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NON-HOMOGENEOUS CONTINUOUS AND DISCRETE GRADIENT SYSTEMS: THE QUASI-CONVEX CASE

H. KHATIBZADEH* AND V. MOHEBBI

(Communicated by Ali Abkar)

Abstract. In this paper, first we study the weak and strong convergence of solutions to the following first order nonhomogeneous gradient system

$$\begin{cases} -x'(t) = \nabla\phi(x(t)) + f(t), & \text{a.e. on } (0,\infty) \\ x(0) = x_0 \in H \end{cases}$$

to a critical point of ϕ, where ϕ is a C^1 quasi-convex function on a real Hilbert space H with Argmin$\phi \neq \varnothing$ and $f \in L^1(0,+\infty;H)$. These results extend the results in the literature to non-homogeneous case. Then the discrete version of the above system by backward Euler discretization has been studied. Beside of the proof of the existence of the sequence given by the discrete system, some results on the weak and strong convergence to the critical point of ϕ are also proved. These results when ϕ is pseudo-convex (therefore the critical points are the same minimum points) may be applied in optimization for approximation of a minimum point of ϕ.
Keywords: Gradient system, quasi-convex, backward Euler discretization, weak convergence, strong convergence.
MSC(2010): Primary: 34D20; Secondary: 37C75, 93D09.

1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $|.|$. We denote the weak convergence in H by $\rightharpoonup$ and the strong convergence by $\to$. A function $\psi : H \to \mathbb{R}$ is called convex if and only if

$$\psi(\lambda x + (1-\lambda)y) \leq \lambda\psi(x) + (1-\lambda)\psi(y), \quad \forall x,y \in H, \quad \forall \lambda \in [0,1].$$

A function $\phi : H \to \mathbb{R}$ is said to be quasi-convex iff

$$\phi(\lambda x + (1-\lambda)y) \leq \max\{\phi(x), \phi(y)\}, \quad \forall x,y \in H, \quad \forall \lambda \in [0,1],$$

or equivalently every sub-level set of ϕ is convex. A function $\phi : H \to \mathbb{R}$ is called pseudo-convex iff

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$f(y) > f(x)$ implies that there exist $\beta(x,y) > 0$ and $0 < \delta(x,y) \leq 1$ such that $f(y) - f(tx + (1-t)y) \geq t\beta(x,y)$, $\forall t \in (0, \delta(x,y))$.
A differentiable function ϕ on H is quasi-convex iff

$$\phi(x) \geq \phi(y) \Rightarrow \langle \nabla\phi(x), y - x\rangle \leq 0.$$

A differentiable function ϕ is pseudo-convex iff

$$\phi(x) > \phi(y) \Rightarrow \langle \nabla\phi(x), y - x\rangle < 0.$$

Obviously for a differentiable function, convexity implies pseudo-convexity and pseudo-convexity implies quasi-convexity. We refer the reader to the interesting book by Chambini and Martein [3] for the definitions of convexity and its extensions, their examples and properties. Throughout the paper, we assume that $\phi : H \to \mathbb{R}$ is a continuously differentiable and quasi-convex function with $\text{Argmin}\phi \neq \varnothing$ and $\nabla\phi$ is Lipschitz continuous on bounded subsets of H.
Consider the following nonhomogeneous evolution system

$$\begin{cases} -x'(t) = \nabla\phi(x(t)) + f(t), & \text{a.e. on } (0, +\infty) \\ x(0) = x_0 \in H. \end{cases} \tag{1.1}$$

When $f(t) \equiv 0$, Goudou and Munier [5] proved the weak convergence of solutions to (1.1) to a critical point of ϕ. They also proved the strong convergence of solutions to (1.1) under additional conditions on ϕ. In Section 2, we consider (1.1) with condition $f \in L^1((0, +\infty); H)$. We prove the weak and strong convergence of solutions to (1.1) to a critical point of ϕ. These results extend the similar classical results on the asymptotic behavior of non-homogeneous gradient systems associated with convex functions which have been also extended to non-smooth convex functions (see Bruck [2] as well as Morosanu [7] for a complete bibliography). In Section 3, we consider the following backward Euler discretization of (1.1)

$$\begin{cases} x_{n-1} - x_n = \lambda_n \nabla\phi(x_n) + f_n, \\ x_0 = x \in H. \end{cases} \tag{1.2}$$

We prove the existence of the sequence $\{x_n\}$ as well as some similar results on the weak and strong convergence of solutions of equation (1.2) with condition $\sum_{n=1}^{+\infty} |f_n| < +\infty$. The generated sequence by (1.2) is called the proximal point algorithm that has been studied initially by Martinet [6] and Rockafellar [9] in the convex case even for non-smooth convex functions or more generally maximal monotone operators to approximate a minimum point of a convex function or a zero of a maximal monotone operator.

