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ON Φ-τ-QUASINORMAL SUBGROUPS OF FINITE GROUPS

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Abstract. Let τ be a subgroup functor and H a p-subgroup of a finite group G. Let $\bar{G} = G/H_G$ and $\bar{H} = H/H_G$. We say that H is Φ-τ-quasinormal in G if for some S-quasinormal subgroup $\bar{T}$ of $\bar{G}$ and some τ-subgroup $\bar{S}$ of $\bar{G}$ contained in $\bar{H}$, $\bar{H}\bar{T}$ is S-quasinormal in $\bar{G}$ and $\bar{H}\cap\bar{T} \leq \bar{S}\Phi(\bar{H})$. In this paper, we study the structure of a group G under the condition that some primary subgroups of G are Φ-τ-quasinormal in G. Some new characterizations about p-nilpotency and solubility of finite groups are obtained.

Keywords: S-quasinormal subgroups; p-nilpotent subgroups, subgroup functor, soluble group.
MSC(2010): Primary: 20D20; Secondary: 20D10.

1. Introduction

Throughout this paper, all groups considered are finite and G always denotes a group, π denotes a set of primes and p denotes a prime. Let H_{sG} be the subgroup of H generated by all those subgroups of H which are S-quasinormal in G and H^{sG} the intersection of all such S-quasinormal subgroups of G containing H. All unexplained notation and terminology are standard, as in [6] and [8].

For a class of groups $\mathfrak{F}$, a chief factor L/K of G is said to be $\mathfrak{F}$-central in G if $L/K \rtimes G/C_G(L/K) \in \mathfrak{F}$. A normal subgroup N of G is called $\mathfrak{F}$-hypercentral in G if either $N = 1$ or every chief factor of G below N is $\mathfrak{F}$-central in G. Let $Z_{\mathfrak{F}}(G)$ denote the $\mathfrak{F}$-hypercentre of G, that is, the product of all $\mathfrak{F}$-hypercentral normal subgroups of G. We use $\mathfrak{S}$ and $\mathfrak{U}$ to denote the formations of all soluble groups and supersoluble groups, respectively. It is well known that $\mathfrak{U}$ and $\mathfrak{S}$ are all S-closed saturated formations.

A function τ which assigns to each group G a set of subgroups $\tau(G)$ of G is called a subgroup functor (see [11]) if $1 \in \tau(G)$ and $\theta(\tau(G)) = \tau(\theta(G))$ for any

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isomorphism $\theta : G \to G^*$. If $H \in \tau(G)$, then we say that H is a τ-subgroup of G.

In recent years, many scholars study the structure of the finite group by discussing the generalized normality of subgroups. Recall that a subgroup H of G is S-quasinormal in G if H permutes with every Sylow subgroup of G. A subgroup H of G is said to be s-semipermutable in G [3] if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$. A subgroup H of G is said to s-embedded in G [10] if G has two S-quasinormal subgroups T and S such that $HT = H^{sG}$ and $H \cap T \leq S$ (see [5, Example 4.1]), where $S \leq H$. A subgroup H of G is said to be nearly s-embedded in G [1] if G has an S-quasinormal subgroup T and an s-semipermutable subgroup S contained in H such that $HT = H^{sG}$ and $H \cap T \leq S$ (see [5, Example 4.6]). A subgroup H of G is said to be nearly SS-embedded in G [12] if G has an S-quasinormal subgroup T such that HT is S-quasinormal in G and $H \cap T \leq S$ (see [5, Example 4.15]), where S is the subgroup of H generated by all those subgroups of H which are S-quasinormal embedded in G. A subgroup H of a group G is called $S\Phi$-supplemented [15] in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$. Naturally, it is necessary to unify the above-mentioned generalized normal subgroups and discuss the influence on the structure of a finite group by connecting these subgroups with Frattini subgroup. Hence we give the following notion.

