Title:
Inequalities for the polar derivative of a polynomial with $S$-fold zeros at the origin

Author(s):
E. Khojastehnezhad and M. Bidkham
INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL WITH S-FOLD ZEROS AT THE ORIGIN

E. KHOJASTEHNEZHAD AND M. BIDKHAM* 

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Abstract. Let \( p(z) \) be a polynomial of degree \( n \) and for a complex number \( \alpha \), let \( D_\alpha p(z) = np(z) + (\alpha - z)p'(z) \) denote the polar derivative of the polynomial \( p(z) \) with respect to \( \alpha \). Dewan et al proved that if \( p(z) \) has all its zeros in \( |z| \leq k, (k \leq 1), \) with \( s \)-fold zeros at the origin then for every \( \alpha \in \mathbb{C} \) with \( |\alpha| \geq k \),
\[
\max_{|z|=1} |D_\alpha p(z)| \geq \frac{(n + sk)(|\alpha| - k)}{1 + k} \max_{|z|=1} |p(z)|.
\]
In this paper, we obtain a refinement of the above inequality. Also as an application of our result, we extend some inequalities for polar derivative of a polynomial of degree \( n \) which does not vanish in \( |z| < k \), where \( k \geq 1 \), except \( s \)-fold zeros at the origin.

Keywords: Polynomial, inequality, maximum modulus, polar derivative, restricted zeros.


1. Introduction and statement of results

According to a well known result as Bernstein’s inequality on the derivative of a polynomial \( p(z) \) of degree \( n \), we have
\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}
\]
The result is best possible and equality holds for a polynomial having all its zeros at the origin (see \([4, 16]\)).
The inequality (1.1) can be sharpened, by considering the class of polynomials having no zeros in \( |z| < 1 \).
In fact, P. Erdős conjectured and later Lax \([12]\) proved that if \( p(z) \neq 0 \) in
\[ |z| < 1, \text{ then } (1.1) \text{ can be replaced by} \]
\[
\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}
\]

If \( p(z) \) has all its zeros in \(| z | \leq 1\), then it was shown by Turan [17] that
\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.3}
\]

As an extension of inequality (1.3) Malik [14], proved that if \( p(z) \) has all its zeros in \(| z | \leq k, k \leq 1\), then
\[
\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.4}
\]

Aziz and Shah [3, Theorem 3] generalized (1.4) and proved that if \( p(z) \) has all its zeros in \(| z | \leq k, k \leq 1\) with \( s \)-fold zeros at the origin, then
\[
\max_{|z|=1} |p'(z)| \geq \frac{n + sk}{1+k} \max_{|z|=1} |p(z)|. \tag{1.5}
\]

Govil [9] improved inequality (1.4) and proved that if \( p(z) \) is a polynomial of degree \( n \) having all its zeros in \(| z | \leq k, k \leq 1\), then
\[
\max_{|z|=1} |p'(z)| \geq \frac{n + sk}{1+k} \max_{|z|=1} |p(z)|. \tag{1.6}
\]

As an improvement of inequality (1.2) Dewan and Hans [6] proved that if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \(| z | < 1\), then for any complex number \( \beta \) with \(| \beta | \leq 1 \) and \(| z | = 1\),
\[
|zp'(z) + \frac{n\beta}{2} p(z)| \leq \frac{n}{2} \left\{ (1 + \frac{\beta}{2}) \max_{|z|=1} |p(z)| - (1 + \frac{\beta}{2} - \frac{\beta^2}{4}) \min_{|z|=1} |p(z)| \right\}. \tag{1.7}
\]

Let \( \alpha \) be a complex number. For a polynomial \( p(z) \) of degree \( n \), \( D_\alpha p(z) \), the polar derivative of \( p(z) \) is defined as
\[
D_\alpha p(z) = np(z) + (\alpha - z)p'(z).
\]

It is easy to see that \( D_\alpha p(z) \) is a polynomial of degree at most \( n - 1 \) (for more information, see [1, 5, 8]) and that \( D_\alpha p(z) \) generalizes the ordinary derivative in the sense that
\[
\lim_{\alpha \to \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z). \tag{1.8}
\]