Our results extend the results of Goudou and Munier [5] to non-homogeneous case and the classical results when ϕ is a convex function to quasi-convex case in discrete and continuous cases. The results are also applicable even for one dimensional differential and difference equations (where of course weak and strong convergence coincide). Consider the following nonlinear differential equation

$$\begin{cases} -x'(t) = \frac{2x(t)}{((x(t))^2+1)^2} + \frac{2}{(t+1)^3} - \frac{2(t+1)^6}{((t+1)^4+1)^2}, \\ x(0) = 1, \end{cases}$$

which is in the form (1.1) with $\phi(x) = \frac{x^2}{x^2+1}$ and $f(t) = \frac{2}{(t+1)^3} - \frac{2(t+1)^6}{((t+1)^4+1)^2}$. One can easily see that $x(t) = \frac{1}{(t+1)^2}$ is a solution of that which converges to zero as predicated by Theorem 2.3. Also consider the following nonlinear difference equation

$$\begin{cases} x_{n-1} - x_n = \lambda_n \frac{2x_n}{(x_n^2+1)^2} + \frac{1}{n(n+1)} - \frac{(n+1)^2}{((n+1)^2+1)^2}, \\ x_0 = 1. \end{cases}$$

Obviously if $\lambda_n = \frac{1}{2(n+1)}$, then $x_n = \frac{1}{(n+1)}$ is a solution of the above difference equation, that converges to zero as $n \to +\infty$.

2. **Continuous case**

In this section, we concentrate on the asymptotic behavior of solutions to the gradient system (1.1). When $\nabla\phi$ is Lipschitz and $f \in L^1((0, +\infty); H)$ an application of Banach contraction principle on $W^{1,1}_{\text{loc}}(0, +\infty; H)$ implies the existence and uniqueness of a solution $x \in W^{1,1}_{\text{loc}}(0, +\infty; H)$. We prove the weak convergence of solutions to (1.1) to a critical point of the quasi-convex function ϕ. Then with some additional assumptions on ϕ, we prove the strong convergence of solutions to (1.1).

Lemma 2.1. *Suppose that $x(t)$ is a solution to (1.1). If $\text{Argmin}\phi \neq \varnothing$, then $\lim_{t\to+\infty} |x(t) - x^*|$ exists for each $x^* \in \text{Argmin}\phi$.*

Proof. Since $x^* \in \text{Argmin}\phi$, we have $\phi(x^*) \leq \phi(x(t))$ for all $t \geq 0$. By the quasi-convexity of ϕ, we have

$$\langle \nabla\phi(x(t)), x^* - x(t)\rangle \leq 0.$$

Therefore

$$\begin{aligned} \frac{d}{dt}|x(t) - x^*|^2 &= 2\langle x'(t), x(t) - x^*\rangle \\ &= 2\langle -\nabla\phi(x(t)) - f(t), x(t) - x^*\rangle \\ &\leq 2|f(t)||x(t) - x^*|, \end{aligned} \tag{2.1}$$

for almost every $t \in (0, +\infty)$. Integrating (2.1) over $[0, t]$ and then using a Gronwall type lemma (see [7, Lemma 2.1, p. 47]), we get

$$|x(t) - x^*| \leq |x(0) - x^*| + \int_0^t |f(s)| ds,$$

which implies the boundedness of $x(t)$.

Let $M := \sup_{t \geq 0} |x(t) - x^*|$. Now integrating (2.1) from s to $t > s$, we get

$$|x(t) - x^*|^2 - |x(s) - x^*|^2 \leq 2M \int_s^t |f(\tau)| d\tau.$$

Taking limsup as $t \to +\infty$ and liminf as $s \to +\infty$, we get that $\lim_{t \to +\infty} |x(t) - x^*|$ exists.

□

Lemma 2.2. *Suppose that $x(t)$ is a solution to (1.1) and* $\text{Argmin}\phi \neq \varnothing$*, then* $\lim \phi(x(t))$ *exists.*

Proof. Since $\nabla\phi$ is bounded on bounded subsets of H, by equation (1.1) and Lemma 2.1, we have

$$\begin{aligned} \frac{d}{dt}\phi(x(t)) &= \langle \nabla\phi(x(t)), x'(t) \rangle = \langle \nabla\phi(x(t)), -\nabla\phi(x(t)) - f(t) \rangle \\ &= -|\nabla\phi(x(t))|^2 - \langle \nabla\phi(x(t)), f(t) \rangle \leq |\nabla\phi(x(t))||f(t)| \\ &= |\nabla\phi(x(t)) - \nabla\phi(x^*)||f(t)| \leq L|x(t) - x^*||f(t)| \leq LM|f(t)|, \end{aligned}$$

for almost every $t \in (0, +\infty)$, where $M = \sup_{t \geq 0} |x(t) - x^*|$, L is the Lipschitz constant of $\nabla\phi$ and x^* is a critical point of ϕ. Now $\lim \phi(x(t))$ exists; because $f \in L^1((0, +\infty); H)$. □