Definition 1.1. Let τ be a subgroup functor and H a p-subgroup of a finite group G. Let $\bar{G} = G/H_G$ and $\bar{H} = H/H_G$. We say that H is Φ-τ-quasinormal in G if for some S-quasinormal subgroup $\bar{T}$ of $\bar{G}$ and some τ-subgroup $\bar{S}$ of $\bar{G}$ contained in $\bar{H}$, $\bar{H}\bar{T}$ is S-quasinormal in $\bar{G}$ and $\bar{H} \cap \bar{T} \leq \bar{S}\Phi(\bar{H})$.

It is easy to see that the above mentioned subgroups are Φ-τ-quasinormal in G. But the following examples show that the converse does not hold in general.

Example 1.2. *Let $G = A_5$ be the alternating group of degree 5 and $\tau(G)$ be the set of all subgroups of G. We take $H = \langle(123)\rangle$. Then H is a τ-subgroup of G. Since $H_{sG} = H_G = \Phi(H) = 1$ and H is not s-semipermutable in G, H is not the above mentioned subgroups, but clearly, H is Φ-τ-quasinormal in G.* □

Now we introduce some properties of subgroup functors (see [11, Definition 1.3]) which will be used in our results. If τ is a subgroup functor, then we say that τ is:

(1) inductive if for any group G, whenever $H \in \tau(G)$ is a p-group and $N \trianglelefteq G$, then $HN/N \in \tau(G/N)$.
(2) hereditary if for any group G, whenever $H \in \tau(G)$ is a p-group and $H \leq E \leq G$, then $H \in \tau(E)$.
(3) regular (respectively, quasiregular) if for any group G, whenever $H \in \tau(G)$ is a p-group and N is a minimal normal subgroup (respectively,

an abelian minimal normal subgroup) of G, then $|G : N_G(H \cap N)|$ is a power of p.

(4) Φ-regular (respectively, Φ-quasiregular) if any primitive group G, whenever $H \in \tau(G)$ is a p-group and N is a minimal normal subgroup (respectively, an abelian minimal normal subgroup) of G, then $|G : N_G(H \cap N)|$ is a power of p.

2. Preliminaries

Lemma 2.1 ([9, Chapter I. Lemma 5.34])**.** *Let $H, K \leq G$ and $N \trianglelefteq G$.*

(1) *If H is S-quasinormal in G, then HN/N is S-quasinormal in G/N.*
(2) *If H/N is S-quasinormal in G/N, then H is S-quasinormal in G.*
(3) *If H and K are S-quasinormal in G, then $H \cap K$ is S-quasinormal in G.*
(4) *If H is S-quasinormal in G, then $H \cap K$ is S-quasinormal in K.*
(5) *If H is a p-group, then H is S-quasinormal in G if and only if $O^p(G) \leq N_G(H)$.*
(6) *If H is S-quasinormal in G, then H/H_G is nilpotent.*
(7) *If H is an S-quasinormal nilpotent subgroup of G, then every Sylow subgroup of H is also S-quasinormal in G.*

Lemma 2.2. *Let H be a p-subgroup of G and τ an inductive subgroup functor. Suppose that H is Φ-τ-quasinormal in G.*

(1) *If $N \trianglelefteq G$ and either $N \leq H$ or $(|H|, |N|) = 1$, then HN/N is Φ-τ-quasinormal in G/N.*
(2) *If τ is hereditary and $H \leq K \leq G$, then H is Φ-τ-quasinormal in K.*

Proof. Let $\bar{G} = G/H_G$ and $\bar{H} = H/H_G$. Since H is Φ-τ-quasinormal in G, $\bar{G}$ has an S-quasinormal subgroup $\bar{T}$ and a τ-subgroup $\bar{S}$ contained in $\bar{H}$ such that $\bar{H}\bar{T}$ is S-quasinormal in $\bar{G}$ and $\bar{H} \cap \bar{T} \leq \bar{S}\Phi(\bar{H})$, where $\bar{S} = S/H_G$ and $\bar{T} = T/H_G$.

