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INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL WITH S-FOLD ZEROS AT THE ORIGIN

E. KHOJASTEHNEZHAD AND M. BIDKHAM*

(Communicated by Ali Abkar)

Abstract. Let $p(z)$ be a polynomial of degree n and for a complex number α, let $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ denote the polar derivative of the polynomial p(z) with respect to α. Dewan et al proved that if $p(z)$ has all its zeros in $|z| \le k$, $(k \le 1)$, with s-fold zeros at the origin then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$,

$$\max_{|z|=1} |D_\alpha p(z)| \ge \frac{(n+sk)(|\alpha|-k)}{1+k} \max_{|z|=1} |p(z)|.$$

In this paper, we obtain a refinement of the above inequality. Also as an application of our result, we extend some inequalities for polar derivative of a polynomial of degree n which does not vanish in $|z| < k$, where $k \ge 1$, except s-fold zeros at the origin.

Keywords: Polynomial, inequality, maximum modulus, polar derivative, restricted zeros.
MSC(2010): Primary: 30A10; Secondary: 30C10, 30D15.

1. Introduction and statement of results

According to a well known result as Bernstein's inequality on the derivative of a polynomial $p(z)$ of degree n, we have

$$\max_{|z|=1} |p'(z)| \le n \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is best possible and equality holds for a polynomial having all its zeros at the origin (see [4, 16]).
The inequality (1.1) can be sharpened, by considering the class of polynomials having no zeros in $|z| < 1$.
In fact, P. Erdös conjectured and later Lax [12] proved that if $p(z) \ne 0$ in

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$|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

If $p(z)$ has all its zeros in $|z| \leq 1$, then it was shown by Turan [17] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.3}$$

As an extension of inequality (1.3) Malik [14], proved that if $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.4}$$

Aziz and Shah [3, Theorem 3] generalized (1.4) and proved that if $p(z)$ has all its zeros in $|z| \leq k \leq 1$ with s-fold zeros at the origin, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n+sk}{1+k} \max_{|z|=1} |p(z)|. \tag{1.5}$$

Govil [9] improved inequality (1.4) and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k}\{\max_{|z|=1} |p(z)| + \frac{1}{k^{n-1}} \min_{|z|=k} |p(z)|\}. \tag{1.6}$$

As an improvement of inequality (1.2) Dewan and Hans [6] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for any complex number β with $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} |zp'(z) + \frac{n\beta}{2}p(z)| \leq \frac{n}{2}\{&(|1+\frac{\beta}{2}| + |\frac{\beta}{2}|) \max_{|z|=1} |p(z)| - \\ &(|1+\frac{\beta}{2}| - |\frac{\beta}{2}|) \min_{|z|=1} |p(z)|\}. \end{aligned} \tag{1.7}$$

Let α be a complex number. For a polynomial $p(z)$ of degree n, $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ (for more information, see [1, 5, 8]) and that $D_\alpha p(z)$ generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} [\frac{D_\alpha p(z)}{\alpha}] = p'(z). \tag{1.8}$$

For the polar derivative $D_\alpha p(z)$, Aziz and Rather [2] generalized the inequality (1.4) to the polar derivative of a polynomial. In fact, they proved that if all zeros of $p(z)$ lie in $|z| \leq k$, $k \leq 1$, then for every real or complex number α with $|\alpha| \geq k$, we get

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{1+k}(|\alpha| - k) \max_{|z|=1} |p(z)|. \tag{1.9}$$

As a refinement to inequality (1.9), Govil [10] proved that if $p(z)$ is a polynomial of degree n having all zeros in $|z| \leq k$, where $k \leq 1$, then for every real or complex number α with $|\alpha| \geq k$, we have

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{n}{1+k}\{(|\alpha| - k) \max_{|z|=1} |p(z)| + \frac{(|\alpha|+1)}{k^{n-1}} \min_{|z|=k} |p(z)|\}. \tag{1.10}$$

As an improvement and generalization of (1.9), Dewan et al [7, Theorem 2] proved that if $p(z)$ has all its zeros in $|z| \leq k \leq 1$ with s-fold zeros at the origin, then