For the polar derivative \( D_\alpha p(z) \), Aziz and Rather [2] generalized the inequality (1.4) to the polar derivative of a polynomial. In fact, they proved that if all
zeros of \( p(z) \) lie in \(|z| \leq k, k \leq 1\), then for every real or complex number \( \alpha \) with \(|\alpha| \geq k\), we get

\[
(1.9) \quad \max_{|z|=1} |D_{\alpha} p(z)| \geq \frac{n}{1+k} (|\alpha| - k) \max_{|z|=1} |p(z)|.
\]

As a refinement to inequality (1.9), Govil [10] proved that if \( p(z) \) is a polynomial of degree \( n \) having all zeros in \(|z| \leq k\), where \( k \leq 1\), then for every real or complex number \( \alpha \) with \(|\alpha| \geq k\), we have

\[
(1.10) \quad \max_{|z|=1} |D_{\alpha} p(z)| \geq \frac{n}{1+k} (|\alpha| - k) \max_{|z|=1} |p(z)| + \frac{(|\alpha| + 1) \min_{|z|=k} |p(z)|}{k}.
\]

As an improvement and generalization of (1.9), Dewan et al [7, Theorem 2] proved that if \( p(z) \) has all its zeros in \(|z| < k\) with \( s\)-fold zeros at the origin, then

\[
(1.11) \quad \max_{|z|=1} |D_{\alpha} p(z)| \geq \frac{(n + sk)(|\alpha| - k)}{1+k} \max_{|z|=1} |p(z)|.
\]

As an improvement and generalization to the inequalities (1.7) and (1.4), Liman et al [13] proved that if \( p(z) \) is a polynomial of degree \( n \) having no zeros in \(|z| < 1\), then for all real or complex numbers \( \alpha, \beta \) with \(|\alpha| \geq 1\), \(|\beta| \leq 1\) and \(|z| = 1\),

\[
|zD_{\alpha} p(z) + n\beta \frac{|\alpha| - 1}{2} \max_{|z|=1} |p(z)| |z + \beta \frac{|\alpha| - 1}{2}| \leq \frac{n}{2} \{ (|\alpha + \beta \frac{|\alpha| - 1}{2}| + |z + \beta \frac{|\alpha| - 1}{2}|) \max_{|z|=1} |p(z)| \}
\]

\[
(1.12) \quad -(|\alpha + \beta \frac{|\alpha| - 1}{2}| - |z + \beta \frac{|\alpha| - 1}{2}|) \min_{|z|=1} |p(z)|).
\]

Our first result, Theorem 1.1, is a generalization and refinement of inequalities (1.10) and (1.11) respectively.

**Theorem 1.1.** Let \( p(z) \) be a polynomial of degree \( n \), having all its zeros in \(|z| \leq k\), where \( k \leq 1\), with \( s\)-fold zeros at the origin, then

\[
|zD_{\alpha} p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1+k} \max_{|z|=1} |p(z)| |z + (n + sk) \frac{(|\alpha| - k)}{1+k}| \min_{|z|=1} |p(z)| |z + (n + sk) \frac{(|\alpha| - k)}{1+k}| |z^n \min_{|z|=k} |p(z)| |
\]

\[
(1.13) \quad k^{-n} n \alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1+k} |z^n \min_{|z|=k} |p(z)| |
\]

for every real or complex numbers \( \beta, \alpha \) with \(|\beta| \leq 1\), \(|\alpha| \geq k\) and \(|z| \geq 1\).

According to Lemma 2.2, if \( p(z) \) is a polynomial of degree \( n \), having all its zeros in \(|z| \leq k\), \( k \leq 1\), with \( s\)-fold zeros at the origin, then for \(|z| = 1\),

\[
|D_{\alpha} p(z)| \geq \frac{(n + sk)(|\alpha| - k)}{1+k} |p(z)|,
\]
also for every complex number $\beta$ with $|\beta| \leq 1$, by choosing suitable argument of $\beta$ we have

$$|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| = |zD_\alpha p(z)| - |\beta| \frac{(n + sk)(|\alpha| - k)}{1 + k} |p(z)|.$$  

(1.14)

Combining (1.13) and (1.14), we have

$$|zD_\alpha p(z)| - |\beta| \frac{(n + sk)(|\alpha| - k)}{1 + k} |p(z)| \geq k^{-n} n \alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} \min_{|z|=k} |p(z)|,$$

or

$$|D_\alpha p(z)| - |\beta| \frac{(n + sk)(|\alpha| - k)}{1 + k} |p(z)| \geq k^{-n} \left(n|\alpha| - |\beta| \frac{(n + sk)(|\alpha| - k)}{1 + k}\right) \min_{|z|=k} |p(z)|.$$

Letting $|\beta| \to 1$, we have the following result which is a refinement and extension of inequalities (1.10) and (1.11).

