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SOME RESULTS ON PRE-MONOTONE OPERATORS

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ABSTRACT. In this paper, some properties of pre-monotone operators are proved. It is shown that in a reflexive Banach space, a full domain multivalued σ-monotone operator with sequentially norm×weak* closed graph is norm×weak* upper semicontinuous. The notion of σ-convexity is introduced and the relations between the σ-monotonicity and σ-convexity is investigated. Moreover, some results on the sum and difference of two σ-monotone operators is considered.
Keywords: Generalized monotone operators, upper semicontinuous, convex function.
MSC(2010): Primary: 47H05; Secondary: 49J53, 47H04.

1. Introduction

Throughout this paper, X is a Banach space with norm $\|\cdot\|$, X^* is the topological dual of X. Also X and X^* are paired by $\langle\cdot,\cdot\rangle$. We denote by $\to$, $\xrightarrow{w}$ and $\xrightarrow{w^*}$ the strong, weak and weak* convergence of nets, respectively. We also write $\mathbb{R}_+ = [0, +\infty)$.

Let $T : X \multimap X^*$ be a multivalued operator. For convenience, we will use $D(T) := \{x \in X : T(x) \neq \emptyset\}$ and $R(T) := T(X)$ to denote the *domain* and *range* of T. The *graph* of T is

$$\operatorname{gr}(T) := \{(x, x^*) \in X \times X^* : x^* \in T(x)\},$$

and T is *monotone* if

$$\langle x^* - y^*, x - y\rangle \geq 0$$

for all $(x, x^*) \in \operatorname{gr} T$ and $(y, y^*) \in \operatorname{gr} T$. We recall that T is *maximal monotone* if it is monotone and its graph is not properly included in any other monotone graph.

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The notion of monotone operator has been found appropriate in various branches of Mathematics such as Operator Theory, Partial Differential Equations, Differentiability Theory of Convex Functions, Numerical Analysis and has brought a new life to Nonlinear Functional Analysis and Nonlinear Operator Equations. In particular, monotone operators are a powerful tool for the study of variational inequalities. All these reasons have convinced many mathematicians, to start research in this rich and important branch of mathematics.

Monotone operators have been generalized in many ways; see [7]. One of these generalizations is the so-called σ-monotone operators which is introduced and studied by Iusem, Kassay and Sosa in the seminal paper [8]. Pre-monotone operators included many important classes of operators such as monotone and ε-monotone operators (see [2, 4] and [8]).

In this paper we introduce the notion of σ-convexity and we investigate its relation with σ-monotonicity. We also show that, given two maximal σ-monotone operators T and S, a weak condition on the mutual position of their domains implies that $T + S$ is weak* closed valued. Moreover, we will study conditions under which difference of two maximal σ-monotone operators is maximal σ-monotone.

The paper is organized as follows. The current section contains some notations, and a short review of basic concepts of σ-monotone operators. In Section 2, we will introduce the notion of σ-convexity and study the relation between σ-monotonicity and σ-convexity. Some more results on σ-monotonicity are investigated in Section 3. In fact, we show that in a reflexive Banach space if the graph of a σ-monotone operator, with full domain, is closed then the operator is upper semicontinuous. Section 4 is devoted to some results on the sum of σ-monotone operators. Finally, in Section 5, it is shown that under some mild additional assumptions the difference of two maximal σ-monotone operators is a maximal σ-monotone operator.

We recall the notion of σ-monotonicity based on some ideas from [2–4].

Definition 1.1 ([4, Definition 2.1]). Given an operator $T : X \multimap X^*$ and a map $\sigma : D(T) \to \mathbb{R}_+$.

(i) T is said to be σ-*monotone*, if

$$\langle x^* - y^*, x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\|, \tag{1.1}$$

for every $(x, x^*), (y, y^*) \in \mathrm{gr}(T)$.

(ii) T is called *pre-monotone*, if it is σ-monotone for some $\sigma : D(T) \to \mathbb{R}_+$.

(iii) A σ-monotone operator T is called *maximal σ-monotone*, if for every σ'-monotone operator T' with $\mathrm{gr}(T) \subseteq \mathrm{gr}(T')$, where σ' is an extension of σ, one has $T = T'$.

Example 1.2. Consider the functions $\varphi, \sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} x \sin^2 x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

and

$$\sigma(x) = \max\{\varphi(x), \max_{z \leq x} \varphi(z) - \varphi(x)\}.$$

In [4, Example 2.8], it is shown that φ is σ-monotone, but not ε-monotone. Another example can be found in [8, Example 3.6].

