ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 43 (2017), No. 6, pp. 2057–2066

Title:

Singular values of convex functions of matrices

Author(s):

Z. Ulukök

Published by the Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 43 (2017), No. 6, pp. 2057–2066
Online ISSN: 1735-8515

SINGULAR VALUES OF CONVEX FUNCTIONS OF MATRICES

Z. ULUKÖK

(Communicated by Abbas Salemi)

Abstract. Let $A_i, B_i, X_i, i = 1, \ldots, m$, be n-by-n matrices such that $\sum_{i=1}^m |A_i|^2$ and $\sum_{i=1}^m |B_i|^2$ are nonzero matrices and each X_i is positive semidefinite. It is shown that if f is a nonnegative increasing convex function on $[0, \infty)$ satisfying $f(0) = 0$, then

$$2s_j\left(f\left(\frac{\left|\sum_{i=1}^m A_i^* X_i B_i\right|}{\sqrt{\left\|\sum_{i=1}^m |A_i|^2\right\| \left\|\sum_{i=1}^m |B_i|^2\right\|}}\right)\right) \le s_j\left(\oplus_{i=1}^m f(2X_i)\right)$$

for $j = 1, \ldots, n$. Applications of our results are given.
Keywords: Singular value, arithmetic-geometric mean, direct sum, positive semidefinite matrix, convex function.
MSC(2010): Primary: 15A18; Secondary: 15A42, 15A60.

1. **Introduction**

The set of $n \times n$ complex matrices is denoted by $\mathbb{M}_n(\mathbb{C})$. The singular values $s_1(A), \ldots, s_n(A)$ of $A \in \mathbb{M}_n(\mathbb{C})$ are the eigenvalues of the matrix $|A| = (A^*A)^{1/2}$ arranged in decreasing order and repeated according to multiplicities. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be Hermitian if $A^* = A$, where A^* denotes the conjugate transpose of A. A Hermitian matrix A is said to be positive semidefinite or nonnegative definite written as $A \ge 0$, if $x^*Ax \ge 0$ for all $x \in \mathbb{C}^n$. If $s_1(A) \le 1$ for a matrix $A \in \mathbb{M}_n(\mathbb{C})$, then A is called a contraction.

Article electronically published on 30 November, 2017.
Received: 23 May 2016, Accepted: 24 December 2016.

The direct sum of matrices $A_1, \ldots, A_m$ in $\mathbb{M}_n(\mathbb{C})$, denoted by $\oplus_{i=1}^m A_i$, is the block diagonal matrix defined by $\oplus_{i=1}^m A_i = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_m \end{pmatrix}$. When $m = 2$, we write $A_1 \oplus A_2$ instead of $\oplus_{i=1}^2 A_i$.

The symbol $\|\cdot\|$ shows the spectral norm on $\mathbb{M}_n(\mathbb{C})$, that is the norm defined by $\|A\| = \sup\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}$.

The well-known arithmetic-geometric mean inequality for singular values was proved in [4]. It was shown that if $A, B \in \mathbb{M}_n(\mathbb{C})$ are positive semidefinite, then

$$s_j(AB) \le \frac{1}{2} s_j\left(A^2 + B^2\right), \ j = 1, 2, \ldots, n. \tag{1.1}$$

Inequality (1.1) can be stated in another form: If $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$s_j(A^*B) \le \frac{1}{2} s_j(AA^* + BB^*), \ j = 1, 2, \ldots, n. \tag{1.2}$$

On the other hand, Zhan in [11] proved that

$$s_j(A - B) \le s_j(A \oplus B), \ j = 1, 2, \ldots, n \tag{1.3}$$

for any positive semidefinite matrices $A, B \in \mathbb{M}_n(\mathbb{C})$. It is pointed out in [12] that the two inequalities (1.1) and (1.3) are equivalent.

Tao in [10] gave a new equivalent form of the two inequalities:

$$2 s_j(K) \le s_j \begin{bmatrix} M & K \\ K^* & N \end{bmatrix} \tag{1.4}$$

for any positive semidefinite block matrix $\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$, where $M \in \mathbb{M}_m(\mathbb{C})$, $N \in \mathbb{M}_n(\mathbb{C})$,.

Also, Albadawi in [1] proved that if A, B, X are bounded linear operators on a complex separable Hilbert space $\mathbb{H}$ such that X is positive, then

$$s_j(AXB^*) \le \frac{1}{2} \|X\| \, s_j(A^*A + B^*B). \tag{1.5}$$

for $j = 1, \ldots$.