**Corollary 1.2.** If $p(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$, with $s$-fold zeros at the origin, then we have

$$\max_{|z|=1} |D_\alpha p(z)| \geq \frac{(n + sk)(|\alpha| - k)}{1 + k} \max_{|z|=1} |p(z)| + \frac{(n - s)|\alpha| + (n + sk)}{(1 + k)k^{n-1}} \min_{|z|=k} |p(z)|.$$  

(1.15)

Dividing two sides of inequality (1.15) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following refinement and generalization of the inequalities (1.5) and (1.6), respectively.

**Corollary 1.3.** If $p(z)$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$, with $s$-fold zeros at the origin, then we have

$$\max_{|z|=1} |p'(z)| \geq \frac{n + sk}{1 + k} \max_{|z|=1} |p(z)| + \frac{n - s}{(1 + k)k^{n-1}} \min_{|z|=k} |p(z)|.$$  

(1.16)

Next, as an application of Theorem 1.1, we prove the following generalization of inequality (1.12).

**Theorem 1.4.** Let $p(z)$ be a polynomial of degree $n$ that does not vanish in $|z| < k$, $k \geq 1$, except at $s$-fold zeros at the origin, then for all $\alpha$, $\beta \in \mathbb{C} with
If we take $s = 0$, $k = 1$ in Theorem 1.4, then the inequality (1.17) reduces to the inequality (1.12).

Dividing two sides of inequality (1.17) by $|z|$ and letting $|z| \to 1$, we have the following generalization of the inequality (1.7).

**Corollary 1.5.** Let $p(z)$ be a polynomial of degree $n$ that does not vanish in $|z| < k$, $k \geq 1$, except at $s$-fold zeros at the origin, then for all $z$ in $C$ with $|z| = 1$ and $|z| = 1$, we have

$$|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z) | \leq \frac{1}{2} |$$

$$\left\{ k^{-n} |n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} | + k^{-s} |(n - s)z + s\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} | \right\} \max_{|z|=k} |p(z)|$$

$$- \left\{ k^{-n} |n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} | - k^{-s} |(n - s)z + s\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} | \right\} \min_{|z|=k} |p(z)| |.$$  

(1.17)

Theorem 1.4 simplifies to the following result by taking $\beta = 0$.

**Corollary 1.6.** Let $p(z)$ be a polynomial of degree $n$ that does not vanish in $|z| < k$, $k \geq 1$, except at $s$-fold zeros at the origin, then for any $\alpha$ in $C$ with $|\alpha| \geq k$ and $|z| = 1$, we have

$$|zD_\alpha p(z) | \leq \frac{1}{2} \left\{ nk^{-n} |\alpha| + k^{-s} |(n - s)z + s\alpha | \right\} \max_{|z|=k} |p(z)| -$$

$$\left\{ nk^{-n} |\alpha| - k^{-s} |(n - s)z + s\alpha | \right\} \min_{|z|=k} |p(z)| |.$$  

(1.19)

2. **Lemmas**

For the proofs of these theorems, we need the following lemmas. The first lemma is due to Laguerre [11, 15].
Lemma 2.1. If all the zeros of an \( n \)th degree polynomial \( p(z) \) lie in a circular region \( C \) and \( w \) is any zero of \( D_\alpha p(z) \), then at most one of the points \( w \) and \( \alpha \) may lie outside \( C \).

Lemma 2.2. If \( p(z) \) is a polynomial of degree \( n \), having all its zeros in the closed disk \( |z| \leq 1 \), with \( s \)-fold zeros at the origin, then for each real or complex number \( w \) with \( \beta \leq 1, |\alpha| \geq k \) and \( |z| = 1 \), we have

\[
|D_\alpha p(z)| \geq \frac{(n + sk)(|\alpha| - k)}{1 + k} |p(z)|.
\] (2.1)

The above lemma is due to K.K. Dewan and A. Mir [7].