Remark 1.3 ([4, Remark 2.2]).

(i) The notion of pre-monotone operators for the finite-dimensional case was introduced and studied in [8].

(ii) An operator $T : X \multimap X^*$ is σ-monotone if and only if

$$\langle x^* - y^*,\ x - y\rangle \geq -\sigma(y)\|x - y\|, x, y \in D\left(T\right),\ x^* \in T\left(x\right),\ y^* \in T\left(y\right).$$

(iii) Every globally bounded operator is pre-monotone, [8, Proposition 3.3(i)].

(iv) A σ-monotone operator is maximal σ-monotone if and only if, for every σ'-monotone operator T' which is with $\mathrm{gr}(T) \subseteq \mathrm{gr}(T')$ and $\sigma'(x) \leq \sigma(x)$ for all $x \in D(T)$, one has $T = T'$.

Definition 1.4 ([4, Definition 2.4])**.** Let A be a subset of X. Given a mapping $\sigma : A \to \mathbb{R}_+$, two pairs $(x, x^*), (y, y^*) \in A \times X^*$ are *σ-monotonically related* if

$$\langle x^* - y^*, x - y\rangle \geq -\min\{\sigma(x), \sigma(y)\}\|x - y\|.$$

Proposition 1.5 ([4, Proposition 2.5])**.** *The σ-monotone operator $T : X \multimap X^*$ is maximal σ-monotone if and only if, for every point $(x_0, x_0^*) \in X \times X^*$ and every extension σ' of σ to $D(T) \cup \{x_0\}$ such that (x_0, x_0^*) is σ'-monotonically related to all pairs $(y, y^*) \in \mathrm{gr}(T)$, we have $(x_0, x_0^*) \in \mathrm{gr}(T)$.*

Given an operator $T : X \multimap X^*$, in [4] the function $\sigma_T : D(T) \to \mathbb{R}_+ \cup \{\infty\}$ is defined by

$$\sigma_T(y) = \inf\{a \in \mathbb{R}_+ : \langle x^* - y^*, x - y\rangle \geq -a \left\|x - y\right\|, \forall (x, x^*) \in \mathrm{gr}(T), y^* \in T(y)\}.$$

Note that if the operator T is pre-monotone, then

$$\sigma_T = \inf\{\sigma : T \text{ is } \sigma\text{-monotone,}\}$$

and thus σ_T is finite, and T is σ_T-monotone. In this case, it is obvious that

$$\sigma_T(y) = \max\left\{\sup\{\frac{\langle x^* - y^*, y - x\rangle}{\|y - x\|} : x \neq y, (x, x^*) \in \mathrm{gr}(T), y^* \in T(y)\}, 0\right\}.$$

Proposition 1.6 ([4, Proposition 2.6])**.** *For an operator $T : X \multimap X^*$,*

(i) *σ_T is finite and T is σ_T-monotone if and only if T is σ-monotone for some σ.*

(ii) *σ_T is finite and T is maximal σ_T-monotone if and only if T is maximal σ-monotone for some σ.*

2. Continuity and σ-monotonicity

In this section, the interrelation between σ-monotonicity and continuity is investigated. First we recall the following result from [4].

Proposition 2.1 ([4, Proposition 2.7])**.** *Let $T : X \multimap X^*$ be a maximal σ-monotone operator. Then T has convex and weak* closed values. If σ is defined and upper semicontinuous on $\mathrm{cl}(D(T))$, then $\mathrm{gr}(T)$ is sequentially norm$\times$weak* closed.*

Example 2.2. Let $\varphi, \sigma : \mathbb{R} \to \mathbb{R}$ be as in Example 1.2. Define $T, \sigma_1 : \mathbb{R} \to \mathbb{R}$ by

$$T(x) = \begin{cases} \varphi(x) & \text{if } \ x \neq \frac{\pi}{2}, \\ \frac{\pi}{4} & \text{if } \ x = \frac{\pi}{2}, \end{cases} \quad \text{and} \quad \sigma_1(x) = \begin{cases} \sigma(x) & \text{if } \ x \neq \frac{\pi}{2}, \\ \frac{\pi}{4} & \text{if } \ x = \frac{\pi}{2}. \end{cases}$$

Alizadeh et al. in [4, Example 2.8] proved that T is σ_1-monotone, while its graph is not closed. So upper semicontinuity of σ in Proposition 2.1 is essential.