Zou in [13] obtained a new equivalent form of the arithmetic-geometric mean inequality for singular values. It says that if $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that X is positive semidefinite, then

$$s_j(A^*XB) \le s_j\left((AA^* + BB^*)^{1/2} X (AA^* + BB^*)^{1/2}\right) \tag{1.6}$$

for $j = 1, \ldots, n$.

In this article, we introduce inequalities for the singular values of convex functions of matrices and we give applications of our results for partitioned

2×2 positive semidefinite block matrices. Moreover, special cases of our results are proposed.

2. Main results

In this section, based on the arithmetic-geometric mean inequality (1.2) we establish inequalities for singular values of convex functions of matrices and for 2×2 block matrices. To start our analysis we start with the following basic lemmas that we need.

Lemma 2.1 ([3])**.** *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j\left(AA^*\right) = s_j\left(A^*A\right)$$

for $j = 1, 2, \ldots, n$.

Lemma 2.2 ([3])**.** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$s_j\left(AB\right) \leq \|A\|\, s_j\left(B\right)$$

for $j = 1, 2, \ldots, n$.

Lemma 2.3. *Let $A \in \mathbb{M}_n\left(\mathbb{C}\right)$ and let f be a nonnegative increasing function on I. Then*

$$f\left(s_j\left(A\right)\right) = s_j\left(f\left(|A|\right)\right)$$

for $j = 1, 2, \ldots, n$.

Lemma 2.4. *Let $A, B \in \mathbb{M}_n\left(\mathbb{C}\right)$. Then*

$$s_j\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = s_j\left(\left(A + B \oplus \left(A - B\right)\right)\right)$$

for $j = 1, 2, \ldots, n$.

Lemma 2.5. *Let $A, X \in \mathbb{M}_n\left(\mathbb{C}\right)$ such that A is positive semidefinite with spectrum contained in an interval I and X is contraction. If f is an increasing convex function on I such that $0 \in I$ and $f\left(0\right) \leq 0$, then*

$$s_j\left(f\left(X^*AX\right)\right) \leq s_j\left(X^*f\left(A\right)X\right)$$

for $j = 1, 2, \ldots, n$.

The following is our first main result.

Theorem 2.6. *Let $A, B, X \in \mathbb{M}_n(\mathbb{C})$ such that A, B are nonzero matrices and X is positive semidefinite. If f is a nonnegative increasing convex function on $[0, \infty)$ satisfying $f\left(0\right) = 0$, then*

$$2s_j\left(f\left(\frac{|A^*XB|}{\|A\|\,\|B\|}\right)\right) \leq s_j\left(f\left(2X\right)\right) \tag{2.1}$$

for $j = 1, \ldots, n$.

Proof. Let $\mathcal{A} = \frac{A}{\|A\|}$ and $\mathcal{B} = \frac{B}{\|B\|}$. Then

$$\|\mathcal{A}\| = \|\mathcal{B}\| = 1 \tag{2.2}$$

and the partitioned matrix $\frac{1}{\sqrt{2}}\begin{bmatrix} |\mathcal{A}^*| & 0 \\ 0 & |\mathcal{B}^*| \end{bmatrix}$ is a contraction. For $j = 1, \ldots, n$, we have

$$s_j\left(f\left(|\mathcal{A}^* X \mathcal{B}|\right)\right) = s_j\left(f\left(\frac{|A^* X B|}{\|A\|\,\|B\|}\right)\right), \tag{2.3}$$

$$s_j\left(2X\right) = s_j\left(2X \oplus 0\right), \tag{2.4}$$

and

$$s_j\left(f\left(|\mathcal{A}^* X \mathcal{B}|\right)\right) = s_j\left(f\left(|\mathcal{A}^* X \mathcal{B}| \oplus 0\right)\right). \tag{2.5}$$