(1) Let $\widehat{G} = G/(HN)_G$, $\widehat{HN} = HN/(HN)_G$, $\widehat{T} = T(HN)_G/(HN)_G$ and $\widehat{S} = S(HN)_G/(HN)_G$. Clearly, $H_G \leq (HN)_G$. Since τ is injective, $\widehat{S} \in \tau(\widehat{G})$. By Lemma 2.1(1), $\widehat{T}$ and $\widehat{HN}\widehat{T}$ is S-quasinormal in $\widehat{G}$. Since $(|N|, |H|) = 1$, $(|NH \cap T : T \cap N|, |NH \cap T : T \cap H|) = 1$. Hence $NH \cap T = (N \cap T)(H \cap T)$. It follows that $\widehat{HN} \cap \widehat{T} = HN/(HN)_G \cap T(HN)_G/(HN)_G = (H \cap T)(HN)_G/(HN)_G \leq (S(HN)_G/(HN)_G) \Phi(HN/(HN)_G)) = \widehat{S}\Phi(\widehat{HN})$. Therefore, HN/N is Φ-τ-quasinormal in G/N.

(2) It is easy to see that $H_G \leq H_K$. Let $\widetilde{K} = K/H_K$, $\widetilde{H} = H/H_K$, $\widetilde{T} = TH_K/H_K \cap K/H_K$ and $\widetilde{S} = SH_K/H_K$. Since τ is hereditary and inductive, $\widetilde{S} \in \tau(\widetilde{K})$. By Lemma 2.1(2)(5), $\widetilde{T}$ and $\widetilde{H}\widetilde{T} = (H/H_K)(TH_K/H_K \cap K/H_K) = H(T \cap K)/H_K = (HT \cap K)/H_K$ is S-quasinormal in $\widetilde{K}$. It is easy to see that

$\widetilde{H} \cap \widetilde{T} = H/H_K \cap TH_K/H_K = (H \cap T)H_K/H_K \leq (SH_K/H_K)\Phi(H/H_K) = \widetilde{S}\Phi(\widetilde{H})$. Hence H is Φ-τ-quasinormal in K. □

The next lemma is clear.

Lemma 2.3. *Let p be a prime divisor of $|G|$ with $(|G|, p-1) = 1$.*

(1) *If G has a cyclic Sylow p-subgroup, then G is p-nilpotent.*
(2) *If N is a normal subgroup of G such that $|N|_p \leq p$ and G/N is p-nilpotent, then G is p-nilpotent.*

Lemma 2.4 ([18, Lemma 2.8])**.** *Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = MP$, where p is a prime divisor of $|G|$. Then $P \cap M \trianglelefteq G$.*

Lemma 2.5 ([4, Theorem 2.12])**.** *Let P be a normal p-subgroup of G and D a Thompson critical subgroup of P* (see [7, p. 186])*. If $D \leq Z_{\mathfrak{U}}(G)$, then $P \leq Z_{\mathfrak{U}}(G)$.*

Lemma 2.6 ([17, Theorem B])**.** *Let $\mathfrak{F}$ be any formation and E a normal subgroup of G. If $F^*(E) \leq Z_{\mathfrak{F}}(G)$, then $E \leq Z_{\mathfrak{F}}(G)$.*

The following well-known facts about the generalized Fitting subgroup are needful in our proof (see [10, Lemma 2.14] or [14, Chapter X])

Lemma 2.7. *Let N be a subgroup of G.*

(1) *If $N \trianglelefteq G$, then $F^*(N) = F^*(G) \cap N$.*
(2) *$F(G) \leq F^*(G)$. If $F^*(G)$ is soluble, then $F(G) = F^*(G)$.*
(3) *If N is a p-group contained in $Z(G)$, then $F^*(G/N) = F^*(G)/N$.*
(4) *If N is soluble in G, then $F^*(G)/\Phi(N) = F^*(G/\Phi(N))$.*
(5) *$F^*(G) = F(G)E(G)$, $F(G) \cap E(G) = Z(E(G))$, $[F(G), E(G)] = 1$ and $E(G)/Z(E(G))$ is the direct product of simple non-abelian groups, where $E(G)$ is the layer of G.*

3. Main results

Theorem 3.1. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E such that $(|E|, p-1) = 1$. Suppose that τ is a Φ-regular inductive subgroup functor and every τ-subgroup of G contained in P is subnormally embedded in G. If every maximal subgroup of P is Φ-τ-quasinormal in G, then E is p-nilpotent.*

Proof. Suppose that the theorem is false and let (G, E) is a counterexample with $|G| + |E|$ minimal. Then: (1) $O_{p'}(E) = 1$.