Note that in Proposition 2.1 we observed that if T is maximal σ-monotone and σ is upper semicontinuous, then $\mathrm{gr}(T)$ is sequentially norm$\times$weak* closed. As for monotone operators (see [5]), in general $\mathrm{gr}(T)$ is only sequentially norm$\times$weak* closed, not norm$\times$weak* closed. Moreover, in Example 2.2 it is shown that upper semicontinuity of σ cannot be omitted from the statement of Proposition 2.1. Now in the following we show that closedness of the graph of a σ-monotone and single-valued operator T implies the continuity of σ_T. Next we give an elementary proof for the local boundedness of T in $\mathbb{R}$, however, it is also true for a general Banach space, ([8, Proposition 3.5], see also [2]).

Proposition 2.3. *Suppose that $T : \mathbb{R} \to \mathbb{R}$ is σ-monotone. Then T is locally bounded. Moreover, if $\mathrm{gr}(T)$ is closed, then T is continuous.*

Proof. First we show that T is locally bounded on $\mathbb{R}$. Assume that $a < b$. Note that, for each $y \in \mathbb{R}$,

$$\sigma_T(y) = \max\left\{ \sup_{x \leq y}\{T(x) - T(y)\}, \sup_{x \geq y}\{T(y) - T(x)\} \right\}.$$

Thus $\sigma_T(b) \geq \sup_{x \leq b}\{T(x) - T(b)\}$ and so $T(x) \leq \sigma_T(b) + T(b)$ for all $x \leq b$. i.e. T is bounded above on $(-\infty, b]$. Likewise, $\sigma_T(a) \geq \sup_{a \leq x}\{T(a) - T(x)\}$.

Therefore, $T(x) \geq T(a) - \sigma_T(a)$, that is T is bounded below on $[a, +\infty)$. Hence T is bounded on every interval $[a, b]$. Now assume that $\mathrm{gr}(T)$ is closed but T is not continuous. Then there exists a sequence $\{x_n\}$ in $\mathbb{R}$ converging to some x, such that $\{T(x_n)\}$ does not converge to $T(x)$. Thus there exists $\varepsilon > 0$ such that $|T(x_n) - T(x)| \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$. Since T is locally bounded, there would be a subsequence (which we denote again by $\{T(x_n)\}$ for simplicity) converging to a point $a \in \mathbb{R}$ such that $|a - T(x)| \geq \varepsilon$. This means that $(x_n, T(x_n)) \to (x, a) \neq (x, T(x))$, which contradicts the fact that T is closed. $\square$

The last proposition can be generalized as the following. Note that, upper semicontinuity (usc, for short) is equivalent to the continuity when T is single-valued, and then we have the same result as Proposition 2.3. The following proposition and its proof are due to Professor Nicolas Hadjisavvas (private communication), a proof for finite dimensional spaces can be found in [8, Proposition 3.5].

Proposition 2.4. *Suppose X is a reflexive Banach space and $T : X \multimap X^*$ is σ-monotone with $D(T) = X$. Then T is locally bounded. Moreover, if $\mathrm{gr}(T)$ is sequentially norm×weak* closed, then T is norm×weak* usc.*

Proof. The first part is an immediate consequence of [3, Corollary 3.11]. As for the second part, assume that T is not usc at some point x_0. Then there exists an weakly open set $V \subseteq X^*$ such that $T(x_0) \subseteq V$ and for every $\varepsilon > 0$, $T(B(x_0, \varepsilon)) \nsubseteq V$. By taking $\varepsilon = 1/n$ we can construct a sequence $\{x_n\} \subseteq X$ with $\|x_n - x_0\| < \frac{1}{n}$ and $x_n^* \in T(x_n) \cap V^c$. Since T is locally bounded, $\{x_n^*\}$ is a bounded sequence. Since any closed ball in X^* is weak* compact (Alaoglu) there exists a subsequence $\{x_{n_k}^*\}$ weakly converging to some $x^* \in X^*$ (Eberlein-Smulian). We have $(x_{n_k}, x_{n_k}^*) \to (x_0, x^*)$ (norm×weak*) so $x^* \in T(x_0)$ by the closedness assumption. This contradicts the assumption that $x_{n_k}^* \notin V$. $\square$

Proposition 2.5. *Suppose that $T : \mathbb{R} \to \mathbb{R}$ is σ-monotone and $\mathrm{gr}(T)$ is closed. Then σ_T is continuous.*

Proof. For the continuity of σ_T it is enough to show that $\sup_{x \leq y}\{T(x) - T(y)\}$ and $\sup_{x \geq y}\{T(y) - T(x)\}$ are continuous as functions of y. By Proposition 2.3, T is continuous. So it is enough to prove that $f(y) = \sup_{x \leq y} T(x)$ is continuous. The continuity of T implies that T is locally uniformly continuous. Let $y_0 \in \mathbb{R}$. For a given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|T(x) - T(y_0)| < \frac{\varepsilon}{2}, \ \forall x \in [y_0 - \delta, y_0 + \delta]. \tag{2.1}$$