Now,

$$\begin{aligned}
s_j\left(f\left(|\mathcal{A}^* X \mathcal{B}|\right)\right) &= s_j\left(f\left(|\mathcal{A}^* X \mathcal{B}| \oplus 0\right)\right) \quad \text{(by Lemma 2.3 and since } f(0)=0\text{)} \\
&\leq f\left(s_j\left(\frac{X^{1/2}\left|\mathcal{A}^*\right|^2 X^{1/2} + X^{1/2}\left|\mathcal{B}^*\right|^2 X^{1/2}}{2} \oplus 0\right)\right) \\
&\qquad \text{(by inequality (1.2))} \\
&= f\left(s_j\left(\frac{1}{2}\begin{bmatrix} X^{1/2} & X^{1/2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} |\mathcal{A}^*|^2 & 0 \\ 0 & |\mathcal{B}^*|^2 \end{bmatrix}\begin{bmatrix} X^{1/2} & 0 \\ X^{1/2} & 0 \end{bmatrix}\right)\right) \\
&= f\left(s_j\left(\frac{1}{\sqrt{2}}\begin{bmatrix} |\mathcal{A}^*| & 0 \\ 0 & |\mathcal{B}^*| \end{bmatrix}\begin{bmatrix} X & X \\ X & X \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} |\mathcal{A}^*| & 0 \\ 0 & |\mathcal{B}^*| \end{bmatrix}\right)\right) \\
&\qquad \text{(by Lemma 2.1)} \\
&\leq \frac{1}{2}\left\|\begin{bmatrix} |\mathcal{A}^*| & 0 \\ 0 & |\mathcal{B}^*| \end{bmatrix}\right\|^2 s_j\left(f\begin{bmatrix} X & X \\ X & X \end{bmatrix}\right) \\
&\qquad \text{(by Lemma 2.2)} \\
&= \frac{1}{2}\max\left(\|\mathcal{A}\|^2, \|\mathcal{B}\|^2\right) s_j\left(f\begin{bmatrix} 2X & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&\qquad \text{(by Lemma 2.4)} \\
&= \frac{1}{2} s_j\left(f\left(2X \oplus 0\right)\right) \quad \text{(by identity (2.2))} \\
&= \frac{1}{2} f\left(s_j\left(2X \oplus 0\right)\right) \quad \text{(by Lemma 2.3)} \\
&= \frac{1}{2} f\left(s_j\left(2X\right)\right) \quad \text{(by identity (2.4))} \\
&= \frac{1}{2} s_j\left(f\left(2X\right)\right) \quad \text{(by Lemma 2.3).} \qquad (2.6)
\end{aligned}$$

Thus, the result follows from identites (2.3), (2.4), and inequality (2.6). $\square$

An extension of Theorem 2.6 for sums and products of several matrices can be stated as follows. Its proof follows by applying Theorem 2.6 to the matrices $\mathcal{A} = \begin{bmatrix} A_1 & 0 \\ \vdots & \vdots \\ A_m & 0 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} B_1 & 0 \\ \vdots & \vdots \\ B_m & 0 \end{bmatrix}$, and $\mathcal{X} = \oplus_{i=1}^m X_i$.

Corollary 2.7. *Let $A_i, B_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1, \ldots, m$, such that $\sum_{i=1}^m |A_i|^2$ and $\sum_{i=1}^m |B_i|^2$ are nonzero matrices and each X_i is positive semidefinite. If f is a nonnegative increasing convex function on $[0, \infty)$ satisfying $f(0) = 0$, then*

$$2s_j \left(f \left(\frac{\left|\sum_{i=1}^m A_i^* X_i B_i\right|}{\sqrt{\left\|\sum_{i=1}^m |A_i|^2\right\| \left\|\sum_{i=1}^m |B_i|^2\right\|}} \right) \right) \le s_j \left(\oplus_{i=1}^m f(2X_i) \right) \tag{2.7}$$

for $j = 1, \ldots, n$.

Another application of Theorem 2.6 can be seen in the following result.

Corollary 2.8. *Let $A_i, B_i, K, M, N \in \mathbb{M}_n(\mathbb{C})$, $i = 1, 2$, such that $|A_1|^2 + |A_2|^2$ and $|B_1|^2 + |B_2|^2$ are nonzero matrices and $L = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ is positive semidefinite. If f is a nonnegative increasing convex function on $[0, \infty)$ satisfying $f(0) = 0$, then*

$$2s_j \left(f \left(\frac{|A_1^* M B_1 + A_2^* K^* B_1 + A_1^* K B_2 + A_2^* N B_2|}{\sqrt{\left\||A_1|^2 + |A_2|^2\right\| \left\||B_1|^2 + |B_2|^2\right\|}} \right) \right) \le s_j \left(f(2L) \right)$$

for $j = 1, \ldots, n$.