Suppose that $O_{p'}(E) \neq 1$. Let $M/O_{p'}(E)$ be a maximal subgroup of $PO_{p'}(E)/O_{p'}(E)$. Then $M = P_1O_{p'}(E)$ for some maximal subgroup P_1 of P. By the Lemma 2.2(1) and the hypothesis, $P_1O_{p'}(E)/O_{p'}(E)$ is Φ-τ-quasinormal in $E/O_{p'}(E)$. This shows that $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis of the theorem. The choice of (G, E) implies that $E/O_{p'}(E)$ is p-nilpotent, and so E is p-nilpotent, a contradiction. Hence $O_{p'}(E) = 1$.

(2) G has a unique minimal normal subgroup N contained in E, E/N is p-nilpotent and $G = NM$, where M is a maximal subgroup of G.

Let N be a minimal normal subgroup of G contained in E and H/N be a maximal subgroup of PN/N. Then there exists a maximal subgroup P_1 of P such that $H = P_1N$ and $P_1 \cap N = P \cap N$. Set $\bar{G} = G/(P_1)_G$ and $\bar{P}_1 = P_1/(P_1)_G$. By the hypothesis, $\bar{G}$ has an S-quasinormal subgroup $\bar{T}$ and a τ-subgroup $\bar{S}$ contained in $\bar{P}_1$ such that $\bar{P}_1\bar{T}$ is S-quasinormal in $\bar{G}$ and $\bar{P}_1 \cap \bar{T} \leq \bar{S}\Phi(\bar{P}_1)$, where $\bar{S} = S/(P_1)_G$ and $\bar{T} = T/(P_1)_G$. Let $\widehat{G} = G/(P_1N)_G$, $\widehat{P_1N} = P_1N/(P_1N)_G$, $\widehat{T} = T(P_1N)_G/(P_1N)_G$ and $\widehat{S} = S(P_1N)_G/(P_1N)_G$. Since $(|P_1N \cap T : P_1 \cap T|, |P_1N \cap T : N \cap T|) = 1$, $P_1N \cap T = (P_1 \cap T)(N \cap T)$. By using a similar discussion as in the proof of Lemma 2.2(1), we have that H/N is Φ-τ-quasinormal in G/N. The choice of (G, E) implies that E/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G contained in E and $N \nleq \Phi(G)$. Then there exists a maximal subgroup M of G such that $G = NM$.

(3) $O_p(E) = 1$.

Suppose that $O_p(E) \neq 1$. Then by (2), $N \leq O_p(E)$ and $G = N \rtimes M$. Since $O_p(G) \leq C_G(N)$, $O_p(G) \cap M$ is normal G and so $O_p(E) \cap M$ is normal G. If $O_p(E) \cap M \neq 1$, then $N \leq O_p(E) \cap M$, a contradiction. Thus $O_p(E) \cap M = 1$. It follows that $O_p(E) = O_p(E) \cap NM = N$ and it is easy to see that $C_E(N) = N$. Let $K = M \cap E$. Then $E = N \rtimes K$. Let K_p be a Sylow p-subgroup of K such that $P = NK_p$ and M_p a Sylow p-subgroup of M containing K_p. Then $G_p = NM_p$ is a Sylow p-subgroup of G. Let N_1 be a maximal subgroup of N such that N_1 is normal in G_p. Then $G_1 = N_1M_p$ is a maximal subgroup of G_p, $P_1 = N_1K_p$ is a maximal subgroup of P and $P = NP_1$. If $(P_1)_G \neq 1$, then by (2), $N \leq P_1$ and so $P = P_1$, a contradiction. Hence $(P_1)_G = 1$. By the hypothesis, G has an S-quasinormal subgroup T and a τ-subgroup S contained in P_1 such that P_1T is S-quasinormal in G and $P_1 \cap T \leq S\Phi(P_1)$. Since τ is a Φ-regular inductive subgroup functor, $|G/M_G : N_{G/M_G}(SM_G/M_G \cap NM_G/M_G)|$ is a power of p. If $SM_G \cap NM_G \neq M_G$, then $(SM_G/M_G \cap NM_G/M_G)^{G/M_G} = (SM_G/M_G \cap NM_G/M_G)^{G_pM_G/M_G} \leq G_1M_G/M_G$ and so $N \leq G_1M_G$. Hence $N = N \cap G_1M_G = N \cap N_1M_pM_G = N_1$, a contradiction. Thus $SM_G \cap NM_G = M_G$. Obviously, $SN \cap M_G = 1$ because $E \cap M_G = 1$. Hence $SM_G \cap NM_G = (S \cap N)M_G = M_G$ and so $S \cap N \leq M_G \cap N = 1$.