Set $A = [y_0 - \frac{\delta}{2}, y_0 + \frac{\delta}{2}]$ and take $y \in A$. It follows from (2.1) that

$$\left| \sup_{x \in A, x \leq y} T(x) - \sup_{x \in A, x \leq y_0} T(x) \right| < \varepsilon. \tag{2.2}$$

Note that

$$f(y) = \sup_{x \leq y} T(x) = \max\left\{ \sup_{x < y_0 - \frac{\delta}{2}} T(x), \sup_{y_0 - \frac{\delta}{2} \leq x \leq y} T(x) \right\}$$

and

$$f(y_0) = \sup_{x \leq y_0} T(x) = \max\left\{ \sup_{x < y_0 - \frac{\delta}{2}} T(x), \sup_{y_0 - \frac{\delta}{2} \leq x \leq y_0} T(x) \right\}.$$

For simplicity, we set

$$a = \sup_{x < y_0 - \frac{\delta}{2}} T(x),\ b = \sup_{y_0 - \frac{\delta}{2} \le x \le y} T(x) \text{ and } c = \sup_{y_0 - \frac{\delta}{2} \le x \le y_0} T(x).$$

Therefore $f(y) = \max\{a, b\}$ and $f(y_0) = \max\{a, c\}$. Using (2.2) we infer that $|b - c| < \varepsilon$, i.e.

$$-\varepsilon + c < b < \varepsilon + c$$

which implies

$$\max\{a, c - \varepsilon\} < \max\{a, b\} < \max\{a, c + \varepsilon\}. \tag{2.3}$$

On the other hand,

$$-\varepsilon + \max\{a, c\} = \max\{a - \varepsilon, c - \varepsilon\} \le \max\{a, c - \varepsilon\} \tag{2.4}$$

and

$$\max\{a, c + \varepsilon\} \le \max\{a + \varepsilon, c + \varepsilon\} = \max\{a, c\} + \varepsilon. \tag{2.5}$$

Combining (2.3), (2.4) and (2.5) we obtain

$$-\varepsilon + \max\{a, c\} < \max\{a, b\} < \max\{a, c\} + \varepsilon,$$

so $|f(y) - f(y_0)| < \varepsilon$. This means that f is continuous. In a similar way one can get $\sup_{x \ge y}\{T(y) - T(x)\}$ is continuous. □

At this stage the next question naturally arises: Can we extend the above result to more general spaces? For instance, given a pre-monotone operator T with norm×weak* closed graph, is σ_T upper semicontinuous?

3. σ-convexity and σ-monotonicity

In this section, we investigate the notion of σ-convexity which generalizes the concepts of ε-convexity [9] and convexity. First we recall from [9] that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is ε-convex if it satisfies the following inequality for every $a, b \in X$, and $\lambda \in (0, 1)$:

$$f(\lambda a + (1 - \lambda)b) \le \lambda f(a) + (1 - \lambda)f(b) + \lambda(1 - \lambda)\varepsilon\|a - b\|.$$

Definition 3.1. Given $\sigma : \operatorname{dom} f \to \mathbb{R}_+$, we say that the function $f : X \to \mathbb{R} \cup \{+\infty\}$ is σ-convex if

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y) + \lambda(1 - \lambda)\min\{\sigma(x), \sigma(y)\}\|x - y\| \tag{3.1}$$

for all $x, y \in \operatorname{dom} f$ and $\lambda \in (0, 1)$.

We need to recall the following definition.

Definition 3.2. Given a proper function $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x, z \in X$,

(i) the *Clarke-Rockafellar generalized directional derivative* of f at x in the direction of z is defined by

$$f^{\uparrow}(x,z) = \sup_{\delta>0} \limsup_{\substack{(y,\alpha)\xrightarrow{f} x \\ \lambda\searrow 0}} \inf_{u\in B(z,\delta)} \frac{f(y+\lambda u)-\alpha}{\lambda},$$

where $(y,\alpha)\xrightarrow{f} x$ means that $y\to x$, $\alpha\to f(x)$ and $\alpha\geq f(y)$.

