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ON NORMALIZERS OF MAXIMAL SUBFIELDS OF DIVISION ALGEBRAS

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Abstract. Here, we investigate a conjecture posed by Amiri and Ariannejad claiming that if every maximal subfield of a division ring D has trivial normalizer, then D is commutative. Using Amitsur classification of finite subgroups of division rings, it is essentially shown that if D is finite dimensional over its center then it contains a maximal subfield with non-trivial normalizer if and only if D^* contains a non-abelian soluble subgroup. This result generalizes a theorem of Mahdavi-Hezavehi and Tignol about cyclicity of division algebras of prime index.
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MSC(2010): Primary: 11R52; Secondary: 20D10.

1. **Introduction**

Let D be a division algebra over its center F and $D^* = D \setminus \{0\}$ be its multiplicative group. Recall that by Wedderburn's little theorem [14], D is commutative whenever D has a finite cardinal number. This theorem became the first result connecting the algebraic structure of D to that of D^* (since the group theoretic properties of D^* are effectively employed in the proof of this result) and a major turning point in the theory of non-commutative algebras. More precisely, since Wedderburn's theorem characterizes a class of division rings in terms of a group theoretic (more exactly a set theoretic) property, one way to generalize this theorem is to determine what groups can be considered as the multiplicative group of a suitable division ring. For a historical background and a self-contained account of the recent developments on this topic see [4]. Recently in [1] Amiri and Ariannejad presented a new proof for Wedderburn's little theorem based on the concept of Frobenius kernel in Frobenius representation theorem in the theory of finite groups. In the last part of their note, the authors posed the following conjecture:

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Conjecture 1.1. *D is commutative if and only if $N_{D^*}(K^*) = K^*$ for each maximal subfield K of D, where $N_{D^*}(K^*)$ is the normalizer of K^* in D^*.*

Let K be a maximal subfield of D. We say that K has *trivial normalizer* if $N_{D^*}(K^*) = K^*$. Otherwise, we say that K has *non-trivial normalizer*. Note that in this terminology, Conjecture 1.1 essentially claims that D is non-commutative if and only if there exists a maximal subfield of D with non-trivial normalizer. The aim of this note is to investigate Conjecture 1.1 in the level of division algebras; Division rings that are finite dimensional over their centers as a vector space.

Recall that if D is a division algebra, with $\dim_F D = n^2$ for some positive integer n (cf. [3, Section 5, Corollary 5]), then n is called the index of D and we write $\mathrm{ind}(D) = n$. Given a maximal subfield K of a division algebra D, by Skolem-Noether Theorem [3, p. 39], the map $\sigma : N_{D^*}(K^*) \longrightarrow \mathrm{Gal}(K/F)$, $u \mapsto \sigma_u$ where $\sigma_u(k) = u^{-1}ku$ for all $k \in K$, is a group epimorphism. On the other hand, since K is a maximal subfield of D we have $K = C_D(K)$ and hence $\ker\sigma = K^*$. Consequently, $\mathrm{Gal}(K/F) \cong N_{D^*}(K^*)/K^*$. So, in the level of division algebras, Conjecture 1.1 is equivalent to that D is non-commutative if and only if it contains a maximal subfield K with non-trivial $\mathrm{Gal}(K/F)$. In the special case in which the index of D is a prime number, we observe that D contains a maximal subfield with non-trivial normalizer if and only if D is a cyclic division algebra (i.e., D has a maximal subfield that is cyclic Galois over F). So Conjecture 1.1 contains the following well-known conjecture, appointed to Albert, as a special case:

Conjecture 1.2. Every division algebra of prime index is cyclic.

Conjecture 1.2 is one of oldest and most challenging problem in the theory of division algebras which is remained unsolved to this day. However, in this direction, in [11] Mahdavi-Hezavehi and Tignol presented some equivalent cyclicity conditions in terms of the group theoretic properties of D^*. In particular, they proved that a division algebra D of prime index is cyclic if and only if D^* contains a non-abelian soluble subgroup. The main object of this note is to establish a similar result for Conjecture 1.1. Indeed, we will prove that a division algebra D of arbitrary index $n > 1$, contains a maximal subfield with non-trivial normalizer if and only if D^* contains a non-abelian soluble subgroup.