We claim that $S = 1$. Assume that $S \neq 1$. Since S is subnormally embedded in G, there exists a subnormal subgroup V of G such that S is a Sylow p-subgroup of V. Without loss of generality, we may assume that $V \leq E$. Let L be a minimal subnormal subgroup of G contained in V. Then $L \cap N$ is subnormal in G. If $L \cap N = 1$, then by (2), $L \cong LN/N \leq E/N$ is p-nilpotent, and so by (1), L is a p-group. Then $L \leq O_p(E) = N$, a contradiction. Hence

$L \cap N \neq 1$. Then $L \leq N$ and so $L^G = N$. It implies that $L \cap S = 1$. By (1), $L \leq O_{p'}(E) = 1$, which is impossible. Hence $S = 1$.

Clearly, $N_1 = P_1 \cap N$. If $T = 1$, then P_1 is S-quasinormal in G and so N_1 is S-quasinormal in G. Since N_1 is normal in G_p, N_1 is normal in G, and so $|N| = p$. By Lemma 2.3(2), E is p-nilpotent, a contradiction. Hence $T \neq 1$. Assume that $N \nleq T$. Then $(T \cap E)_G = 1$. By Lemma 2.1(4) and (6), $T \cap E$ is a nilpotent group. By (1), $O_{p'}(T \cap E) = 1$. Hence $T \cap E = T \cap P$ is a p-group. It deduces that $P_1T \cap E = P_1(T \cap E)$ is an S-quasinormal p-subgroup of G. If $T \cap E = 1$, then P_1 is S-quasinormal in G, which is impossible. Thus $T \cap E \neq 1$. Then $P_1 \leq P_1(T \cap E) \leq O_p(E) = N$, and so $P = N$. It follows that $T \cap P$ is S-quasinormal in E and $P_1 \cap T = 1$. Thus $T \cap P$ is normal in E and $|P \cap T| = p$. Since $(|E|, p-1) = 1$, $E/C_E(P \cap T) = 1$, and thereby $P \cap T \leq Z(E)$. By Lemma [13, Chapter VI, Theorem 14.3], $P \cap T = 1$, which is a contradiction. Therefore we assume that $N \leq T$. Then $N \cap P_1 \leq \Phi(P_1)$. This deduces that $P_1 = P_1 \cap NK_p = K_p(P_1 \cap N) = K_p$, which contradicts $N \cap K_p = 1$. Hence $O_p(E) = 1$.

(4) $N \cap P < P$.