Lemma 2.3. If \( p(z) \) is a polynomial of degree \( n \) with \( s \)-fold zeros at the origin, then for all \( \alpha, \beta \in \mathbb{C} \) with \( \beta \leq 1, |\alpha| \geq k \) and \( |z| = 1 \), we have

\[
|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| \leq \nonumber \\
|na + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} |k^{-n} \max_{|z|=k} |p(z)|.\nonumber
\]

Proof. Let \( M = \max_{|z|=k} |p(z)| \), if \( |\alpha| < 1 \), then \( |\alpha p(z)| < |M(\frac{z}{k})^n| \) for \( |z| = k \). Therefore it follows from Rouche’s theorem that the polynomial \( G(z) = M(\frac{z}{k})^n - \lambda p(z) \) has all its zeros in \( |z| < k \) with \( s \)-fold zeros at the origin. By applying Lemma 2.2, to the polynomial \( G(z) \), we have for every real or complex number \( \alpha \) with \( |\alpha| \geq k \) and for \( |z| = 1 \),

\[
|zD_\alpha G(z)| \geq \frac{(n + sk)(|\alpha| - k)}{1 + k} |G(z)|,
\]

or

\[
|naMk^{-n} - \lambda zD_\alpha p(z)| \geq \frac{(n + sk)(|\alpha| - k)}{1 + k} |Mk^{-n} - \lambda p(z)|.
\]

On the other hand by Lemma 2.1, all the zeros of \( D_\alpha G(z) = naMk^{-n}z^{-1} - \lambda D_\alpha p(z) \) lie in \( |z| < k \), where \( |\alpha| \geq k \). Therefore for any \( \beta \) with \( |\beta| \leq 1 \), Rouche’s theorem implies that all the zeros of

\[
n\alpha Mk^{-n}z^n - \lambda zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} (Mk^{-n}z^n - \lambda p(z)),
\]

lie in \( |z| < 1 \). This implies that the polynomial

\[
T(z) = (na + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}) Mk^{-n}z^n - \lambda (zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)),
\]

(2.3)
will have no zeros in $|z| \geq 1$. This implies that for every real or complex number $\beta$ with $|\beta| < 1$ and $|z| = 1$,

\begin{equation}
|zD_\alpha p(z) + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k} p(z)| \leq |n\alpha + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k}| k^{-n} M.
\end{equation}

If the inequality (2.4) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

\[ |n\alpha + \beta\frac{(n + sk)(|\alpha| - k)}{1 + k}| k^{-n} M < |z_0 D_\alpha p(z_0) + \beta\frac{(n + sk)(|\alpha| - k)}{1 + k} p(z_0)|. \]

Take

\[ \lambda = \frac{(n\alpha + \beta\frac{(n + sk)(|\alpha| - k)}{1 + k})k^{-n} M}{z_0 D_\alpha p(z_0) + \beta\frac{(n + sk)(|\alpha| - k)}{1 + k} p(z_0)}, \]

then $|\lambda| < 1$ and with this choice of $\lambda$, we have $T(z_0) = 0$ for $|z_0| \geq 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. For $\beta$ with $|\beta| = 1$, the inequality (2.4) follows by continuity. This completes the proof of Lemma 2.3. \qed

**Lemma 2.4.** If $p(z)$ is a polynomial of degree $n$ with $s$-fold zeros at the origin, then for all $\alpha$, $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $|\alpha| \geq k$ and $|z| = 1$, we have

\begin{align*}
k^n z |zD_\alpha p(z) + \beta\frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| + \\
|zD_\alpha Q(z) + \beta\frac{(n + sk)(|\alpha| - k)}{1 + k} Q(z)| \leq \\
\{k^n |n\alpha + \beta\frac{(n + sk)(|\alpha| - k)}{1 + k}| + \\
k^n |(n - s)z + s\alpha + \beta\frac{(n + sk)(|\alpha| - k)}{1 + k}| \} \max_{|z| = k} |p(z)|,
\end{align*}

where $Q(z) = z^{n+s} p\left(\frac{k^s}{z}\right)$.

**Proof.** Let $M = \max_{|z|=k} |p(z)|$. For $\lambda$ with $|\lambda| > 1$, it follows from Rouche’s theorem that the polynomial $G(z) = p(z) - \lambda M k^{-s} z^s$ has no zeros in $|z| < k$, except at $s$-fold zeros at the origin. Consequently the polynomial

\[ H(z) = z^{n+s} G\left(\frac{k^s}{z}\right), \]

has all its zeros in $|z| \leq k$ with $s$-fold zeros at the origin, also $k^{n+s} |G(z)| = |H(z)|$ for $|z| = k$. Since all the zeros of $H(z)$ lie in $|z| \leq k$, therefore, for $\delta$ with $|\delta| > 1$, by Rouche’s theorem all the zeros of $k^{n+s} G(z) + \delta H(z)$ lie in $|z| \leq k$. 