Proof. Let $A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}$. Then

$$2s_j \left(f \left(\frac{|A_1^* M B_1 + A_2^* K^* B_1 + A_1^* K B_2 + A_2^* N B_2|}{\sqrt{\left\||A_1|^2 + |A_2|^2\right\| \left\||B_1|^2 + |B_2|^2\right\|}} \right) \right)$$

$$
\begin{aligned}
&= 2s_j\left(f\left(\frac{|A^*LB|}{\|A\|\,\|B\|}\oplus 0\right)\right)\\
&= 2s_j\left(f\left(\frac{|A^*LB|}{\|A\|\,\|B\|}\right)\oplus f(0)\right)\\
&= 2s_j\left(f\left(\frac{|A^*LB|}{\|A\|\,\|B\|}\right)\oplus 0\right)\ \ (\text{since } f(0)=0)\\
&\le s_j\left(f(2L)\oplus 0\right)\ \ (\text{by Theorem 2.6})\\
&= s_j\left(f(2L)\right)
\end{aligned}
$$

for $j = 1, \ldots, n$. □

Now, we give three applications of Corollary 2.8 by using some special positive semidefinite 2×2 block matrices.

Corollary 2.9. *Let $A_i, B_i, X, Y \in \mathbb{M}_n(\mathbb{C})$, $i = 1, 2$, such that $|A_1|^2 + |A_2|^2$ and $|B_1|^2 + |B_2|^2$ are nonzero matrices. If f is a nonnegative increasing convex function on $[0, \infty)$ satisfying $f(0) = 0$, then*

$$
2s_j\left(f\left(\frac{\left|\begin{array}{c} A_1^*X^*XB_1 + A_2^*Y^*XB_1 \\ +A_1^*X^*YB_2 + A_2^*Y^*YB_2 \end{array}\right|}{\sqrt{\left\||A_1|^2+|A_2|^2\right\|\left\||B_1|^2+|B_2|^2\right\|}}\right)\right) \le s_j\left(f\left(2\left(XX^*+YY^*\right)\right)\right)
$$

for $j = 1, \ldots, n$.

Proof. Let $L = \begin{bmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y \end{bmatrix}$ in Corollary 2.8. Then $L = P^*P$ is positive semidefinite, where $P = \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix}$. It follows from Corollary 2.8 that

$$
2s_j\left(f\left(\frac{\left|\begin{array}{c} A_1^*X^*XB_1 + A_2^*Y^*XB_1 \\ +A_1^*X^*YB_2 + A_2^*Y^*YB_2 \end{array}\right|}{\sqrt{\left\||A_1|^2+|A_2|^2\right\|\left\||B_1|^2+|B_2|^2\right\|}}\right)\right) \le s_j\left(f(2L)\right) \tag{2.8}
$$

for $j = 1, \ldots, n$. Also,

$$
\begin{aligned}
s_j(f(2L)) &= f(2s_j(P^*P))\\
&= f(2s_j(PP^*))\\
&= f(2s_j(((XX^*+YY^*)\oplus 0)))\\
&= s_j(f(2(XX^*+YY^*))))
\end{aligned}
\tag{2.9}
$$

for $j = 1, \ldots, n$. Now, the result follows from inequality (2.8) and identity (2.9). □

Corollary 2.10. *Let $A_i, B_i, X, Y \in \mathbb{M}_n(\mathbb{C})$, $i = 1, 2$, such that $|A_1|^2 + |A_2|^2$ and $|B_1|^2 + |B_2|^2$ are nonzero matrices and X, Y are positive semidefinite. If f is a nonnegative increasing convex function on $[0, \infty)$ satisfying $f(0) = 0$, then*

$$2s_j\left(f\left(\frac{\left|\begin{array}{c} A_1^*(X+Y)B_1 + A_2^*(X-Y)B_1 \\ +A_1^*(X-Y)B_2 + A_2^*(X+Y)B_2 \end{array}\right|}{\sqrt{\left\||A_1|^2+|A_2|^2\right\|\left\||B_1|^2+|B_2|^2\right\|}}\right)\right) \le s_j\left(f(2X) \oplus f(2Y)\right)$$

for $j = 1, \ldots, n$.