Assume that $N \cap P = P$, then $P \leq N$. If $N < E$, the choice of the (G, E) shows N is p-nilpotent. Then by (1), N is a p-group, which contradicts (3). Hence $E = N$. Let P_1 be a maximal subgroup of P. Obviously, $(P_1)_G = 1$. Hence by the hypothesis, G has an S-quasinormal subgroup T and a τ-subgroup S contained in P_1 such that P_1T is S-quasinormal in G and $P_1 \cap T \leq S\Phi(P_1)$. Assume that $S \neq 1$. Since τ is Φ-regular and inductive, $|G : N_G(SM_G)|$ is a power of p. It follows that $N \leq S^G M_G = S^{G_p} M_G \leq G_p M_G$, where G_p is a Sylow p-subgroup of G containing P. Then $N = N \cap G_p M_G = N \cap G_p$ because $N \cap M_G = 1$. It follows that N is a p-group. This contradicts (3). Hence $S = 1$. It is easy to see that $N \nleq T$. If $N \cap T = 1$, then $P_1 = P_1T \cap N$ is S-quasinormal in G, and so $P_1 \leq O_p(E) = 1$ by (3). Then $|P| = p$. By Lemma 2.3(1), N is p-nilpotent, a contradiction. Hence $N \cap T \neq 1$. Assume that $(N \cap T)_G \neq 1$, then $N \leq T$, a contradiction. Hence $(N \cap T)_G = 1$. By Lemma 2.1(4) and (6), $T \cap N$ is a nilpotent group. By (1) and (3), $T \cap N \leq F(E) = 1$. Then $P_1 = P_1T \cap N$ is S-quasinormal in G. This contradiction shows that (4) holds.

(5) Final contradiction.

By (4), P has a maximal subgroup P_1 such that $N \cap P \leq P_1$. Clearly, $(P_1)_G = 1$. By hypothesis, G has an S-quasinormal subgroup T and a τ-subgroup S contained in P_1 such that P_1T is S-quasinormal in G and $P_1 \cap T \leq S\Phi(P_1)$. By (2), $SN \cap M_G = 1$. Thus $SM_G \cap NM_G = (S \cap N)M_G$. Since τ is Φ-regular and inductive, $|G : N_G(SM_G \cap NM_G)|$ is a power of p. It follows that $N \leq (S \cap N)^G M_G = (S \cap N)^{G_p} M_G \leq G_p M_G$, where G_p is a Sylow p-subgroup of G contained P. Then $N = N \cap G_p M_G = N \cap G_p$, which means that N is a p-group. This contradics (3). Thus $S \cap N = 1$. By using a similar dicussion as in (3), we have $S = 1$. If $T = 1$, then P_1 is S-quasinormal in G, and so $P_1 \leq O_p(E) = 1$, a contradiction. Hence $T \neq 1$. By applying a similar argument as in (3), we derive that $(T \cap E)_G \neq 1$. Then $N \leq (T \cap E)_G \leq T$. Hence $N \cap P_1 \leq \Phi(P)$ and so $N \cap P \leq \Phi(P)$. Then by [13, Chapter IV, Satz 4.7], N is p-nilpotent, a contradiction too. The proof of the theorem ends. □

Theorem 3.2. *Let τ be a Φ-regular inductive subgroup functor and E be a normal subgroup of G such that G/E is supersoluble. Suppose that every τ-subgroup of G contained in E is subnormally embedded in G. If every maximal subgroup of every noncyclic Sylow subgroup of E is Φ-τ-quasinormal in G, then G is supersoluble.*

Proof. Suppose that it is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. (1) If p is the largest prime dividing $|E|$ and P is a Sylow p-subgroup of E, then $P \trianglelefteq G$.

Let q be the smallest prime dividing $|E|$ and Q a Sylow q-subgroup of E. If Q is cyclic, then E is q-nilpotent by Lemma 2.3(1). Now assume that Q is non-cyclic. It is easy to see that (G, E) satisfies the hypothesis of Theorem 3.1, hence E is q-nilpotent too. By Lemma 2.2(1), we can deduce that E is a Sylow tower group of supersoluble type by analogy. Hence $P \trianglelefteq E$ and so $P \trianglelefteq G$.

(2) G has a unique minimal normal subgroup N contained in P such that G/N is supersoluble.