(ii) the *Clarke-Rockafellar subdifferential* of f at $x\in \operatorname{dom} f$ is defined by

$$\partial^{CR} f(x) := \{x^*\in X^* : \langle x^*, z\rangle \leq f^{\uparrow}(x,z)\ \forall z\in X\}.$$

(iii) the *Clarke directional derivative* of f at x in the direction of $z\in X$ is defined by

$$f^{o}(x,z) = \limsup_{\substack{y\xrightarrow{f} x \\ \lambda\searrow 0}} \frac{f(y+\lambda z)-f(y)}{\lambda}.$$

(iv) the *Clarke's subdifferential* of f at $x\in \operatorname{dom} f$ is defined by

$$\partial^{C} f(x) = \{x^*\in X^* : \langle x^*, z\rangle \leq f^{o}(x,z)\ \forall z\in X\}.$$

Remark 3.3. If f is lower semicontinuous at x, then the Clarke-Rockafellar generalized directional derivative at x in the direction of $z\in X$ reduces to

$$f^{\uparrow}(x,z) = \sup_{\delta>0} \limsup_{\substack{y\xrightarrow{f} x \\ \lambda\searrow 0}} \inf_{u\in B(z,\delta)} \frac{f(y+\lambda u)-f(y)}{\lambda}.$$

Here, $y\xrightarrow{f} x$ means that $y\to x$ and $f(y)\to f(x)$. Further, if f is locally Lipschitz, then $f^{\uparrow}(x,z) = f^{o}(x,z)$.

Lemma 3.4. *Suppose that $f : X\to \mathbb{R}\cup\{+\infty\}$ is lower semicontinuous and σ-convex, where σ is upper semicontinuous, then*

$$\partial^{CR} f(x) \subseteq \big\{x^*\in X^* : \langle x^*, z\rangle \leq f(x+z)-f(x)+\min\{\sigma(x),\sigma(z+x)\}\|z\|, \forall z\in X\big\}.$$

Proof. By the σ-convexity of f, we get

$$f(y+\lambda u) \leq \lambda f(y+u) + (1-\lambda) f(y) + \lambda(1-\lambda)\min\{\sigma(y),\sigma(u+y)\}\|u\|,$$

for each $y, u \in X$ and $\lambda \in (0,1)$. Fix z and x in X. For an arbitrary $\delta > 0$ and each $y \in B(x,\delta)$, we obtain

$$\begin{aligned}\limsup_{\substack{y \xrightarrow{f} x \\ \lambda \searrow 0}} \inf_{u \in B(z,\delta)} \frac{f(y+\lambda u) - f(y)}{\lambda} &\le \limsup_{\substack{y \xrightarrow{f} x \\ \lambda \searrow 0}} \frac{f(y+\lambda(z+x-y)) - f(y)}{\lambda} \\ &\le \limsup_{\substack{y \xrightarrow{f} x \\ \lambda \searrow 0}} [f(x+z) - f(y) + (1-\lambda)\min\{\sigma(x+z), \sigma(y)\}\|x+z-y\|] \\ &\le f(x+z) - f(x) + \min\{\sigma(x), \sigma(z+x)\}\|z\|.\end{aligned}$$

Therefore,

$$f^{\uparrow}(x,z) \le f(x+z) - f(x) + \min\{\sigma(x), \sigma(z+x)\}\|z\|.$$

By the definition of the Clarke-Rockafellar's subdifferential, the proof is complete. □

Proposition 3.5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and σ-convex and let σ be upper semicontinuous. Then $\partial^{CR} f$ is 2σ-monotone.*

Proof. Select $x, y \in X$, $x^* \in \partial^{CR} f(x)$ and $y^* \in \partial^{CR} f(y)$. Applying Lemma 3.4 we obtain

$$\langle x^*, y-x \rangle \le f(y) - f(x) + \min\{\sigma(x), \sigma(y)\}\|y-x\|,$$

and

$$\langle y^*, x-y \rangle \le f(x) - f(y) + \min\{\sigma(x), \sigma(y)\}\|y-x\|.$$

These inequalities imply that $\partial^{CR} f$ is 2σ-monotone. □

Proposition 3.6. *Let f be locally Lipschitz on X and $\sigma : \operatorname{dom} f \to \mathbb{R}$ be lower semicontinuous. If $\partial^C f$ is σ-monotone, then f is σ-convex.*