Hence by Lemma 2.2 for every real or complex number $\alpha$ with $|\alpha| \geq k$, and $|z| = 1$, we have

\[
\frac{(n + sk)(|\alpha| - k)}{1 + k}|k^{n+s}G(z) + \delta H(z)| \leq |zD_\alpha(k^{n+s}G(z) + \delta H(z))|.
\]

Now using a similar argument as that used in the proof of Lemma 2.3, we get for every real or complex number $\beta$ with $|\beta| \leq 1$ and $|z| \geq 1$,

\[
k^{n+s}|zD_\alpha G(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}G(z)| \leq |zD_\alpha H(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}H(z)|.
\]

(2.6)

Therefore by using the equality

\[
H(z) = z^{n+s}G\left(\frac{k^2}{z}\right) = z^{n+s}\frac{k^2}{z} - \bar{\lambda} Mk^s z^n
\]

\[
= Q(z) - \bar{\lambda} Mk^s z^n,
\]

and $G(z)$ in (2.6), we get

\[
k^{n+s}|(zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}p(z)) - \lambda((n - s)z + s\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k})Mk^{-s}z^s| \leq
\]

\[
|zD_\alpha Q(z) + \frac{\beta(n + s)(|\alpha| - 1)}{2}Q(z) - \bar{\lambda}(n\alpha + \frac{\beta(n + s)(|\alpha| - 1)}{2})k^sMz^n|.
\]

This implies

\[
k^{n+s}|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}p(z)| - \lambda((n - s)z + s\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k})Mk^{-s}z^s| \leq
\]

\[
|(zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}Q(z)) - \bar{\lambda}(n\alpha + \beta \frac{(n + sk)(|\alpha| - 1)}{2})k^sMz^n|.
\]

(2.7)
As $|Q(z)| = k^{n+s}|p(z)|$ for $|z| = k$, i.e., $\max_{|z|=k} |Q(z)| = k^{n+s} \max_{|z|=k} |p(z)| = k^{n+s}M$, by applying Lemma 2.3 to $Q(z)$, we obtain for $|z| = 1$,
\[
|zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} Q(z)| < \\
|\lambda||n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} k^{-n} \max_{|z|=k} |Q(z)|| \\
= |\lambda||n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} k^n M.
\]
Thus taking suitable choice of argument of $\lambda$, we get
\[
|zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} Q(z) - \\
\bar{\lambda}(n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} M k^n z^n)| \\
= |\lambda||n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} k^n M - \\
|zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} Q(z)|.
\]
By combining right hand side of (2.7) and (2.8), we get for $|z| = 1$ and $|\beta| \leq 1$,
\[
k^{n+s}|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| - \\
|\lambda((n - s)z + s\alpha + \beta \frac{(n + s)(|\alpha| - 1)}{2}) M k^n z^n| \leq |\lambda| \\
|\lambda|\left| n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} k^n M - |zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} Q(z)|, \right|
\]
i.e.,
\[
k^{n+s}|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| + \\
|zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} Q(z)| \leq \\
|\lambda||k^n (n - s)z + s\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} M|.$
Inequalities for the polar derivative of a polynomial

Letting $|\lambda| \to 1$, we have

\[
k^{n+s} |zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| + n \alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} | + k^n |(n - s)z + s\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} | M.
\]

(2.9)

This gives the result. □

The following lemma is due to Zireh [18].

**Lemma 2.5.** If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n$, having all its zeros in $|z| < k$, $(k > 0)$, then $m < k^n |a_n|$, where $m = \min_{|z|=k} |p(z)|$.