Proof. Let $L = \begin{bmatrix} \frac{X+Y}{2} & \frac{X-Y}{2} \\ \frac{X-Y}{2} & \frac{X+Y}{2} \end{bmatrix}$ and $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & I_n \\ -I_n & I_n \end{bmatrix}$. Then $L = U^*(X \oplus Y)U$ and so L is positive semidefinite. It follows from Corollary 2.8 that

$$2s_j\left(f\left(\frac{\left|\begin{array}{c} A_1^*(X+Y)B_1 + A_2^*(X-Y)B_1 \\ +A_1^*(X-Y)B_2 + A_2^*(X+Y)B_2 \end{array}\right|}{2\sqrt{\left\||A_1|^2+|A_2|^2\right\|\left\||B_1|^2+|B_2|^2\right\|}}\right)\right) \le s_j\left(f(2L)\right) \tag{2.10}$$

for $j = 1, \ldots, n$. Since U is unitary and $L = U^*(X \oplus Y)U$, we have

$$\begin{aligned} s_j(f(2L)) &= f\left(2s_j\left(U^*(X \oplus Y)U\right)\right) \\ &= s_j\left(f(2X) \oplus f(2Y)\right) \end{aligned} \tag{2.11}$$

for $j = 1, \ldots, n$. Now, the result follows from inequality (2.10) and identity (2.11). ∎

Corollary 2.11. *Let $A_i, B_i, X, Y \in \mathbb{M}_n(\mathbb{C})$, $i = 1, 2$ such that $|A_1|^2 + |A_2|^2$, $|B_1|^2 + |B_2|^2$ are nonzero matrices and X, Y are positive semidefinite. If m is a positive integer, then*

$$2s_j\left(f\left(\frac{\left|\begin{array}{c} A_1^*X^{1/2}\mathbb{Z}^{m-1}X^{1/2}B_1 + A_2^*Y^{1/2}\mathbb{Z}^{m-1}X^{1/2}B_1 \\ +A_1^*X^{1/2}\mathbb{Z}^{m-1}Y^{1/2}B_2 + A_2^*Y^{1/2}\mathbb{Z}^{m-1}Y^{1/2}B_2 \end{array}\right|}{\sqrt{\left\||A_1|^2+|A_2|^2\right\|\left\||B_1|^2+|B_2|^2\right\|}}\right)\right) \le s_j\left(f(2\mathbb{Z}^m)\right)$$

for $j = 1, \ldots, n$, where $\mathbb{Z} = X + Y$.

Proof. Let $L = (PP^*)^m$, where $P = \begin{bmatrix} X^{1/2} & 0 \\ Y^{1/2} & 0 \end{bmatrix}$. Then L is positive semidefinite,

$$\begin{aligned} (P^*P)^r &= (X+Y)^r \oplus 0 \\ &= \mathbb{Z}^r \oplus 0 \quad (\text{for } r = m-1 \text{ and } m) \end{aligned}$$

and

$$\begin{aligned} L &= P\left(P^{*}P\right)^{m-1}P^{*} \\ &= \begin{bmatrix} X^{1/2}\mathbb{Z}^{m-1}X^{1/2} & X^{1/2}\mathbb{Z}^{m-1}Y^{1/2} \\ Y^{1/2}\mathbb{Z}^{m-1}X^{1/2} & Y^{1/2}\mathbb{Z}^{m-1}Y^{1/2} \end{bmatrix}. \end{aligned}$$

It follows from Corollary 2.8 that

$$2s_j\left(f\left(\frac{\left|\begin{array}{c} A_1^{*}X^{1/2}\mathbb{Z}^{m-1}X^{1/2}B_1+A_2^{*}Y^{1/2}\mathbb{Z}^{m-1}X^{1/2}B_1 \\ +A_1^{*}X^{1/2}\mathbb{Z}^{m-1}Y^{1/2}B_2+A_2^{*}Y^{1/2}\mathbb{Z}^{m-1}Y^{1/2}B_2 \end{array}\right|}{\sqrt{\left\||A_1|^2+|A_2|^2\right\|\left\||B_1|^2+|B_2|^2\right\|}}\right)\right)$$

$$\le s_j\left(f\left(2L\right)\right) \tag{2.12}$$

for $j=1,\ldots,n$. Also,

$$\begin{aligned} s_j\left(f\left(2L\right)\right) &= f\left(2s_j\left(\left(PP^{*}\right)^{m}\right)\right) \\ &= f\left(2s_j\left(\left(P^{*}P\right)^{m}\right)\right) \\ &= f\left(2s_j\left(\mathbb{Z}^{m}\oplus 0\right)\right) \\ &= s_j\left(f\left(2\mathbb{Z}^{m}\right)\right) \end{aligned} \tag{2.13}$$

for $j=1,\ldots,n$. Now, the result follows from inequality (2.12) and identity (2.13). □

In order to give our second main result, we need the following lemma.