Proof. Assume that $\partial^C f$ is σ-monotone, $\lambda \in (0,1)$ and $x, y \in X$ with $x \neq y$. Set $x_\lambda = \lambda x + (1-\lambda) y$. It follows from Lebourg's Mean Value Theorem (see in [14, Theorem 4.5]) that there exist $z_1 \in [x, x_\lambda)$ and $z_1^* \in \partial^C f(z_1)$ such that

$$\langle z_1^*, x_\lambda - x \rangle = f(x_\lambda) - f(x). \tag{3.2}$$

Similarly, there exist $z_2 \in (x_\lambda, y]$ and $z_2^* \in \partial^C f(z_2)$ such that

$$\langle z_2^*, x_\lambda - y \rangle = f(x_\lambda) - f(y). \tag{3.3}$$

Since $x_\lambda - x = (1-\lambda)(y-x)$ and $x_\lambda - y = \lambda(x-y)$, multiplying (3.2) and (3.3) by λ and $1-\lambda$, respectively and adding the resulting equalities we get

$$\lambda f(x) + (1-\lambda) f(y) - f(x_\lambda) = \lambda(1-\lambda)\langle z_1^* - z_2^*, x-y \rangle.$$

Now from the *σ-monotonicity of $\partial^C f$* we obtain (3.1). Thus f is σ-convex. □

Example 3.7. Let $\varphi, \sigma : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x) := \begin{cases} x \sin^2 x & x \geq 0, \\ 0 & x < 0, \end{cases} \quad \text{and} \quad \sigma(x) := \max\{\varphi(x), \max_{z \leq x} \varphi(z) - \varphi(x)\}.$$

It follows from Example 1.2 that φ is σ-monotone and it is not ε-monotone for any $\varepsilon > 0$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\psi(x) = \begin{cases} \int_0^x t \sin^2 t dt & x \geq 0, \\ 0 & x < 0. \end{cases}$$

It follows from Proposition 3.6 that ψ is σ-convex, also [10, Theorem 4.4] implies that ψ is not ε-convex for any $\varepsilon > 0$.

4. Sum of σ-monotone operators

In this section we extend some results of [13] for σ-monotone operators.

Roughly speaking, when we say that two operators are σ-monotone we tacitly assume that σ is defined on the union of their domains.

The idea of the proof of the following theorem for monotone operators was first used by A. Verona and M.E. Verona [13] and then by J.M. Borwein [6]. First we recall a useful fact.

Fact 4.1 ([12, Corollary 4])**.** *Let X be a Banach space, $g_1, g_2 : X \to \mathbb{R} \cup \{\infty\}$ be convex, lower semicontinuous functions such that* $\operatorname{dom} g_1 - \operatorname{dom} g_2$ *be absorbing. Then there exists an $n \geq 1$ such that*

$$\{x \in X : g_1(x) \leq n, \|x\| \leq n\} - \{x \in X : g_2(x) \leq n, \|x\| \leq n\}$$

is a neighborhood of 0.

Theorem 4.2. *Let X be a Banach space and let S and $T : X \multimap X^*$ be σ-monotone operators. Suppose that*

$$\text{(CQ)} \qquad 0 \in \operatorname{core}[\operatorname{co} D(T) - \operatorname{co} D(S)].$$

Then there exist $r, c > 0$ such that, for any $x \in D(T) \cap D(S)$, $t^* \in T(x)$ and $s^* \in S(x)$,

$$\max(\|t^*\|, \|s^*\|) \leq c(r + \|x\|)(2r + \|t^* + s^*\|).$$

Proof. Consider the function

$$\rho_T(x) = \sup\Big\{\frac{\langle z^*, x - z\rangle}{1 + \|z\|} : (z, z^*) \in \operatorname{gr}(T)\Big\}.$$

ρ_T is convex and lower semicontinuous as supremum of affine functions. If $x \in D(T)$, $x^* \in T(x)$, then for all $z \in D(T)$ and $z^* \in T(z)$ we have

$$\begin{aligned}\frac{\langle z^*, x-z\rangle}{1+\|z\|} &= \frac{\langle z^*-x^*, x-z\rangle}{1+\|z\|} + \frac{\langle x^*, x-z\rangle}{1+\|z\|}\\ &\leq \frac{\min\{\sigma(x), \sigma(z)\}}{1+\|z\|}\|x-z\| + \|x^*\|\frac{\|x-z\|}{1+\|z\|}\\ &\leq (\|x^*\| + \min\{\sigma(x), \sigma(z)\})\Big(\frac{\|x\|}{1+\|z\|} + \frac{\|z\|}{1+\|z\|}\Big)\\ &< (\|x^*\| + \sigma(x))(\|x\|+1),\end{aligned}$$

which shows that $\rho_T(x) < +\infty$, that is $D(T) \subseteq \operatorname{dom}\rho_T$. Since ρ_T is convex we conclude that $\operatorname{co} D(T) \subseteq \operatorname{dom}\rho_T$. Likewise, we get $\operatorname{co} D(S) \subseteq \operatorname{dom}\rho_S$. Thus

$$\operatorname{co} D(T) - \operatorname{co} D(S) \subseteq \operatorname{dom}\rho_T - \operatorname{dom}\rho_S. \tag{4.1}$$