Theorem 2.3. *Suppose that $x(t)$ is a solution to (1.1). If* $\text{Argmin}\phi \neq \varnothing$*, then there is $x^* \in H$ such that $x(t) \rightharpoonup x^*$ as $t \to +\infty$ and $\nabla\phi(x^*) = 0$.*

Proof. We consider two following cases:
1) $\lim \phi(x(t)) = \inf \phi$. Since $x(t)$ is bounded by Lemma 2.1, there exist a sequence $\{t_n\}$ and $x^* \in H$ such that $x(t_n) \rightharpoonup x^*$, as $n \to +\infty$. By Lemma 2.2 and weak lower semicontinuity of ϕ (by Mazur's Lemma), we have

$$\phi(x^*) \leq \liminf \phi(x(t_n)) = \lim \phi(x(t)) = \inf \phi.$$

Therefore $x^* \in \text{Argmin}\phi$, which implies by Lemma 2.1 and Opial's Lemma [5], $x(t) \rightharpoonup x^* \in \text{Argmin}\phi$.
2) $\lim \phi(x(t)) > \inf \phi$. Then there exist $r > 0$, $t_0 > 0$ and $\tilde{x} \in \text{Argmin}\phi$ such that for all $t \geq t_0$ and every $y \in \bar{B}_r(\tilde{x})$, $\phi(y) \leq \phi(x(t))$. In turn by quasi-convexity of ϕ, $\langle y - x(t), \nabla\phi(x(t)) \rangle \leq 0$. Now if $\nabla\phi(x(t)) \neq 0$ letting

$y = \tilde{x} + r\frac{\nabla\phi(x(t))}{|\nabla\phi(x(t))|}$, we have

$$\begin{aligned} r|\nabla\phi(x(t))| &\leq \langle x(t) - \tilde{x}, \nabla\phi(x(t))\rangle \\ &\leq -\frac{1}{2}\frac{d}{dt}|x(t) - \tilde{x}|^2 + M|f(t)|, \quad \text{a.e. } t \in (0, +\infty), \end{aligned}$$

where $M = \sup_{t\geq 0}|x(t) - \tilde{x}|$. This inequality being obviously true by (2.1) if $\nabla\phi(x(t)) = 0$. Therefore $\nabla\phi(x(\cdot)) \in L^1((0, +\infty); H)$. It implies that $x'(\cdot) \in L^1((0, +\infty); H)$. Therefore there is $x^* \in H$ such that $x(t) \to x^*$, by continuity of $\nabla\phi$, $\nabla\phi(x(t)) \to \nabla\phi(x^*)$. Since $\nabla\phi(x(\cdot)) \in L^1((0, +\infty); H)$, there exists a sequence $t_n \to +\infty$ such that $\nabla\phi(x(t_n)) \to 0$. Therefore $\nabla\phi(x^*) = 0$. □

Remark 2.4. Suppose that the assumptions of Theorem 2.3 are satisfied and ϕ is pseudo-convex. By pseudo-convexity of ϕ, if $\nabla\phi(x^*) = 0$, then $x^* \in \text{Argmin}\phi$. Suppose to the contrary that $\nabla\phi(x^*) = 0$ but $x^* \notin \text{Argmin}\phi$, hence there is an $x' \in H$ such that $\phi(x') < \phi(x^*)$. In turn, by pseudo-convexity of ϕ, we have $\langle x' - x^*, \nabla\phi(x^*)\rangle < 0$. This contradiction shows that $x^* \in \text{Argmin}\phi$. Therefore $x(t) \rightharpoonup x^* \in \text{Argmin}\phi$.

Theorem 2.5. *Suppose that $x(t)$ is a solution to (1.1). If* $\text{Argmin}\phi \neq \varnothing$ *and at least one of the following condition is satisfied:*
a) $x \notin \text{Argmin}\phi$*, where x is a weak cluster point of $x(t)$,*
b) $\text{int}(\text{Argmin}\phi) \neq \varnothing$,
then $x(t) \to x^*$ and x^* is a critical point of ϕ.

