that by [16], an proper (two-sided) ideal P of R is said to be completely prime ideal if the factor ring $\frac{R}{P}$ is a domain (equivalently, $P \neq R$ and for all $a, b \in R, ab \in P \Rightarrow a \in P$ or $b \in P$).
By [17, Proposition 2.2], for any ring R, a proper ideal P of R is completely prime as a right ideal if and only if it is a completely prime ideal. Also [17, Proposition 6.3] states that

$$\{\text{Comonoform right ideals}\} \subseteq \{\text{Completely prime right ideals}\}.$$

If R is commutative, then the sets are same. Indeed we have

$$\{\text{Comonoform right ideals}\} = \{\text{Completely prime right ideals}\} = Spec(R),$$

by [17, Corollaries 2.3 and 6.7].
We remind the reader that a ring R is called right (left) duo if any right (left) ideal of R is a two-sided ideal (or equivalently, if for any $a \in R$, $Ra \subseteq aR$ ($aR \subseteq Ra$)). Next we observe that the above equality is also true for any right duo ring.

Lemma 3.7. *Let R be a right duo ring and P a proper ideal of R. The following statements are equivalent.*

(1) *P is prime.*
(2) *P is completely prime as a right ideal.*
(3) *For any $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$.*
(4) *$\frac{R}{P}$ is a domain.*

We recall that a domain R is said to be right Ore if for every two nonzero elements $x, y \in R$, there exist $r, s \in R$ such that $xr = ys \neq 0$, see [4, Page 112]. Notice that a prime ring is a ring in which 0 is a prime ideal. It is easy to prove that, if R is a right duo ring, then R is a prime ring if and only if R is a domain.
The next lemma is evident, see also [4, Lemma 6.6].

Lemma 3.8. *Let R be a right duo prime ring. Then R is a right Ore domain.*

Proposition 3.9. *For every right duo ring R, we have*

$$Spec(R) = \{P \triangleleft R \mid P \text{ is comonoform right ideal}\}.$$

Corollary 3.10. *For every right duo ring R, we have*

$$\{\textit{Comonoform right ideals}\} = \{\textit{Completely prime right ideals}\} = Spec(R).$$

Proposition 3.11. *Let R be a ring that satisfies the ascending chain condition on every equivalence class of related irreducible right ideals and A be an injective R-module with Krull dimension. Then*

$$n\text{-}dim(A) \leq \sup\{n\text{-}dim(E(\tfrac{R}{P})) \mid P \leq R_R \text{ is comonoform, } E(\tfrac{R}{P}) \text{ has } n\text{-}dim\}.$$

Proof. This proves by Theorem 3.3 and Proposition 3.6. ∎

The following corollary is now immediate.

Corollary 3.12. *Let R be a right Noetherian ring and A be an injective R-module with Krull dimension. Then*

$$n\text{-}dim(A) \leq \sup\{n\text{-}dim(E(\tfrac{R}{P}))|P \leq R_R \text{is comonoform, } E(\tfrac{R}{P}) \text{ has n-dim}\}.$$

Proposition 3.13. *Let R be a right duo ring that satisfies the ascending chain condition on every equivalence class of related irreducible right ideals and A be an injective R-module with Krull dimension. Then*

$$n\text{-}dim(A) \leq \sup\{n\text{-}dim(E(\tfrac{R}{P}))|P \in Spec(R), E(\tfrac{R}{P}) \text{ has n-dim}\}.$$

Proof. This is clear by Propositions 3.9 and 3.11. □

In view of Proposition 3.13, the next result is also immediate.

Corollary 3.14. *Let R be a right Noetherian right duo ring and A be an injective R-module with Krull dimension. Then*

$$n\text{-}dim(A) \leq \sup\{n\text{-}dim(E(\tfrac{R}{P}))|P \in Spec(R), E(\tfrac{R}{P}) \text{ has n-dim}\}.$$

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References

[1] L. Chambless, N-Dimension and N-critical modules, application to artinian modules, *Comm. Algebra* **16** (1980) 1561–1592.

[2] M. Davoudian and O.A.S. Karamzadeh, Artinian serial modules over commutative (or, left Noetherian) rings are at most one step away from being Noetherian, *Comm. Algebra* **44** (2016), no. 9, 3907–3917.

[3] K. R. Goodearl and A.H. Schofield, Non-Artinian essential extensions of simple modules, *Proc. Amer. Math. Soc.* **97** (1986) 233–236.

[4] K.R. Goodearl and R.B. Warfield, An Introduction to Noncommutative Rings, Cambridge Univ. Press, Cambridge, 1989.

[5] J. Hashemi, O.A.S. Karamzadeh and N. Shirali, Rings over which the Krull dimension and the Noetherian dimension of all modules coincide, *Comm. Algebra* **37** (2009) 650–662.

[6] J.P. Jans, On co-Noetherian rings, *J. Lond. Math. Soc. (2)* **1** (1969) 588–590.

[7] N.S. Karamzadeh, O.A.S. Karamzadeh, On Artinian modules over Duo rings, *Comm. Algebra* **38** (2010) 3521–3531.

[8] O.A.S. Karamzadeh, On maximal right ideals which are direct summands, *Bull. Iranian Math. Soc.* **10** (1983), no. 2, 40–46.

[9] O.A.S. Karamzadeh and Sh. Rahimpour, On λ-finitely embedded modules, *Algebra. Colloq.* **12** (2005) 281–292.

[10] O.A.S. Karamzadeh and A.R. Sajedinejad, On the Loewy length and the Noetherian dimension of Artinian modules, *Comm. Algebra* **30** (2002) 1077–1084.

[11] D. Kirby, Dimension and length for Artinian modules, *Q. J. Math.* **41** (1990) 419–429.

[12] T.Y. Lam, Lectures on Modules and Rings, Grad. Texts in Math. Springer Verlag, New York, 1999.

[13] J. Lambek and G. Michler, The torsion theory at a prime ideal of a right Noetherian ring, *J. Algebra* **25** (1973) 364–389.

[14] B. Lemonnier, Dimension de Krull et codeviation, application au theorem d'Eakin, *Comm. Algebra* **6** (1978) 1647–1665.

[15] E. Matlis, Modules with descending chain condition, *Trans. Amer. Math. Soc.* 97 (1960) 495–508.

[16] J.C. McConnel and J.C. Robson, Noncommutative Noetherian Rings, Revised edition, Grad. Stud. Math. 30, Amer. Math. Soc. Providence, RI, 2000.

[17] M.L. Reyes, A one-sided prime ideal principle for noncommutative rings, *J. Algebra Appl.* **9** (2010) 877–919.

[18] R.Y. Sharp, A method for the study of Artinian modules with an application to asymptotic behavior, in: Commutative Algebra (Berkeley, CA, 1987), pp. 443–465, Math. Sci. Res. Inst. Publ. 15, Springer Verlag, New York, 1989.

[19] R.Y. Sharp, Steps in Commutative Algebra, Cambridge Univ. Press, 1990.

[20] P. Vamos, The dual of the notion of finitely generated, *J. Lond. Math. Soc.* **43** (1968) 643–646.

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