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{(n+sk)(|\alpha| - k)}{1+k} \max_{|z|=1} |p(z)|. \tag{1.11}$$

As an improvement and generalization to the inequalities (1.7) and (1.4), Liman et al [13] proved that if $p(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then for all real or complex numbers $\alpha,\ \beta$ with $|\alpha| \geq 1,\ |\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned} |zD_\alpha p(z) + n\beta\frac{|\alpha|-1}{2}p(z)| \leq \frac{n}{2}\{(|\alpha + \beta\frac{|\alpha|-1}{2}| + |z + \beta\frac{|\alpha|-1}{2}|) \max_{|z|=1} |p(z)| \\ -(|\alpha + \beta\frac{|\alpha|-1}{2}| - |z + \beta\frac{|\alpha|-1}{2}|) \min_{|z|=1} |p(z)|\}. \end{aligned} \tag{1.12}$$

Our first result, Theorem 1.1, is a generalization and refinement of inequalities (1.10) and (1.11) respectively.

Theorem 1.1. *Let $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \leq k$, where $k \leq 1$, with s-fold zeros at the origin, then*

$$\begin{aligned} |zD_\alpha p(z) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)| \geq \\ k^{-n}|n\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}||z|^n \min_{|z|=k} |p(z)|, \end{aligned} \tag{1.13}$$

for every real or complex numbers $\beta,\ \alpha$ with $|\beta| \leq 1,\ |\alpha| \geq k$ and $|z| \geq 1$.

According to Lemma 2.2, if $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq k,\ k \leq 1$, with s-fold zeros at the origin, then for $|z| = 1$,

$$|D_\alpha p(z)| \geq \frac{(n+sk)(|\alpha|-k)}{1+k}|p(z)|,$$

also for every complex number β with $|\beta| \leq 1$, by choosing suitable argument of β we have

$$|zD_\alpha p(z) + \beta \frac{(n+sk)(|\alpha|-k)}{1+k} p(z)| = |zD_\alpha p(z)| - |\beta| \frac{(n+sk)(|\alpha|-k)}{1+k} |p(z)|. \tag{1.14}$$

Combining (1.13) and (1.14), we have

$$|zD_\alpha p(z)| - |\beta| \frac{(n+sk)(|\alpha|-k)}{1+k} |p(z)| \geq k^{-n} |n\alpha + \beta \frac{(n+sk)(|\alpha|-k)}{1+k}| \min_{|z|=k} |p(z)|,$$

or

$$|D_\alpha p(z)| - |\beta| \frac{(n+sk)(|\alpha|-k)}{1+k} |p(z)| \geq k^{-n} \left(n|\alpha| - |\beta| \frac{(n+sk)(|\alpha|-k)}{1+k} | \right) \min_{|z|=k} |p(z)|.$$

Letting $|\beta| \to 1$, we have the following result which is a refinement and extension of inequalities (1.10) and (1.11).

Corollary 1.2. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, with s-fold zeros at the origin, then we have*

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{(n+sk)(|\alpha|-k)}{1+k} \max_{|z|=1} |p(z)| + \frac{(n-s)|\alpha| + (n+sk)}{(1+k)k^{n-1}} \min_{|z|=k} |p(z)|. \tag{1.15}$$

Dividing two sides of inequality (1.15) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following refinement and generalization of the inequalities (1.5) and (1.6), respectively.

Corollary 1.3. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, with s-fold zeros at the origin, then we have*

$$\max_{|z|=1} |p'(z)| \geq \frac{n+sk}{1+k} \max_{|z|=1} |p(z)| + \frac{n-s}{(1+k)k^{n-1}} \min_{|z|=k} |p(z)|. \tag{1.16}$$

Next, as an application of Theorem 1.1, we prove the following generalization of inequality (1.12).