Proof. a) Suppose that $x(t_n) \rightharpoonup x \notin \text{Argmin}\phi$. Then

$$\lim_{t\to+\infty}\phi(x(t)) = \liminf_{n\to+\infty}\phi(x(t_n)) \geq \phi(x) > \inf\phi.$$

Therefore the result is concluded by part 2 of the proof of Theorem 2.3.
b) If $\text{int}(\text{Argmin}\phi) \neq \varnothing$, then there exist $\tilde{x} \in \text{Argmin}\phi$, $r > 0$ and $t_0 > 0$ such that for all $t \geq t_0$ and every $y \in \bar{B}_r(\tilde{x})$, $\phi(y) \leq \phi(x(t))$. In turn by quasi-convexity of ϕ, $\langle y - x(t), \nabla\phi(x(t))\rangle \leq 0$. Now if $\nabla\phi(x(t)) \neq 0$ by letting $y = \tilde{x} + r\frac{\nabla\phi(x(t))}{|\nabla\phi(x(t))|}$, we have

$$\begin{aligned} r|\nabla\phi(x(t))| &\leq \langle x(t) - \tilde{x}, \nabla\phi(x(t))\rangle \\ &= \langle x(t) - \tilde{x}, -x'(t) - f(t)\rangle \\ &\leq -\frac{1}{2}\frac{d}{dt}|x(t) - \tilde{x}|^2 + M|f(t)|, \end{aligned}$$

for almost every $t \in (0, +\infty)$, where $M = \sup_{t\geq 0}|x(t) - \tilde{x}|$. This inequality is obviously true by (2.1) if $\nabla\phi(x(t)) = 0$. Therefore $\nabla\phi(x(\cdot)) \in L^1((0, +\infty); H)$. This implies that $x'(\cdot) \in L^1((0, +\infty); H)$. Therefore there is $x^* \in H$ such that $x(t) \to x^*$ as $t \to +\infty$. On the other hand, $\nabla\phi(x(t_n)) \to 0$ for a sequence $t_n \to +\infty$ as $n \to +\infty$. The continuity of $\nabla\phi$ implies that $\nabla\phi(x^*) = 0$. □

Theorem 2.6. *Suppose that $x(t)$ is a solution to (1.1) and $f(t) \equiv 0$. If ϕ is even, then there is $x^* \in H$ such that $x(t) \to x^*$ as $t \to +\infty$, where $\nabla\phi(x^*) = 0$.*

Proof. By Lemma 2.2, we get that $\phi(x(t))$ is nonincreasing. Therefore for all $t \geq s$, $\phi(x(t)) \leq \phi(x(s))$. Quasi-convexity of ϕ implies that

$$\langle \nabla\phi(x(s)), x(t) - x(s)\rangle \leq 0.$$

Since ϕ is even, $\phi(-x(t)) = \phi(x(t))$. Therefore again by quasi-convexity for each $t \geq s$, we get

$$\langle -x(t) - x(s), \nabla\phi(x(s))\rangle \leq 0. \tag{2.2}$$

Summing up the last two inequalities, we get

$$\langle x(s), \nabla\phi(x(s))\rangle \geq 0 \Rightarrow \quad \frac{d}{ds}|x(s)|^2 \leq 0, \quad \text{a.e. } s \in (0, +\infty).$$

Therefore $|x(t)|$ is nonincreasing. By (2.2) for each $t \geq s$, we have

$$\langle x(t) + x(s), x'(s)\rangle \leq 0 \Rightarrow \frac{d}{ds}|x(s)|^2 \leq -2\langle x(t), x'(s)\rangle, \quad \text{a.e. } s \in (0, +\infty)$$

Integrating this inequality, we get

$$|x(t)|^2 \leq |x(s)|^2 - 2|x(t)|^2 + 2\langle x(t), x(s)\rangle, \quad \forall t > s.$$

By the last relation, we get

$$|x(t) - x(s)|^2 = |x(t)|^2 + |x(s)|^2 - 2\langle x(t), x(s)\rangle \leq 2(|x(s)|^2 - |x(t)|^2) \to 0,$$

as $t, s \to +\infty$. Therefore $x(t)$ is a Cauchy net. So $x(t)$ converges to $x^* \in H$, with $\nabla\phi(x^*) = 0$, by Theorem 2.3. ☐

Definition 2.7. Let $f : H \to (-\infty, +\infty]$ be proper, then f is uniformly quasi-convex with modulus η. If η is increasing, η vanishes only at 0 and $(\forall\, x, y \in \operatorname{dom} f\ ,\ \ \forall\alpha \in (0,1))$

$$f(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\eta(|x-y|) \leqslant \max\{f(x), f(y)\}.$$

Example 2.8. We define f and η as follows:

$$f(x) = \begin{cases} x^2 & x \geqslant -1 \\ 4\sqrt{-x} - 3 & -4 \leqslant x \leqslant -1 \\ +\infty & x < -4 \end{cases}$$

and

$$\eta(x) = \frac{x^2}{16 + x^2}, \quad \forall x \in [0, +\infty).$$

f is not convex but it is uniformly quasi-convex with modulus η.