Lemma 2.12. *Let f be a convex function on $[0,\infty)$ such that $f(0)\le 0$. Then*

$$\frac{f\left(x\right)}{y}\le f\left(\frac{x}{y}\right)$$

for all $x\ge 0$ and $0<y\le 1$.

Proof.

$$\begin{aligned} \frac{f\left(x\right)}{y} &= \frac{1}{y}\left(f\left(y\left(\frac{x}{y}\right)+\left(1-y\right)0\right)\right) \\ &\le \frac{1}{y}\left(yf\left(\frac{x}{y}\right)+\left(1-y\right)f\left(0\right)\right) \text{ (since } f \text{ is convex)} \\ &\le f\left(\frac{x}{y}\right) \text{ (since } f(0)\le 0\text{),} \end{aligned}$$

as required. □

Based on Corollary 2.7 and Lemma 2.12 we have the following result.

Theorem 2.13. *Let $A_i, B_i, X_i \in \mathbb{M}_n(\mathbb{C})$, $i = 1, \ldots, m$, such that $\sum_{i=1}^m |A_i|^2$, $\sum_{i=1}^m |B_i|^2$ are contractions and each X_i is positive semidefinite. If f is a nonnegative increasing convex function on $[0, \infty)$ satisfying $f(0) = 0$, then*

$$2s_j\left(f\left(\left|\sum_{i=1}^m A_i^* X_i B_i\right|\right)\right) \leq \sqrt{\left\|\sum_{i=1}^m |A_i|^2\right\| \left\|\sum_{i=1}^m |B_i|^2\right\|} s_j\left(\oplus_{i=1}^m f(2X_i)\right)$$

for $j = 1, \ldots, n$.

Proof. Without loss of generality, assume that the matrices $\sum_{i=1}^m |A_i|^2$, $\sum_{i=1}^m |B_i|^2$ are both nonzero. Since the matrix $|\sum_{i=1}^m A_i^* X_i B_i|$ is positive semidefinite, there exist $U, D \in \mathbb{M}_n(\mathbb{C})$ such that U is unitary and $D = diag(\lambda_1, \ldots, \lambda_n)$ is diagonal such that $|\sum_{i=1}^m A_i^* X_i B_i| = U^* D U$.

Let $\mathcal{A} = \left(\sum_{i=1}^m |A_i|^2\right)^{1/2}$ and $\mathcal{B} = \left(\sum_{i=1}^m |B_i|^2\right)^{1l2}$. Then $\mathcal{A}$ and $\mathcal{B}$ are contractions and so

$$\begin{aligned}
f\left(\frac{|\sum_{i=1}^m A_i^* X_i B_i|}{\sqrt{\left\|\sum_{i=1}^m |A_i|^2\right\| \left\|\sum_{i=1}^m |B_i|^2\right\|}}\right) &= U^* f\left(\frac{D}{\|\mathcal{A}\| \|\mathcal{B}\|}\right) U \\
&= U^* diag\left(f\left(\frac{\lambda_1}{\|\mathcal{A}\| \|\mathcal{B}\|}\right), \ldots, f\left(\frac{\lambda_n}{\|\mathcal{A}\| \|\mathcal{B}\|}\right)\right) U \\
&\geq U^* diag\left(\frac{f(\lambda_1)}{\|\mathcal{A}\| \|\mathcal{B}\|}, \ldots, \frac{f(\lambda_n)}{\|\mathcal{A}\| \|\mathcal{B}\|}\right) U \\
&\qquad \text{(by Lemma 2.12)} \\
&= \frac{U^* f(D) U}{\|\mathcal{A}\| \|\mathcal{B}\|} \\
&= \frac{f\left(\left|\sum_{i=1}^m A_i^* X_i B_i\right|\right)}{\|\mathcal{A}\| \|\mathcal{B}\|}. \qquad (2.14)
\end{aligned}$$

Now, the result follows from Corollary 2.7 and inequality (2.14). □

Applying Corollary 2.7 to the convex functions $f(t) = t^r$, $r \geq 1$ and $f(t) = e^{t^r} - 1$, $r \geq 1$, we have the following two results. Similar results can be obtained by applying our other results to these functions.