Theorem 1.4. *Let $p(z)$ be a polynomial of degree n that does not vanish in $|z| < k$ $k \geq 1$, except at s-fold zeros at the origin, then for all α, $\beta \in \mathbb{C}$ with*

$|\alpha| \geq k,\ |\beta| \leq 1$ *and* $|z| = 1$, *we have*

$$
\begin{aligned}
|zD_\alpha p(z) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)| \leq \frac{1}{2}[\\
\{k^{-n}|n\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}| + \\
k^{-s}|(n-s)z + s\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}|\} \max_{|z|=k}|p(z)| \\
-\{k^{-n}|n\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}| - \\
k^{-s}|(n-s)z + s\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}|\} \min_{|z|=k}|p(z)|]. \qquad (1.17)
\end{aligned}
$$

If we take $s = 0,\ k = 1$ in Theorem 1.4, then the inequality (1.17) reduces to the inequality (1.12)

Dividing two sides of inequality (1.17) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following generalization of the inequality (1.7).

Corollary 1.5. *Let* $p(z)$ *be a polynomial of degree* n *that does not vanish in* $|z| < k,\ k \geq 1$, *except at* s*-fold zeros at the origin, then for all* $\beta \in \mathbb{C}$ *with* $|\beta| \leq 1$ *and* $|z| = 1$, *we have*

$$
\begin{aligned}
|zp'(z) + \beta\frac{n+sk}{1+k}p(z)| \leq \frac{1}{2}[\\
\{k^{-n}|n + \beta\frac{n+sk}{1+k}| + k^{-s}|s + \beta\frac{n+sk}{1+k}|\} \max_{|z|=k}|p(z)| \\
-\{k^{-n}|n + \beta\frac{n+sk}{1+k}| - k^{-s}|s + \beta\frac{n+sk}{1+k}|\} \min_{|z|=k}|p(z)|]. \qquad (1.18)
\end{aligned}
$$

Theorem 1.4 simplifies to the following result by taking $\beta = 0$.

Corollary 1.6. *Let* $p(z)$ *be a polynomial of degree* n *that does not vanish in* $|z| < k,\ k \geq 1$, *except at* s*-fold zeros at the origin, then for any* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geq k$ *and* $|z| = 1$, *we have*

$$
\begin{aligned}
|D_\alpha p(z)| \leq \frac{1}{2}\{nk^{-n}|\alpha| + k^{-s}|(n-s)z + s\alpha|) \max_{|z|=k}|p(z)| - \\
(nk^{-n}|\alpha| - k^{-s}|(n-s)z + s\alpha|) \min_{|z|=k}|p(z)|\}. \qquad (1.19)
\end{aligned}
$$

2. **Lemmas**

For the proofs of these theorems, we need the following lemmas. The first lemma is due to Laguerre [11,15].

Lemma 2.1. *If all the zeros of an n^{th} degree polynomial $p(z)$ lie in a circular region C and w is any zero of $D_\alpha p(z)$, then at most one of the points w and α may lie outside C.*

Lemma 2.2. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed disk $|z| \leq k \leq 1$, with s-fold zeros at the origin, then for each real or complex number α with $|\alpha| \geq k$ and $|z| = 1$, we have*

$$|D_\alpha p(z)| \geq \frac{(n+sk)(|\alpha|-k)}{1+k}|p(z)|. \tag{2.1}$$

The above lemma is due to K.K. Dewan and A. Mir [7].

Lemma 2.3. *If $p(z)$ is a polynomial of degree n with s-fold zeros at the origin, then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\beta| \leq 1$, $|\alpha| \geq k$ and $|z| = 1$, we have*

$$\begin{aligned}|zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)| &\leq \\ |n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|k^{-n}\max_{|z|=k}|p(z)|.\end{aligned} \tag{2.2}$$

Proof. Let $M = \max_{|z|=k}|p(z)|$, if $|\lambda| < 1$, then $|\lambda p(z)| < |M(\frac{z}{k})^n|$ for $|z| = k$. Therefore it follows from Rouche's theorem that the polynomial $G(z) = M(\frac{z}{k})^n - \lambda p(z)$ has all its zeros in $|z| < k$ with s-fold zeros at the origin. By applying Lemma 2.2, to the polynomial $G(z)$, we have for every real or complex number α with $|\alpha| \geq k$ and for $|z| = 1$,

$$|zD_\alpha G(z)| \geq \frac{(n+sk)(|\alpha|-k)}{1+k}|G(z)|,$$

or

$$|n\alpha Mk^{-n}z^n - \lambda zD_\alpha p(z)| \geq \frac{(n+sk)(|\alpha|-k)}{1+k}|Mk^{-n}z^n - \lambda p(z)|.$$