Theorem 2.9. *Suppose that $x(t)$ is a solution to (1.1). If* $\operatorname{Argmin}\phi \neq \varnothing$ *and ϕ is uniformly quasi-convex with modulus η, then $x(t)$ converges strongly to the unique element of* $\operatorname{Argmin}\phi$.

Proof. Let $\tilde{x}$ be the unique element of Argminϕ, then $\phi(\tilde{x}) \leqslant \phi(x(t))$. Uniform quasi-convexity of ϕ shows that

$$\begin{aligned} \langle \tilde{x} - x(t), \nabla\phi(x(t))\rangle \ &\leqslant -\eta(|x(t) - \tilde{x}|) \\ \Longrightarrow 0 \leqslant \eta(|x(t) - \tilde{x}|) &\leqslant \langle x(t) - \tilde{x}, \nabla\phi(x(t))\rangle \end{aligned}$$

$$= \langle x(t) - \tilde{x}, -x'(t) - f(t)\rangle \leq \frac{-1}{2}\frac{d}{dt}|x(t) - \tilde{x}|^2 + M|f(t)|, \quad \text{a.e. } t \in (0, +\infty)$$

where $M = \sup\limits_{t\geqslant 0}|x(t) - \tilde{x}|$ and in turn by taking integral, we have

$$\begin{aligned} 0 \ &\leqslant \ \int_0^{+\infty} \eta(|x(t) - \tilde{x}|)dt \\ &\leqslant \ \frac{1}{2}|x(0) - \tilde{x}|^2 - \lim_{t\to\infty}\frac{1}{2}|x(t) - \tilde{x}|^2 + M\int_0^{+\infty} |f(t)|dt < +\infty. \end{aligned}$$

Since $\lim\limits_{t\to\infty}|x(t) - \tilde{x}|$ exists, it follows that $\lim\limits_{t\to\infty}\eta(|x(t) - \tilde{x}|) = 0$. On the other hand η is an increasing function which vanishes only at 0. Hence, we conclude that $x(t) \longrightarrow \tilde{x}$. □

3. Discrete case

In this section, we concentrate on (1.2), which is a discrete version of (1.1) by backward Euler discretization. Let $\phi : H \longrightarrow (-\infty, +\infty]$ be a continuously differentiable quasi-convex function and $\nabla\phi$ be Lipschitz continuous on bounded subsets of H with a Lipschitz constant $L > 0$. The existence of the sequence satisfying (1.2) is a consequence of the following proposition.

Proposition 3.1. *Let $F : H \to H$ be Lipschitz continuous with Lipschitz constant $L > 0$, $\lambda < \frac{1}{L}$ and $\bar{x} \in H$, then there is a unique element $x^* \in H$ such that $\bar{x} = x^* + \lambda F(x^*)$.*

Proof. Define $G(x) := \lambda F(x) + x$. $G : H \to H$ is continuous and strongly monotone with constant $1 - \lambda L > 0$. By [4, Theorem 11.2], G is surjective. Uniqueness of x^* is deduced by a simple computation. □

Corollary 3.2. *Consider (1.2) and suppose that $\lambda_n < \frac{1}{L}$ and $x_0 \in H$, then the sequence $\{x_n\}$ is well-defined.*

Proof. Since $\nabla\phi$ is Lipschitz continuous with a Lipschitz constant $L > 0$, by setting $F := \nabla\phi$, $\bar{x} := x_{n-1} - f_n$ and $\lambda := \lambda_n$ and then using Proposition 3.1, we can find the unique $x_n \in H$ such that $x_{n-1} - x_n = \lambda_n\nabla\phi(x_n) + f_n$, for all $n \in \mathbb{N}$. Therefore the sequence $\{x_n\}$ is well-defined. □

Lemma 3.3. *Suppose that x_n is a solution to (1.2). If* Argmin$\phi \neq \varnothing$, *then* $\lim_{n\to+\infty}|x_n - x|$ *exists, where* $x \in$ Argminϕ.