2. Main results

Before stating our result, we recall some terminologies and some facts about the multiplicative groups of division algebras. In what follows, Q_8 denotes the Quaternion group, i.e., $Q_8 = \langle i, j \mid i^4 = 1,\ i^2 = j^2,\ j^{-1}ij = i^{-1}\rangle$. As usual, $\mathrm{SL}_2(\mathbb{Z}_5)$ stands for the group consists of 2×2 matrices with entries in the field $\mathbb{Z}_5$ and determinant 1. One can observe that every Sylow 2-subgroup of

$\mathrm{SL}_2(\mathbb{Z}_5)$ is isomorphic to Q_8. Recall that by Amitsur classification of finite subgroups of division rings, the only insoluble subgroup of division rings is $\mathrm{SL}_2(\mathbb{Z}_5)$ (cf. [2, Corollary 4]). A division algebra is called a *crossed product* if it contains a maximal subfield that is Galois over the center. In order to prove our main theorem, we need also the following theorem that characterizes crossed products in terms of their multiplicative groups. For a proof see [6, Theorem 2].

Theorem 2.1. *Let D be a division algebra over its center F. Then D is a crossed product if and only if there exists an irreducible subgroup G (i.e., $D = F[G]$) and a normal abelian subgroup A of G such that $C_G(A) = A$.*

Now, we are in a position to present the main result of this note.

Theorem 2.2. *Let D be a division algebra of index $n > 1$ over its center F. Then the following conditions are equivalent:*

(1) *Every maximal subfield of D has trivial normalizer;*
(2) *Every non-abelian subgroup of D^* contains a non-cyclic free subgroup;*
(3) *Every soluble-by-finite subgroup of D^* is abelian;*
(4) *Every abelian-by-finite subgroup of D^* is abelian;*

Proof. The implications (2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(1). Let K be a maximal subfield of D. Since $N_{D^*}(K^*)/K^* \cong \mathrm{Gal}(K/F)$, we conclude that $N_{D^*}(K^*)$ is abelian-by-finite. Hence, by our assumption $N_{D^*}(K^*)$ is abelian. Therefore, if $a \in N_{D^*}(K^*)$ then $K(a)$ is a subfield containing K. Now, from the maximality of K we have $K(a) = K$ and so $N_{D^*}(K^*) \subseteq K$.

(1)⇒(2). Let G be a non-abelian subgroup of D^* that does not contain any non-cyclic free subgroup. Since G is non-abelian, there are elements $a, b \in G$ such that $ab \neq ba$. So $M = \langle a, b\rangle$ is a finitely generated non-abelian subgroup of D^*. In view of the isomorphism $D \otimes_F D^{\mathrm{op}} \cong M_{n^2}(F)$ (cf. [3, Section 5, Corollary 2]) we can consider M as a linear group. Here, Tit's Alternative [13, Corollary 1] implies that M is a non-abelian soluble-by-finite group. Hence it contains a soluble subgroup H of finite index. Now, by [8, Lemma 3] H contains a normal abelian subgroup B such that $[H : B] < \infty$. But $[M : B] = [M : H][H : B] < \infty$. Now, consider the subgroup $\mathrm{Core}(B) = \cap_{x\in M} xBx^{-1}$ of M. It is well-known and easy to check that $\mathrm{Core}(B) \lhd M$ and $[M : \mathrm{Core}(B)] < \infty$ (see [12, 1.6.9, p. 36]). Since $\mathrm{Core}(B)$ is abelian, it follows that M is abelian-by-finite. So we may find a normal abelian subgroup A of M of finite index which is maximal among the all normal abelian subgroups of finite index. Three cases may occur:

Case 1. $C_M(A) = A$. Let $E = F[M]$ and $L = Z(E)$, the center of E. So $E = L[M]$. Put $\widetilde{E} = C_D(L)$. Since $Z(E) = L$, from [3, Section 7, Cor. 8] it follows that $\widetilde{E} = E \otimes_L C_{\widetilde{E}}(E)$. Now, Theorem 2.1 guarantees that E has a maximal subfield K Galois over L. Let P be an arbitrary maximal subfield of $C_{\widetilde{E}}(E)$. Since $\widetilde{E}$ is a division algebra, K and P are linearly disjoint over L and hence $K \otimes_L P$ is a field isomorphic to KP. Pick $1 \neq \sigma \in \mathrm{Gal}(K/L)$. From field theory, we observe that $\sigma \otimes 1$ is a non-trivial L-automorphism of the field $K \otimes_L P \cong KP \subseteq \widetilde{E}$. Thus the subfield KP of $\widetilde{E}$ has a non-trivial L-automorphism that is evidently an F-automorphism. But, by Centralizer Theorem [3, p. 42–43] we have

$$\begin{aligned}[KP : F] &= [KP : L][L : F] \\ &= \mathrm{ind}(\widetilde{E})[L : F] \\ &= \frac{n}{[L : F]}[L : F] \\ &= n.\end{aligned}$$

Hence KP is a maximal subfield of D with $\mathrm{Gal}(KP/F) \neq 1$. So $N_{D^*}(KP) \neq KP$, contrary to hypothesis.

Case 2. M contains a subgroup isomorphic to Q_8. In this case, if we put $M = Q_8$ and $A = \langle i \rangle$ then $C_M(A) = A$ and the result follows by a similar argument to Case 1.

Case 3. $C_M(A) \neq A$ and M does not contain any subgroup isomorphic to Q_8. Put $N = C_M(A)$. Clearly, N is center-by-finite and hence N' is a finite subgroup of M. But $N' \ncong \mathrm{SL}_2(\mathbb{Z}_5)$, because $\mathrm{SL}_2(\mathbb{Z}_5)$ contains a subgroup isomorphic to Q_8. Hence, by Amitsur classification of finite subgroups of division rings N is soluble. Let $N^0 = N$ and N^j denotes the j-th term of the derived series of N for each $j \geq 1$. Since N is non-abelian soluble, there exists a $s > 1$ such that $N^s = 1$. Suppose that s is minimum with this property. Since N is normal in M, N^{s-1} is a non-trivial abelian normal subgroup of M. But $N^{s-1} \subseteq C_M(A)$. So AN^{s-1} is an abelian normal subgroup of M and hence $AN^{s-1} = A$, because A is maximal. Thus, putting $T = AN^{s-2}$, we have $T' = N^{s-1} \subseteq A \subseteq Z(T)$ and therefore T is non-abelian nilpotent of class 2. Take a maximal normal abelian subgroup B of T. Then $C_T(B) = B$, because T is nilpotent (cf., [12, 5.2.2]). Now, replacing M and A in Case 1 by T and B respectively, this case is reduced to Case 1. □

Theorem 2.2 basically says that if every maximal subfield of D has trivial normalizer then non-abelian subgroups of D^* are "big". As a consequence of Theorem 2.1 we single out the following corollary.

Corollary 2.3. *A division algebra D has a maximal subfield with non-trivial normalizer if and only if D^* contains a non-abelian soluble subgroup.*

Proof. The "if" implication is actually the contrapositive of (1)⇒(3) in Theorem 2.1. For "only if" let K be a maximal subfield of D with $N_{D^*}(K^*) \neq K^*$. Pick $a \in N_{D^*}(K^*) \setminus K^*$. Since $a \notin K = C_D(K)$, it follows that $a^{-1}ka \neq k$ for some $k \in K^*$. Thus the subgroup $G = K^*\langle a \rangle$ is non-abelian. Moreover, since K^* and G/K^* are abelian, we conclude that G is soluble, as desired. $\square$

The next example shows that even for the case of Corollary 2.3, D may have maximal subfields with trivial normalizer.