On the other hand by Lemma 2.1, all the zeros of $D_\alpha G(z) = n\alpha Mk^{-n}z^{n-1} - \lambda D_\alpha p(z)$ lie in $|z| < k$, where $|\alpha| \geq k$. Therefore for any β with $|\beta| \leq 1$, Rouche's theorem implies that all the zeros of

$$n\alpha Mk^{-n}z^n - \lambda zD_\alpha p(z) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}(Mk^{-n}z^n - \lambda p(z)),$$

lie in $|z| < 1$. This implies that the polynomial

$$\begin{aligned}T(z) = (n\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k})Mk^{-n}z^n \\ - \lambda(zD_\alpha p(z) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)),\end{aligned} \tag{2.3}$$

will have no zeros in $|z| \geq 1$. This implies that for every real or complex number β with $|\beta| < 1$ and $|z| = 1$,

(2.4)
$$|zD_\alpha p(z) + \frac{\beta(n+sk)(|\alpha|-k)}{1+k}p(z)| \leq |n\alpha + \frac{\beta(n+sk)(|\alpha|-k)}{1+k}|k^{-n}M.$$

If the inequality (2.4) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$|n\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}|k^{-n}M < |z_0 D_\alpha p(z_0) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z_0)|.$$

Take

$$\lambda = \frac{(n\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k})k^{-n}M}{z_0 D_\alpha p(z_0) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z_0)},$$

then $|\lambda| < 1$ and with this choice of λ, we have $T(z_0) = 0$ for $|z_0| \geq 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. For β with $|\beta| = 1$, the inequality (2.4) follows by continuity. This completes the proof of Lemma 2.3. □

Lemma 2.4. *If $p(z)$ is a polynomial of degree n with s-fold zeros at the origin, then for all $\alpha,\ \beta \in \mathbb{C}$ with $|\beta| \leq 1$, $|\alpha| \geq k$ and $|z| = 1$, we have*

$$\begin{aligned} k^{n+s}|zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)|+& \\ |zD_\alpha Q(z) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z)| &\leq \\ \{k^s|n\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}|+& \\ k^n|(n-s)z + s\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}|\}&\max_{|z|=k}|p(z)|, \end{aligned} \tag{2.5}$$

where $Q(z) = z^{n+s}\overline{p(\frac{k^2}{\overline{z}})}$.

Proof. Let $M = \max_{|z|=k}|p(z)|$. For λ with $|\lambda| > 1$, it follows from Rouche's theorem that the polynomial $G(z) = p(z) - \lambda M k^{-s} z^s$ has no zeros in $|z| < k$, except at s-fold zeros at the origin. Consequently the polynomial

$$H(z) = z^{n+s}\overline{G(\frac{k^2}{\overline{z}})},$$

has all its zeros in $|z| \leq k$ with s-fold zeros at the origin, also $k^{n+s}|G(z)| = |H(z)|$ for $|z| = k$. Since all the zeros of $H(z)$ lie in $|z| \leq k$, therefore, for δ with $|\delta| > 1$, by Rouche's theorem all the zeros of $k^{n+s}G(z) + \delta H(z)$ lie in $|z| \leq k$.