Proof. Let $x \in \text{Argmin}\phi$. Since $\phi(x_n) \geq \phi(x)$, by quasi-convexity of ϕ, we get

$$\langle \nabla\phi(x_n), x - x_n \rangle \leq 0.$$

Equation (1.2) implies that

$$\langle x_{n-1} - x_n - f_n, x - x_n \rangle \leq 0$$
$$\Rightarrow |x - x_n|^2 + \langle x_{n-1} - x, x - x_n \rangle \leq \langle f_n, x - x_n \rangle$$
$$\Rightarrow |x - x_n|^2 + \frac{1}{2}(|x_n - x_{n-1}|^2 - |x - x_{n-1}|^2 - |x - x_n|^2) \leq |f_n||x - x_n|$$
$$\Rightarrow |x - x_n|^2 - |x - x_{n-1}|^2 \leq 2|f_n||x - x_n|. \tag{3.1}$$

First we prove $\{x_n\}$ is bounded. By contradiction if $\{x_n\}$ is unbounded, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $|x_{n_k} - x| \longrightarrow +\infty$ and $|x_{n_{k+1}} - x| > 2|x_{n_k} - x|$ and $|x_{n_k} - x| < |x_n - x| < |x_{n_{k+1}} - x|, \forall n_k < n < n_{k+1}$.
Summing up (3.1) from $n_k + 1$ to n_{k+1}, we get

$$|x_{n_{k+1}} - x|^2 - |x_{n_k} - x|^2 \leq 2|x_{n_{k+1}} - x| \sum_{i=n_k+1}^{n_{k+1}} |f_i|.$$

Dividing both sides of the above inequality by $|x_{n_{k+1}} - x|$, we obtain

$$\frac{3}{2}|x_{n_k} - x| \leq 2 \sum_{i=n_k+1}^{n_{k+1}} |f_i|.$$

By letting $k \to +\infty$, we get a contradiction. Therefore $\{x_n\}$ is bounded. Now summing up (3.1) from $n = k+1$ to $n = m$, where $m > k+1$, we get

$$|x_m - x|^2 \leq |x_k - x|^2 + M \sum_{n=k+1}^{m} |f_n|,$$

where $M := 2\sup_{n\geq 0}|x_n - x|$. Taking limsup when $m \to +\infty$ and liminf when $k \to +\infty$, we get that $\lim_{n\to+\infty} |x_n - x|$ exists. □

Lemma 3.4. *Suppose that x_n is a solution of (1.2), then $\lim \phi(x_n)$ exists.*

Proof. First suppose that $\phi(x_n) > \phi(x_{n-1})$, then

$$\langle \nabla\phi(x_n), x_{n-1} - x_n \rangle \leq 0.$$

By (1.2), we have

$$|x_n - x_{n-1}|^2 \leq |f_n||x_n - x_{n-1}|.$$

Since $x_n \neq x_{n-1}$, by dividing the above inequality by $|x_n - x_{n-1}|$, we obtain

$$|x_n - x_{n-1}| \leq |f_n|.$$

By mean value theorem, boundedness of $\nabla\phi$ on bounded subsets of H and boundedness of the sequence $\{x_n\}$ by Lemma 3.3, we get

$$|\phi(x_n) - \phi(x_{n-1})| \leq K|x_n - x_{n-1}| \leq K|f_n|,$$

for a constant K. Therefore

$$\phi(x_n) - \phi(x_{n-1}) \leq K|f_n|, \tag{3.2}$$

If $\phi(x_n) \leq \phi(x_{n-1})$ obviously we have (3.2) again. It yields the lemma. □

Theorem 3.5. *Suppose that x_n is a solution to (1.2). If* $\mathrm{Argmin}\phi \neq \varnothing$ *and* $\sum_{n=1}^{+\infty} \lambda_n = +\infty$*, then there is $x^* \in H$ such that $x_n \rightharpoonup x^*$ and $\nabla\phi(x^*) = 0$.*

Proof. We consider two following cases:
1) $\lim \phi(x_n) = \inf \phi$. Since $\{x_n\}$ is bounded by Lemma 3.3, there are a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x^* \in H$ such that $x_{n_j} \rightharpoonup x^*$, then

$$\phi(x^*) \leq \liminf \phi(x_{n_j}) = \lim \phi(x_n) = \inf \phi.$$

Therefore $x^* \in \mathrm{Argmin}\phi$. Now, Opial's Lemma and Lemma 3.3 imply that $x_n \rightharpoonup x^* \in \mathrm{Argmin}\phi$ and the first optimality condition implies that $\nabla\phi(x^*) = 0$.
2) $\lim \phi(x_n) > \inf \phi$. Then there exist $r > 0$ and $\tilde{x} \in \mathrm{Argmin}\phi$ such that for each $y \in \bar{B}_r(\tilde{x})$, $\phi(y) < \lim_{n\to+\infty} \phi(x_n)$, which implies that there exists $n_0 > 0$ such that for all $n \geq n_0$, $\phi(y) \leq \phi(x_n)$. Therefore $\langle y - x_n, \nabla\phi(x_n)\rangle \leq 0$, $\forall n \geq n_0$. Now if $\nabla\phi(x_n) \neq 0$, then set $y = \tilde{x} + r\frac{\nabla\phi(x_n)}{|\nabla\phi(x_n)|}$, therefore, we have