The assumption and (4.1) imply that $0 \in \operatorname{core}(\operatorname{dom}\rho_T - \operatorname{dom}\rho_s)$. Therefore

$$\begin{aligned}X &= \bigcup_{n=1}^{\infty} n\,(\operatorname{dom}\rho_T - \operatorname{dom}\rho_S)\\ &= \bigcup_{n=1}^{\infty}\bigcup_{i=1}^{\infty} n(\{x : \rho_T(x) \leq i, \|x\| \leq i\} - \{x : \rho_S(x) \leq i, \|x\| \leq i\}).\end{aligned}$$

On the other hand, $\{x : \rho_T(x) \leq i, \|x\| \leq i\}$ and $\{x : \rho_s(x) \leq i, \|x\| \leq i\}$ are closed and convex. By Fact 4.1 there exist $\varepsilon > 0$ and $r > 0$ such that

$$B(0,\varepsilon) \subseteq (\{x : \rho_T(x) \leq r, \|x\| \leq r\} - \{x : \rho_S(x) \leq r, \|x\| \leq r\}). \tag{4.2}$$

Let now $z \in B(0,\varepsilon)$, $x \in D(T)\cap D(S)$, $t^* \in T(x)$ and $s^* \in S(x)$, then $z = a - b$, where $\rho_T(a) \leq r$, $\|a\| \leq r$, $\rho_S(b) \leq r$, $\|b\| \leq r$. We have

$$\begin{aligned}\langle t^*, z\rangle &= \langle t^*, a-x\rangle + \langle s^*, b-x\rangle + \langle t^*+s^*, x-b\rangle\\ &\leq \rho_T(a)(1+\|x\|) + \rho_S(b)(1+\|x\|) + \|t^*+s^*\|(\|x\|+r)\\ &\leq (r+\|x\|)(2r+\|t^*+s^*\|),\end{aligned}$$

from which we get

$$\|t^*\| \leq \frac{(r+\|x\|)(2r+\|t^*+s^*\|)}{\varepsilon}. \tag{4.3}$$

Likewise

$$\|s^*\| \leq \frac{(r+\|x\|)(2r+\|t^*+s^*\|)}{\varepsilon}. \tag{4.4}$$

Set $c = \frac{1}{\varepsilon}$, now (4.3) and (4.4) imply the desired assertion. □

We recall that a set $A \subseteq X^*$ is *bounded weak* closed* if every bounded weak* convergent net in A has its limit in A.

Corollary 4.3 ([11, Theorem 2.7.11])**.** *A convex set in X^* is weak* closed if and only if it is bounded weak* closed.*

The following result extends a theorem of Verona and Verona to σ-monotone operators. Our proof is very close to the proof of A. Verona and M.E. Verona in [13].

Proposition 4.4. *Let X be any Banach space and let $S, T : X \multimap X^*$ be maximal σ-monotone operators. Suppose that*

$$0 \in \operatorname{core}[\operatorname{co} D(T) - \operatorname{co} D(S)].$$

Then $T(x) + S(x)$ is a weak* closed subset of X^*, for every $x \in D(T) \cap D(S)$.

Proof. Since T and $S : X \multimap X^*$ are maximal σ-monotone, by Proposition 2.1 we infer that $T(x)$ and $S(x)$ are convex. Therefore $T(x) + S(x)$ is also convex. By Corollary 4.3 it suffices to prove that $T(x) + S(x)$ is bounded weak* closed, that is, every bounded weak* convergent net in $T(x) + S(x)$ has a limit in $T(x) + S(x)$.

Let $\{t_i^*\} \subseteq T(x)$ and $\{s_i^*\} \subseteq S(x)$ be nets such that $\{t_i^* + s_i^*\}$ is bounded and weak* convergent to x^*. By Theorem 4.2,

$$\max(\|t_i^*\|, \|s_i^*\|) \le c(r + \|x\|)(2r + \|t_i^* + s_i^*\|).$$

Thus the nets $\{t_i^*\}$ and $\{s_i^*\}$ are bounded. So they are relatively weak* compact. Replacing them by some subnets, if necessary, we may assume that $t_i^* \overset{w^*}{\to} t^*$ and $s_i^* \overset{w^*}{\to} s^*$. Since T and S are maximal σ-monotone, by Proposition 2.1, $T(x)$ and $S(x)$ are weak* closed. Therefore $t^* \in T(x)$ and $s^* \in S(x)$. Then $x^* = t^* + s^* \in T(x) + S(x)$. ☐

5. Difference of two σ-monotone operators

In this section, some results of [1] are extended for σ-monotone operators. Since the difference of two σ-monotone operators is not necessarily σ-monotone, investigating their maximality is very difficult. We study conditions under which the difference of two σ-monotone operators is maximal σ-monotone.