Hence by Lemma 2.2 for every real or complex number α with $|\alpha| \geq k$, and $|z| = 1$, we have

$$\frac{(n+sk)(|\alpha|-k)}{1+k}|k^{n+s}G(z)+\delta H(z)| \leq |zD_\alpha(k^{n+s}G(z)+\delta H(z))|.$$

Now using a similar argument as that used in the proof of Lemma 2.3, we get for every real or complex number β with $|\beta| \leq 1$ and $|z| \geq 1$,

$$k^{n+s}|zD_\alpha G(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}G(z)| \leq$$
$$|zD_\alpha H(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}H(z)|. \tag{2.6}$$

Therefore by using the equality

$$\begin{aligned} H(z) =& z^{n+s}\overline{G(\frac{k^2}{\overline{z}})} = z^{n+s}\overline{p(\frac{k^2}{\overline{z}})} - \overline{\lambda}Mk^s z^n \\ =& Q(z) - \overline{\lambda}Mk^s z^n, \end{aligned}$$

and $G(z)$ in (2.6), we get

$$\begin{aligned} &k^{n+s}|(zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z))- \\ &\lambda((n-s)z+s\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k})Mk^{-s}z^s| \leq \\ &|zD_\alpha Q(z)+\frac{\beta(n+s)(|\alpha|-1)}{2}Q(z)-\overline{\lambda}(n\alpha+\frac{\beta(n+s)(|\alpha|-1)}{2})k^s M z^n|. \end{aligned}$$

This implies

$$\begin{aligned} k^{n+s}|zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)|- \\ |\lambda((n-s)z+s\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k})Mk^{-s}z^s| \leq \\ |(zD_\alpha Q(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z))- \\ \overline{\lambda}(n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k})Mk^s z^n|. \end{aligned} \tag{2.7}$$

As $|Q(z)| = k^{n+s}|p(z)|$ for $|z| = k$, i.e., $\max_{|z|=k}|Q(z)| = k^{n+s}\max_{|z|=k}|p(z)| = k^{n+s}M$, by applying Lemma 2.3 to $Q(z)$, we obtain for $|z| = 1$,

$$\begin{aligned}|zD_\alpha Q(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z)| <& \\ &|\lambda||n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|k^{-n}\max_{|z|=k}|Q(z)| \\ &= |\lambda||n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|k^sM.\end{aligned}$$

Thus taking suitable choice of argument of λ, we get

$$\begin{aligned}|zD_\alpha Q(z)+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}Q(z)-& \\ &\overline{\lambda}(n\alpha+\frac{\beta(n+sk)(|\alpha|-k)}{1+k})Mk^sz^n| \\ &= |\lambda||n\alpha+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}|k^sM- \\ &|zD_\alpha Q(z)+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}Q(z)|.\end{aligned} \tag{2.8}$$

By combining right hand side of (2.7) and (2.8), we get for $|z| = 1$ and $|\beta| \le 1$,

$$\begin{aligned}&k^{n+s}|zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)|- \\ &|\lambda((n-s)z+s\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2})Mk^nz^s| \le |\lambda|\times \\ &|n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|k^sM-|zD_\alpha Q(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z)|,\end{aligned}$$

i.e.,

$$\begin{aligned}k^{n+s}|zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)|+& \\ &|zD_\alpha Q(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z)| \le \\ &|\lambda|\{k^s|n\alpha+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}|+ \\ &k^n|(n-s)z+s\alpha+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}|\}M.\end{aligned}$$

Letting $|\lambda| \to 1$, we have

$$k^{n+s}|zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)|+ |zD_\alpha Q(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z)| \leq \{k^s|n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|+ k^n|(n-s)z+s\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|\}M. \tag{2.9}$$

This gives the result. □

The following lemma is due to Zireh [18].

Lemma 2.5. *If $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n, having all its zeros in $|z|<k$, $(k>0)$, then $m<k^n|a_n|$, where $m=\min_{|z|=k}|p(z)|$.*

3. Proof of the theorems

Proof of Theorem 1.1. If $p(z)$ has a zero on $|z|=k$, then the inequality (1.11) is trivial. Therefore we assume that $p(z)$ has all its zeros in $|z|<k$. Let $m=\min_{|z|=k}|p(z)|$, then $m>0$ and $|p(z)|\geq m$ where $|z|=k$. Therefore, for $|\lambda|<1$, it follows from Rouche's theorem and Lemma 2.5 that the polynomial $G(z)=p(z)-\lambda mk^{-n}z^n$ is of degree n and has all its zeros in $|z|<k$ with s-fold zeros at the origin. By using Lemma 2.1, $D_\alpha G(z)=D_\alpha p(z)-\alpha\lambda mnk^{-n}z^{n-1}$, has all its zeros in $|z|<k$, where $|\alpha|\geq k$. Applying Lemma 2.2 to the polynomial $G(z)$, yields

$$|zD_\alpha G(z)| \geq \frac{(n+sk)(|\alpha|-k)}{1+k}|G(z)|,\ |z|=1. \tag{3.1}$$