Let N be a minimal normal subgroup of G contained in P and T_1/N be a maximal subgroup of a noncyclic Sylow q-subgroup T/N of E/N, where q is a prime divisor of $|E/N|$. Obviously, $(G/N)/(E/N) \cong G/E$ is supersoluble. If $p = q$, then T_1 is a maximal subgroup of T. Assume that $p \neq q$, then there exists a Sylow q-subgroup Q of E such that $T = QN$. Clearly, $Q_1 = Q \cap T_1$ is a maximal subgroup of Q and $T_1 = Q_1N$. By Lemma 2.2(1), T_1/N is Φ-τ-quasinormal in G/N. Hence $(G/N, E/N)$ satisfies the hypothesis. The choice of (G, E) implies that G/N is supersoluble.

(3) $P = N$ and p is the largest prime dividing $|G|$.

Since the class of all supersoluble groups is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$ by (2). Then there exists a maximal subgroup M of G such that $G = N \rtimes M = PM$. It follows from Lemma 2.4 that $P \cap M = 1$. Hence $P = P \cap NM = N(P \cap M) = N$. Assume that p is not the largest prime dividing $|G|$. Let q be the largest prime dividing $|G|$ and Q be a Sylow q-subgroup of G. By (2), G/P is supersoluble. Then $QP/P \trianglelefteq G/P$ and so $QP \trianglelefteq G$. Clearly, P is not cyclic. Hence by Theorem 3.1, PQ is p-nilpotent, and so $Q \trianglelefteq PQ$. Then $Q \trianglelefteq G$. By Lemma 2.2(1), every maximal subgroup of every noncyclic Sylow subgroup of EQ/Q is Φ-τ-quasinormal G/Q. This shows that $(G/Q, EQ/Q)$ satisfies the hypothesis. The choice of (G, E) implies that G/Q is supersoluble. Since p is the largest prime dividing $|E|$, we have that $E \cap Q = 1$. It implies that $G \cong G/E \cap Q \lesssim G/E \times G/Q$ is supersoluble. This contradiction shows that p is the largest prime dividing $|G|$.

(4) Final contradiction.

Let G_p be a Sylow p-subgroup of G, where p is the largest prime dividing $|G|$. Then $P \leq G_p$. Since G/P is supersoluble, $G_p \trianglelefteq G$. It is easy to see that P is not cyclic. Because that $P \cap Z(G_p) \neq 1$, we have that $P \leq Z(G_p)$ by (2) and (3). Let P_1 be a maximal subgroup of P. If $(P_1)_G \neq 1$, then by (3), $P \leq P_1$, a contradiction. Thus $(P_1)_G = 1$. Clearly, $\Phi(P_1) = 1$. By hypothesis, G has an S-quasinormal T and a τ-subgroup S contained in P_1 such that P_1T is S-quasinormal in G and $P_1 \cap T \leq S$. If $T = 1$, then P_1 is S-quasinormal in G. By (3) and Lemma 2.1(5), $P \leq (P_1)^G = (P_1)^{G_p} \leq P_1$. This contradiction shows that $T > 1$. If $P \cap T = 1$, then $P_1 = P_1T \cap P$ is S-quasinormal in G, which is impossible. Hence $P \cap T \neq 1$. By (3), $P \leq (P \cap T)^G = P \cap T$. Thus $P \leq T$ and so $P_1 = S$. Since τ is Φ-regular and inductive, $|G : N_G(SM_G)|$ is a power of p. It follows that $(SM_G)^G = (SM_G)^{G_p} = SM_G$. Therefore $S = P \cap SM_G$ is normal in G. The final contradiction completes the proof of the theorem. □

Corollary 3.3. *Let τ be a Φ-regular inductive subgroup functor and every τ-subgroup of G is subnormally embedded in G. If every maximal subgroup of every noncyclic Sylow subgroup of G is Φ-τ-quasinormal in G, then G is supersoluble.*

Theorem 3.4. *Let τ be a quasiregular heredity inductive subgroup functor and E be a normal subgroup of G. Suppose that every τ-subgroup of G contained in E is subnormally embedded in G. If every maximal subgroup of every noncyclic Sylow subgroup of $F^*(E)$ is Φ-τ-quasinormal in G, then E is soluble.*

Proof. Suppose that it is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Let p be any prime divisor of $|F^*(G)|$ and P the Sylow p-subgroup of $F^*(G)$. We prove theorem via the following steps. (1) $E = G$ and $F^*(G) = F(G)$.