3. Proof of the theorems

**Proof of Theorem 1.1.** If $p(z)$ has a zero on $|z| = k$, then the inequality (1.11) is trivial. Therefore we assume that $p(z)$ has all its zeros in $|z| < k$. Let $m = \min_{|z|=k} |p(z)|$, then $m > 0$ and $|p(z)| \geq m$ where $|z| = k$. Therefore, for $|\lambda| < 1$, it follows from Rouche’s theorem and Lemma 2.5 that the polynomial $G(z) = p(z) - \lambda m k^{-n} z^n$ is of degree $n$ and has all its zeros in $|z| < k$ with $s$-fold zeros at the origin. By using Lemma 2.1, $D_\alpha G(z) = D_\alpha p(z) - \alpha \lambda m k^{-n} z^{n-1}$, has all its zeros in $|z| < k$, where $|\alpha| \geq k$. Applying Lemma 2.2 to the polynomial $G(z)$, yields

\[
|zD_\alpha G(z)| \geq \frac{(n + sk)(|\alpha| - k)}{1 + k} |G(z)|, |z| = 1.
\]

(3.1)

Since $zD_\alpha G(z)$ has all its zeros in $|z| < k \leq 1$, by using Rouche’s theorem, it can be easily verified from (3.1), that the polynomial

\[
zD_\alpha G(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} G(z),
\]

has all its zeros in $|z| < 1$, where $|\beta| < 1$. Substituting for $G(z)$, we conclude that the polynomial

\[
T(z) = (zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)) - \lambda m k^{-n} z^n (\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}),
\]

(3.2)
will have no zeros in $|z| \geq 1$. This implies for every real or complex number $\beta$ with $|\beta| < 1$ and $|z| \geq 1$,

\begin{equation}
|zD_\alpha p(z) + \frac{\beta (n + sk)(|\alpha| - k)}{1 + k} p(z)| \geq m k^{-n} |z^n| |n\alpha + \frac{\beta (n + sk)(|\alpha| - k)}{1 + k}|.
\end{equation}

If the inequality (3.3) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

\[ |z_0 D_\alpha p(z_0) + \frac{\beta (n + sk)(|\alpha| - k)}{1 + k} p(z_0)| < m k^{-n} |z_0^n| |n\alpha + \frac{\beta (n + sk)(|\alpha| - k)}{1 + k}|. \]

Take

\[ \lambda = \frac{z_0 D_\alpha p(z_0) + \beta (n + sk)(|\alpha| - k)}{mk^{-n} z_0^n (n\alpha + \beta (n + sk)(|\alpha| - k))}, \]

then $|\lambda| < 1$ and with this choice of $\lambda$, we have $T(z_0) = 0$ for $|z_0| \geq 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| \geq 1$. For $\beta$ with $|\beta| = 1$, the inequality (3.3) follows by continuity.

This completes the proof of Theorem 1.1. \hfill \Box

**Proof of Theorem 1.4.** Under the assumption of Theorem 1.4, we can write $p(z) = z^n h(z)$, where the polynomial $h(z) \neq 0$ in $|z| < k$, and thus if $m = \min |p(z)|$, then $k^{-m} \leq |h(z)|$ for $|z| \leq k$. Now for $\lambda$ with $|\lambda| < 1$, we have

\[ |\lambda k^{-m}| < k^{-m} \leq |h(z)|, \]

where $|z| = k$.

It follows from Rouche’s theorem that the polynomial $h(z) - \lambda k^{-m} h$ has no zero in $|z| < k$. Hence the polynomial $G(z) = z^n (h(z) - \lambda k^{-m} h) = p(z) - \lambda k^{-m} z^n$, has no zero in $|z| < k$ except $s$-fold zeros at the origin. Therefore the polynomial

\[ H(z) = z^{n+s} G(k^2/z) = Q(z) - \lambda k^s mz^n, \]

will have all its zeros in $|z| \leq k$ with $s$-fold zeros at the origin, where $Q(z) = z^{n+s} p(1/z)$. Also $|H(z)| = k^{n+s}|G(z)|$ for $|z| = k$.