Since $zD_\alpha G(z)$ has all its zeros in $|z|<k\leq 1$, by using Rouche's theorem, it can be easily verified from (3.1), that the polynomial

$$zD_\alpha G(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}G(z),$$

has all its zeros in $|z|<1$, where $|\beta|<1$. Substituting for $G(z)$, we conclude that the polynomial

$$T(z)=(zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z))- \lambda mk^{-n}z^n(n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}), \tag{3.2}$$

will have no zeros in $|z|\ge 1$. This implies for every real or complex number β with $|\beta|<1$ and $|z|\ge 1$,

(3.3)
$$|zD_\alpha p(z)+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}p(z)|\ge mk^{-n}|z^n||n\alpha+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}|.$$

If the inequality (3.3) is not true, then there is a point $z=z_0$ with $|z_0|\ge 1$ such that

$$|z_0D_\alpha p(z_0)+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}p(z_0)|< mk^{-n}|z_0^n||n\alpha+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}|.$$

Take

$$\lambda=\frac{z_0D_\alpha p(z_0)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z_0)}{mk^{-n}z_0^n(n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k})},$$

then $|\lambda|<1$ and with this choice of λ, we have $T(z_0)=0$ for $|z_0|\ge 1$. But this contradicts the fact that $T(z)\ne 0$ for $|z|\ge 1$. For β with $|\beta|=1$, the inequality (3.3) follows by continuity.
This completes the proof of Theorem 1.1. □

Proof of Theorem 1.4. Under the assumption of Theorem 1.4, we can write $p(z)=z^sh(z)$, where the polynomial $h(z)\ne 0$ in $|z|<k$, and thus if $m=\min_{|z|=k}|p(z)|$, then $k^{-s}m\le|h(z)|$ for $|z|\le k$. Now for λ with $|\lambda|<1$, we have

$$|\lambda k^{-s}m|<k^{-s}m\le|h(z)|,$$

where $|z|=k$.

It follows from Rouche's theorem that the polynomial $h(z)-\lambda k^{-s}m$ has no zero in $|z|<k$. Hence the polynomial $G(z)=z^s(h(z)-\lambda k^{-s}m)=p(z)-\lambda k^{-s}mz^s$, has no zero in $|z|<k$ except s-fold zeros at the origin. Therefore the polynomial

$$H(z)=z^{n+s}\overline{G(k^2/\overline{z})}=Q(z)-\overline{\lambda}k^smz^n,$$

will have all its zeros in $|z|\le k$ with s-fold zeros at the origin, where $Q(z)=z^{n+s}\overline{p(1/\overline{z})}$. Also $|H(z)|=k^{n+s}|G(z)|$ for $|z|=k$.

Now, using a similar argument as that used in the proof of Lemma 2.4 (inequality (2.6)), for the polynomials $H(z)$ and $G(z)$, we have

$$k^{n+s}|zD_\alpha G(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}G(z)|\le |zD_\alpha H(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}H(z)|,$$

where $|\alpha|\ge k$, $|\beta|\le 1$ and $|z|=1$. Substituting for $G(z)$ and $H(z)$ in the above inequality, we conclude that for every real or complex numbers α, β, with $|\alpha| \geq k,\ |\beta| \leq 1$ and $|z| = 1$,

$$k^{n+s}|zD_\alpha p(z) - \lambda((n-s)z + s\alpha)k^{-s}mz^s + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}(p(z) - \lambda k^{-s}mz^s)| \leq |zD_\alpha Q(z) - \overline{\lambda}\alpha n k^s m z^n + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}(Q(z) - \overline{\lambda}k^s m z^n)|,$$

i.e.,

$$k^{n+s}|zD_\alpha p(z) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z) - \lambda((n-s)z + s\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2})k^{-s}mz^s| \leq |zD_\alpha Q(z) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z) - \overline{\lambda}(n\alpha + \beta\frac{(n+s)(|\alpha|-1)}{2})k^s m z^n|. \tag{3.4}$$