Corollary 2.14. *Let $A_i, B_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1, \ldots, m$, such that each X_i is positive semidefinite and let $r \geq 1$. Then*

$$s_j\left(\sum_{i=1}^m A_i^* X_i B_i\right) \leq 2^{\frac{r-1}{r}} \sqrt{\left\|\sum_{i=1}^m |A_i|^2\right\| \left\|\sum_{i=1}^m |B_i|^2\right\|} s_j\left(\oplus_{i=1}^m X_i\right) \qquad (2.15)$$

for $j = 1, \ldots, n$.

Corollary 2.15. *Let $A_i, B_i, X_i \in \mathbb{M}_n(\mathbb{C}), i = 1, \ldots, m$, such that $\sum_{i=1}^m |A_i|^2$, $\sum_{i=1}^m |B_i|^2$ nonzero matrices and each X_i is positive semidefinite. Then*

$$2s_j\left(e^{\frac{\left|\sum_{i=1}^m A_i^* X_i B_i\right|^r}{\left(\left\|\sum_{i=1}^m |A_i|^2\right\|\left\|\sum_{i=1}^m |B_i|^2\right\|\right)^{r/2}}} - I_n\right) \le s_j\left(\oplus_{i=1}^m\left(e^{2^r X_i^r} - I_n\right)\right)$$

for $j = 1, \ldots, n$. In particular, if $\sum_{i=1}^m |A_i|^2$ and $\sum_{i=1}^m |B_i|^2$ are contractions, then

$$s_j\left(e^{\left|\sum_{i=1}^m A_i^* X_i B_i\right|^r} - I_n\right) \le \frac{\sqrt{\left\|\sum_{i=1}^m |A_i|^2\right\|\left\|\sum_{i=1}^m |B_i|^2\right\|}}{2} s_j\left(\oplus_{i=1}^m\left(e^{2^r X_i^r} - I_n\right)\right)$$

for $j = 1, \ldots, n$.

Acknowledgements

The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] H. Albadawi, Singular value and arithmetic-geometric mean inequalities for operators, *Ann. Funct. Anal.* **3** (2012), no. 1, 10–18.

[2] J.S. Aujla and F.C. Silva, Weak majorization inequalities and convex functions, *Linear Algebra Appl.* **369** (2003) 217–233.

[3] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.

[4] R. Bhatia and F. Kittaneh, On the singular values of a product of operators, *SIAM J. Matrix Anal. Appl.* **11** (1990), no. 2, 272–277.

[5] R. Bhatia and F. Kittaneh, Notes on matrix arithmetic-geometric mean inequalities, *Linear Algebra Appl.* **308** (2000) 203–211.

[6] K. Fan, On a theorem of Weyl concerning eigenvalues of linear transformations I, *Proc. Natl. Acad. Sci. USA* **35** (1994) 652–655.

[7] O. Hirzallah, Singular values of convex functions of operators and the arithmetic-geometric mean inequality, *J. Math. Anal. Appl.* **433** (2016) 935–947.

[8] R.A. Horn and C.R. Johnson, Topics in Matrix Analysis, Cambridge Univ. Press, Cambridge, 1985.

[9] A.W. Marshall, I. Olkin and B.C. Arnold, Inequalities: Theory of Majorization and Its Applications, Springer Ser. Statist. Springer, 2nd edition, New York, 2011.

[10] Y. Tao, More results on singular value inequalities of matrices, *Linear Algebra Appl.* **416** (2006) 724–729.

[11] X. Zhan, Singular values of differences of positive semidefinite matrices, *SIAM J. Matrix Anal. Appl.* **22** (2000), no. 3, 819–823.

[12] X. Zhan, Matrix Inequalities, Lecture Notes in Math. 1790, Springer-Verlag, Berlin, 2002.

[13] L. Zou, An arithmetic-geometric mean inequality for singular values and its applications, *Linear Algebra Appl.* **528** (2017) 25–32.

(Zübeyde Ulukök) Vadi Park Sit., Gulvatan Sok., Yazir, Selçuklu, 42250, Konya, Turkey.

E-mail address: zbydmatrix@gmail.com.tr