Example 2.4. Consider the finite fields $\mathbb{F}_q$ and $\mathbb{F}_{q^n}$ with q and q^n elements, respectively. Moreover assume that n and $q-1$ are relatively prime. From field theory we know that $\mathbb{F}_{q^n}/\mathbb{F}_q$ is a cyclic extension of degree n. Take a generator σ of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Then by Hilbert construction [7, Proposition 14.2], $D = \mathbb{F}_{q^n}((x,\sigma))$ is a division algebra of index n and center $F = \mathbb{F}_q((x^n))$. In this setting, $K = \mathbb{F}_{q^n}((x^n))$ and $L = \mathbb{F}_q((x))$ are two distinct maximal subfields of D. We observe that K/F is Galois cyclic and hence $N_{D^*}(K^*)/K^* \cong \mathrm{Gal}(K/F) \cong \mathbb{Z}_n$. At the other extreme, L/F is a radical extension (since x is radical over F). But, F does not contain any n-th root of unity, because $(n, q-1) = 1$. This immediately implies that $\mathrm{Gal}(L/F) = 1$ and so L has trivial normalizer.

Finally, we conclude this note with a remark about the importance of the above conjectures. There are various open problems that are strongly related to Conjecture 1.2. For instance, the problem of the existence of maximal subgroups in division algebras and non-triviality of certain quotients of K_1 groups of division algebras would have positive answer if Conjecture 1.2 is true (cf., [5] and [10]). One can also observe that the validity of Conjecture 1.2 leads to a complete classification of radicable division algebras [9]. So, an affirmative answer to Conjecture 1.1 would be an umbrella for several open problems in the theory of division algebras.

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References

[1] M. Amiri and M. Ariannejad, Frobenius kernel and Wedderburn's little theorem, *Bull. Iranian Math. Soc.* **40** (2014), no. 4, 961–965.

[2] S.A. Amitsur, Finite subgroups of division rings, *Trans. Amer. Math. Soc.* **80** (1955) 361–386.

[3] P.K. Draxl, Skew Fields, Cambridge Univ. Press, Cambridge, 1983.

[4] R. Hazrat, M. Mahdavi-Hezavehi and M. Motiee, Multiplicative groups of division rings, *Math. Proc. R. Ir. Acad.* **114A** (2014), no. 1, 37–114.

[5] R. Hazrat and A.R. Wadsworth, On maximal subgroups of the multiplicative group of a division algebra, *J. Algebra* **322** (2009), no. 7, 2528–2543.

[6] T. Keshavarzipour and M. Mahdavi-Hezavehi, Crossed product conditions for central simple algebras in terms of irreducible subgroups, *J. Algebra* **315** (2007), no. 2, 738–744.

[7] T.Y. Lam, A First Course in Noncommutative Rings, Grad. Texts in Math. 131, Springer-Verlag, 2nd edition, New York, 2001.
[8] M. Mahdavi-Hezavehi, Free subgroups in maximal subgroups of $\mathrm{GL}_1(D)$, *J. Algebra* **241** (2001), no. 2, 720–730.
[9] M. Mahdavi-Hezavehi and M. Motiee, Division algebras with radicable multiplicative groups, *Comm. Algebra* **39** (2011), no. 11, 4084–4096.
[10] M. Mahdavi-Hezavehi and M. Motiee, A criterion for the triviality of $G(D)$ and its applications to the multiplicative structure of D, *Comm. Algebra* **40** (2012), no. 7, 2645–2670.
[11] M. Mahdavi-Hezavehi and J.-P. Tignol, Cyclicity conditions for division algebras of prime degree, *Proc. Amer. Math. Soc.* **131** (2003), no. 12, 3673–3676.
[12] D.J.S. Robinson, A Course in The Theory of Groups, Grad. Texts in Math. 80, Springer-Verlag, 2nd edition, New York, 1996.
[13] J. Tits, Free subgroups in linear groups, *J. Algebra* **20** (1972) 250–270.
[14] J.H.M. Wedderburn, A theorem on finite algebras, *Trans. Amer. Math. Soc.* **6** (1905), no. 3, 349–352.

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