Since all the zeros of $Q(z)$ lie in $|z| \leq 1$ with s-fold zeros at the origin, and $|Q(z)| = k^{n+s}|p(z)|$ for $|z| = k$, therefore by applying Theorem 1.1 to $Q(z)$, we have

$$|zD_\alpha Q(z) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z)| \geq k^{-n}|n\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}| \min_{|z|=k}|Q(z)| = |n\alpha + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}|k^s m.$$

Hence for an appropriate choice of the argument of λ, we have

$$|zD_\alpha Q(z) + \frac{\beta(n+sk)(|\alpha|-k)}{1+k}Q(z) - \overline{\lambda}(n\alpha + \frac{\beta(n+sk)(|\alpha|-k)}{1+k})k^s m z^n| = |zD_\alpha Q(z) + \frac{\beta(n+sk)(|\alpha|-k)}{1+k}Q(z)| - |\lambda||n\alpha + \frac{\beta(n+sk)(|\alpha|-k)}{1+k}|k^s m, \tag{3.5}$$

where $|z| = 1$. Combining the right hand sides of (3.4) and (3.5), we can rewrite (3.4) as

$$\begin{aligned}
k^{n+s}|zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)|-&\\
&|\lambda||(n-s)z+s\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|k^n m\le\\
&|zD_\alpha Q(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z)|-\\
&|\lambda||n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|k^s m. \qquad (3.6)
\end{aligned}$$

where $|z|=1$. Equivalently

$$\begin{aligned}
k^{n+s}|zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)|\le&\\
&|zD_\alpha Q(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z)|-\\
&|\lambda|\{k^s|n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|-\\
&k^n|(n-s)z+s\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|\}m.
\end{aligned}$$

As $|\lambda|\to 1$ we have

$$\begin{aligned}
k^{n+s}|zD_\alpha p(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)|\le&\\
&|zD_\alpha Q(z)+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}Q(z)|-\\
&\{k^s|n\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|-\\
&k^n|(n-s)z+s\alpha+\beta\frac{(n+sk)(|\alpha|-k)}{1+k}|\}m.
\end{aligned}$$

It implies for every real or complex number β with $|\beta|\le 1$ and $|z|=1$,

$$\begin{aligned}
&2|zD_\alpha p(z)+\beta\frac{(n+s)(|\alpha|-1)}{2}p(z)|\le\\
&|zD_\alpha p(z)+\frac{\beta(n+s)(|\alpha|-1)}{2}p(z)|+|zD_\alpha Q(z)+\frac{\beta(n+s)(|\alpha|-1)}{2}Q(z)|\\
&-\{|n\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}|-|(n-s)z+s\alpha+\beta\frac{(n+s)(|\alpha|-1)}{2}|\}m.
\end{aligned}$$

This in conjunction with Lemma 2.4 gives for $|\beta| \leq 1$ and $|z| = 1$,

$$2k^{n+s}|zD_\alpha p(z) + \beta\frac{(n+sk)(|\alpha|-k)}{1+k}p(z)| \leq$$
$$\{k^s|n\alpha + \frac{\beta(n+sk)(|\alpha|-k)}{1+k}|+$$
$$k^n|(n-s)z + s\alpha + \frac{\beta(n+sk)(|\alpha|-k)}{1+k}|\}M-$$
$$\{k^s|n\alpha + \frac{\beta(n+sk)(|\alpha|-k)}{1+k}|-$$
$$k^n|(n-s)z + s\alpha + \frac{\beta(n+sk)(|\alpha|-k)}{1+k}|\}m.$$

This completes the proof of Theorem 1.4. □

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