$$r|\nabla\phi(x_n)| \leq \langle x_n - \tilde{x}, \nabla\phi(x_n)\rangle.$$

Multiplying both sides by λ_n and using (1.2), we get

$$\begin{aligned} r\lambda_n|\nabla\phi(x_n)| &\leq \langle x_n - \tilde{x}, x_{n-1} - x_n - f_n\rangle \\ &\leq \frac{1}{2}|x_{n-1} - \tilde{x}|^2 - \frac{1}{2}|x_n - \tilde{x}|^2 + M|f_n|, \end{aligned}$$

where $M = \sup_{n\geq 0} |x_n - \tilde{x}|$. The previous inequality being true by (3.1) if $\nabla\phi(x_n) = 0$. Summing up the above inequality from $n = 1$ to $+\infty$, by the hypothesis on $\{f_n\}$, we get

$$\sum_{n=1}^{\infty} \lambda_n|\nabla\phi(x_n)| < +\infty. \tag{3.3}$$

Equation (1.2) and summability assumption on the sequence $\{|f_n|\}$ imply that

$$\sum_{n=1}^{+\infty} |x_n - x_{n-1}| < +\infty.$$

It follows that $x_n \to x^* \in H$. On the other hand, by (3.3) and the assumptions on $\{\lambda_n\}$, $\liminf_{n\to\infty} |\nabla\phi(x_n)| = 0$. Continuity of $\nabla\phi$ implies that $\nabla\phi(x^*) = 0$. □

Remark 3.6. By Theorem 3.5 and Remark 2.4, if ϕ is pseudo-convex with $\mathrm{Argmin}\phi \neq \varnothing$, then (1.2) gives an algorithm to approximate a minimum point of ϕ, which extends inexact proximal point algorithm [9] for pseudo-convex functions.

Theorem 3.7. *Suppose that x_n is a solution to* (1.2). *If* $\mathrm{Argmin}\phi \neq \varnothing$ *and at least one of the following conditions is satisfied:*
a) $x \notin \mathrm{Argmin}\phi$, *where x is a weak cluster point of x_n,*
b) $\mathrm{int}(\mathrm{Argmin}\phi) \neq \varnothing$,
then $\{x_n\}$ converges strongly to a critical point of ϕ.

Proof. a) If $x \notin \mathrm{Argmin}\phi$, then $\lim \phi(x_n) > \inf \phi$. Now the result is concluded similar to the proof of case 2 of Theorem 3.5.
b) If $\mathrm{int}(\mathrm{Argmin}\phi) \neq \varnothing$, then there exist $\tilde{x} \in \mathrm{Argmin}\phi$ and $r > 0$ such that for each $y \in \bar{B}_r(\tilde{x})$, $\phi(y) \leq \lim_{n\to+\infty} \phi(x_n)$, which implies that there exists $n_0 > 0$ such that for all $n \geq n_0$, $\phi(y) \leq \phi(x_n)$. Therefore $\langle y - x_n, \nabla\phi(x_n)\rangle \leq 0, \quad \forall n \geq n_0$. Now if $\nabla\phi(x_n) \neq 0$, then set $y = \tilde{x} + r\frac{\nabla\phi(x_n)}{|\nabla\phi(x_n)|}$, therefore, we have

$$\begin{aligned} r\lambda_n|\nabla\phi(x_n)| &\leq \lambda_n\langle x_n - \tilde{x}, \nabla\phi(x_n)\rangle = \langle x_n - \tilde{x}, x_{n-1} - x_n - f_n\rangle \\ &\leq \frac{1}{2}|x_{n-1} - \tilde{x}|^2 - \frac{1}{2}|x_n - \tilde{x}|^2 + M|f_n|, \end{aligned}$$

where $M = \sup_{n\geq 0}|x_n - \tilde{x}|$. The previous inequality is true by (3.1) if $\nabla\phi(x_n) = 0$. Summing up the last inequality from $n = 1$ to $+\infty$, we get $\lambda_n|\nabla\phi(x_n)| \in l^1$. This implies that $|x_n - x_{n-1}| \in l^1$, hence $x_n \to x^* \in H$ and $\nabla\phi(x^*) = 0$, by a proof similar to the one in the continuous case (see Theorem 3.3). □