Theorem 5.1. *Let $S : X \multimap X^*$ be a maximal σ-monotone operator and let $T : X \multimap X^*$ be monotone. If $D(T) = X$ and $S - T$ is σ-monotone, then $S - T$ is also maximal σ-monotone.*

Proof. Let $(y, y^*) \in X \times X^*$ be σ-monotonically related to $\operatorname{gr}(S - T)$. For any $(x, x^*) \in \operatorname{gr}(S)$ and $(x, z^*) \in \operatorname{gr}(T)$, we have $(x, x^* - z^*) \in \operatorname{gr}(S - T)$. Then

$$\langle x^* - z^* - y^*, x - y\rangle \ge -\min\{\sigma(x), \sigma(y)\}\|x - y\|.$$

By the monotonicity of T and that $D(T) = X$, there exists $t^* \in T(y)$ such that

$$\begin{aligned}\langle x^* - t^* - y^*, x - y\rangle &= \langle x^* - z^* - y^*, x - y\rangle + \langle z^* - t^*, x - y\rangle \\ &\ge -\min\{\sigma(x), \sigma(y)\}\|x - y\|.\end{aligned}$$

It follows that $(y, y^* + t^*)$ is σ-monotonically related to $\mathrm{gr}(S)$. Maximality of S and Proposition 1.5 imply that $(y, y^* + t^*) \in \mathrm{gr}(S)$. Hence $(y, y^*) \in \mathrm{gr}(S - T)$, and so $S - T$ is maximal σ-monotone. $\square$

Example 5.2. Define $T, S : \mathbb{R} \to \mathbb{R}$ by

$$T(x) = \begin{cases} 0 & x = 0, \\ \emptyset & \text{otherwise,} \end{cases} \quad \text{and } S(x) = \begin{cases} \{0\} & x < 0, \\ [0, \infty) & x = 0, \\ \emptyset & x > 0, \end{cases}$$

and let $\sigma \equiv 0$. Then S is maximal σ-monotone (indeed monotone), T is monotone and $S - T$ is σ-monotone (indeed monotone) but not maximal, since $\mathrm{gr}(S - T) = \{0\} \times \mathbb{R}$. Therefore, the condition that $D(T) = X$ in the preceding theorem is essential.

Corollary 5.3. *Let $S : X \multimap X^*$ be a maximal σ-monotone operator and let $T : X \multimap X^*$ be a positive linear relation. Suppose that $D(T) = X$ and $S - T$ is σ-monotone. Then $S - T$ is maximal σ-monotone.*

Proof. Since any positive linear operator is monotone, all conditions of Theorem 5.1 are satisfied and hence the proof is complete. $\square$

Example 5.4. Let $S : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ be such that $S(x) := 2x$ for all $x \in \mathbb{R}$ and $\sigma \equiv 0$. Suppose that the mapping $T : \mathbb{R} \to \mathbb{R}$ is defined by $T(x) := \frac{x}{2} + 1$ for all positive real number, $T(x) := x + 1$ otherwise. It is easy to see that S is maximal σ-monotone, T is monotone but it is not positive and linear while $S - T$ is maximal σ-monotone.

The linear relation $T : X \multimap X^*$ is said to be a *skew linear relation* if $\langle x^*, x \rangle = 0$ for any $(x, x^*) \in \mathrm{gr}(T)$, [1].

Corollary 5.5. *Let $S : X \multimap X^*$ be maximal σ-monotone, $T : X \multimap X^*$ be skew linear and $D(T) = X$. Then $S \pm T$ is maximal σ-monotone.*

Proof. Since T is a skew linear relation, $-T$ is skew linear too. Then $\pm T$ and also $S - (\pm T)$ are σ-monotone. Therefore $S \pm T$ is maximal σ-monotone by Theorem 5.1. $\square$

In accordance with the above corollary, the following result is clear.

Corollary 5.6. *Let $S : X \multimap X^*$ be maximal σ-monotone and $T : X \to X^*$ be skew linear. Then $S \pm T$ is maximal σ-monotone.*

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