Now, using a similar argument as that used in the proof of Lemma 2.4 (inequality (2.6)), for the polynomials $H(z)$ and $G(z)$, we have

\[ k^{n+s} |zD_\alpha G(z) + \beta (n + sk)(|\alpha| - k) G(z)| \leq |zD_\alpha H(z) + \beta (n + sk)(|\alpha| - k) H(z)|, \]

where $|\alpha| \geq k$, $|\beta| \leq 1$ and $|z| = 1$. Substituting for $G(z)$ and $H(z)$ in the above inequality, we conclude that for every real or complex numbers $\alpha$, $\beta$, 

\[ k^{n+s} |zD_\alpha G(z) + \beta (n + sk)(|\alpha| - k) G(z)| \leq |zD_\alpha H(z) + \beta (n + sk)(|\alpha| - k) H(z)|, \]
with \(|\alpha| \geq k, \ |\beta| \leq 1\) and \(|z| = 1\),

\[
k^{n+s}|zD_\alpha p(z) - \lambda((n-s)z + s\alpha)k^{-s}mz^s +
\frac{\beta(n + sk)(|\alpha| - k)}{1 + k}(p(z) - \lambda k^{-s}mz^s)| \leq
\]

\[
|zD_\alpha Q(z) - \lambda nk^s m z^n + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}(Q(z) - \lambda k^s m z^n)|,
\]

i.e.,

\[
k^{n+s}|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}p(z) - \\
\lambda((n-s)z + s\alpha + \beta \frac{(n + s)(|\alpha| - 1)}{2}k^{-s}mz^s)| \leq
\]

\[
|zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}Q(z) - \\
\lambda(n\alpha + \beta \frac{(n + s)(|\alpha| - 1)}{2})k^s m z^n|.
\]

(3.4)

Since all the zeros of \(Q(z)\) lie in \(|z| \leq 1\) with \(s\)-fold zeros at the origin, and \(|Q(z)| = k^{n+s}|p(z)|\) for \(|z| = k\), therefore by applying Theorem 1.1 to \(Q(z)\), we have

\[
|zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}Q(z)| \geq
\]

\[
k^{-n}|n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}| \min_{|z| = k} |Q(z)| = \\
|n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}|k^s m.
\]

Hence for an appropriate choice of the argument of \(\lambda\), we have

\[
|zD_\alpha Q(z) + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k}Q(z) - \\
\lambda(n\alpha + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k})k^s m z^n|
\]

\[
= |zD_\alpha Q(z) + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k}Q(z)| - \\
|\lambda||n\alpha + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k}|k^s m,
\]

(3.5)

where \(|z| = 1\). Combining the right hand sides of (3.4) and (3.5), we can rewrite (3.4) as
\[ k^{n+s}|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| - \lambda |(n - s)z + s\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}| |k^n m \leq |zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} Q(z)| - \lambda |n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}| |k^n m. \]

(3.6)

where \(|z| = 1\). Equivalently

\[ k^{n+s}|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| \leq |zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} Q(z)| - \lambda |k^n|n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}| - k^n |(n - s)z + s\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}| m. \]

As \(|\lambda| \to 1\) we have

\[ k^{n+s}|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| \leq |zD_\alpha Q(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} Q(z)| - \lambda |k^n|n\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}| - k^n |(n - s)z + s\alpha + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k}| m. \]

It implies for every real or complex number \(\beta\) with \(|\beta| \leq 1\) and \(|z| = 1\),

\[ 2|zD_\alpha p(z) + \beta \frac{(n + s)(|\alpha| - 1)}{2} p(z)| \leq |zD_\alpha p(z) + \frac{\beta(n + s)(|\alpha| - 1)}{2} p(z)| + |zD_\alpha Q(z) + \frac{\beta(n + s)(|\alpha| - 1)}{2} Q(z)| - \{|n\alpha + \beta \frac{(n + s)(|\alpha| - 1)}{2}| - |(n - s)z + s\alpha + \beta \frac{(n + s)(|\alpha| - 1)}{2}|| m. \]
This in conjunction with Lemma 2.4 gives for $|\beta| \leq 1$ and $|z| = 1$,

$$2k^{n+s}|zD_\alpha p(z) + \beta \frac{(n + sk)(|\alpha| - k)}{1 + k} p(z)| \leq$$

$$\{k^n|n\alpha + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k}| +$$

$$k^n|[n - s]z + s\alpha + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k}]M -$$

$$\{k^n|n\alpha + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k}| -$$

$$k^n|[n - s]z + s\alpha + \frac{\beta(n + sk)(|\alpha| - k)}{1 + k}]m.$$ 

This completes the proof of Theorem 1.4. □

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(Elahe Khojastehnezhad) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SEMNAN, SEMNAN, IRAN.

E-mail address: khojastehnejadelahe@gmail.com

(Mahmood Bidkham) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SEMNAN, SEMNAN, IRAN.

E-mail address: mdbidkham@gmail.com