Theorem 3.8. *Suppose that x_n is a solution to* (1.2) *and ϕ is uniformly quasi-convex with modulus η. If* $\mathrm{Argmin}\phi \neq \varnothing$ *and* $\sum_{n=1}^{\infty} \lambda_n = +\infty$, *then $\{x_n\}$ converges strongly to the unique minimum point of ϕ.*

Proof. If $\tilde{x}$ is the unique element of $\mathrm{Argmin}\phi$, then $\phi(\tilde{x}) \leqslant \phi(x_n)$, for all $n > 0$. Hence by uniformly quasi-convexity of ϕ, we have:

$$\begin{aligned} &\langle \tilde{x} - x_n, \nabla\phi(x_n)\rangle \leqslant -\eta(|x_n - \tilde{x}|) \\ \Longrightarrow \quad & 0 \leqslant \eta(|x_n - \tilde{x}|) \leqslant \langle x_n - \tilde{x}, \frac{x_{n-1} - x_n - f_n}{\lambda_n}\rangle \\ &= \frac{1}{\lambda_n}\langle x_n - \tilde{x}, x_{n-1} - x_n\rangle - \frac{1}{\lambda_n}\langle x_n - \tilde{x}, f_n\rangle \\ \Longrightarrow\ & 0 \leqslant \lambda_n\eta(|x_n - \tilde{x}|) \leqslant \frac{1}{2}\{|x_{n-1} - \tilde{x}|^2 - |x_n - \tilde{x}|^2\} + M|f_n|, \end{aligned}$$

where $M = \sup_{n\geq 0}|x_n - \tilde{x}|$. Summing up the both sides of the last inequality from $n = 1$ to $n = m$, we obtain

$$0 \leqslant \sum_{n=1}^{m} \lambda_n\eta(|x_n - \tilde{x}|) \leqslant \frac{1}{2}\{|x_0 - \tilde{x}|^2 - |x_m - \tilde{x}|^2\} + M\sum_{n=1}^{m} |f_n|.$$

Since $\lim\limits_{n\to\infty}|x_n - \tilde{x}|$ exists, we have

$$\sum_{n=1}^{\infty} \lambda_n\eta(|x_n - \tilde{x}|) < +\infty.$$

On the other hand, since $\sum_{n=1}^{+\infty} \lambda_n = +\infty$, thus $\lim_{n\to+\infty} \eta(|x_n - \tilde{x}|) = 0$ and hence, we deduce that $x_n \to \tilde{x}$, which is the unique element of Argminϕ. □

Theorem 3.9. *Suppose that x_n is a solution to (1.2) with $f_n \equiv 0$, and ϕ is even. If* Argmin$\phi \neq \varnothing$, *then $x_n \to x^*$, which is a critical point of ϕ.*

Proof. Since ϕ is quasi-convex and by Lemma 3.4, $\{\phi(x_n)\}$ is nonincreasing, we have

$$\forall k \geq n \ \ , \ \phi(x_k) \leq \phi(x_n) \Rightarrow \langle x_k - x_n, \nabla\phi(x_n)\rangle \leq 0.$$

On the other hand, since ϕ is even, $\phi(-x_k) = \phi(x_k)$. Therefore, we have

$$\langle -x_k - x_n, \nabla\phi(x_n)\rangle \leq 0. \tag{3.4}$$

By adding the last two inequalities, we obtain

$$\langle x_n, \nabla\phi(x_n)\rangle \geq 0.$$

By equation (1.2),

$$\langle x_n, \frac{x_{n-1} - x_n}{\lambda_n}\rangle \geq 0 \ \ \Rightarrow |x_n| \leq |x_{n-1}|,$$

hence $\lim_{n\to\infty} |x_n|$ exists. By (3.4), for all $k \geq n$, we have

$$\langle x_k + x_n, \frac{x_{n-1} - x_n}{\lambda_n}\rangle \geq 0$$

$$\Longrightarrow |x_k + x_n|^2 \leq |x_k + x_{n-1}|^2 \ \ \Rightarrow 4|x_k|^2 \leq |x_k + x_n|^2, \ \ \forall k \geq n.$$

Now using the parallelogram identity, we have

$$\begin{aligned} |x_n - x_k|^2 &= 2|x_k|^2 + 2|x_n|^2 - |x_n + x_k|^2 \\ &\leq 2|x_k|^2 + 2|x_n|^2 - 4|x_k|^2 \\ &= 2(|x_n|^2 - |x_k|^2), \ \ \forall k \geq n \end{aligned}$$

Since $\lim_{n\to\infty} |x_n|$ exists thus $\lim_{n\to\infty} |x_n - x_k| = 0$, which implies that $\{x_n\}$ is a Cauchy sequence and therefore $x_n \to x^* \in H$ and $\nabla\phi(x^*) = 0$, by Theorem 3.5. □

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