By Lemma 2.2(2), (E, E) satisfies the hypothesis of the theorem. If $E \neq G$, the choice of (G, E) implies that E is soluble, a contradiction. Hence $E = G$. By Lemma 2.2(2), $F^*(G)$ satisfies the hypothesis of Corollary 3.3, so $F^*(G)$ is soluble. Then by Lemma 2.7(2), $F^*(G) = F(G)$.

(2) Every proper normal subgroup of G containing $F(G)$ is soluble and $G = F(G)O^p(G)$.

Let M be a proper normal subgroup of G containing $F(G)$. By Lemma 2.7(1), $F^*(G) = F(G) \leq F(M) \leq F^*(M) \leq F^*(G)$. Thus $F^*(G) = F^*(M)$. It follows that (M, M) satisfies the hypothesis. The choice of (G, E) implies that M is soluble. Assume that $G \neq F(G)O^p(G)$. Then $F(G)O^p(G)$ is soluble, and hence G is soluble, a contradiction. Thus $G = F(G)O^p(G)$.

(3) $\Phi(P) = 1$ and G has no normal subgroup of prime order contained in $F(G)$.

Assume that $\Phi(P) \neq 1$. By Lemma 2.7(4), $F^*(G)/\Phi(P) = F^*(G/\Phi(P))$. Therefore, by Lemma 2.2(1), $(G/\Phi(P), G/\Phi(P))$ satisfies the hypothesis of the theorem. The choice of (G,E) implies that $G/\Phi(P)$ is soluble and so G is soluble, a contradiction. Hence $\Phi(P)=1$. This implies that $F(G)$ is elementary abelian.

Let L be a normal subgroup of G contained in $F(G)$ and $|L|=p$. Then $F^*(G)=F(G)\leq C_G(L)$. If $C_G(L)<G$, then $C_G(L)$ is soluble by (2). Since $G/C_G(L)$ is cyclic, G is soluble. This contradiction implies that $C_G(L)=G$ and so $L\leq Z(G)$. By Lemma 2.7(3), $F^*(G)/L=F^*(G/L)$. Therefore, by Lemma 2.2(1), $(G/L,G/L)$ satisfies the hypothesis of the theorem. The choice of (G,E) implies that G/L is soluble, and thereby G is soluble, a contradiction.

(4) $P\cap\Phi(G)\neq 1$.

Assume that $P\cap\Phi(G)=1$. Then $P=R_1\times R_2\times\cdots\times R_m$, where R_i $(i=1,2,\ldots,m)$ is a minimal normal subgroup of G (see [8, Theorem 1.8.17]). We claim that $|R_i|=p$ for all $i\in\{1,\ldots,m\}$. Assume that $|R_i|>p$ for some i. Without loss of generality, let $|R_1|>p$. Let R_1^* be a maximal subgroup of R_1 such that R_1^* is normal in G_p, where G_p is a Sylow p-subgroup of G. Then $R_1^*\neq 1$ is not normal in G and $P_1=R_1^*\times R_2\times\cdots\times R_m$ is a maximal subgroup of P. Put $T=R_2\times\cdots\times R_m$. Clearly, $(P_1)_G=T$ and $\Phi(P_1/T)=1$. Therefore by the hypothesis, there exists an S-quasinormal subgroup K/T in G/T and some τ-subgroup S/T of G/T contained in P_1/T such that P_1K/T is S-quasinormal in G/T and $(P_1/T)\cap(K/T)\leq S/T$. By Lemma 2.1(2)(3), $R_1\cap K$ and $R_1\cap P_1K$ is S-quasinormal in G. Hence by (2), (3) and Lemma 2.1(5), $R_1\cap K$ and $R_1\cap P_1K$ is normal in G. Clearly, $P_1K=R_1^*K$. If $R_1\cap K=1$, then $R_1^*=R_1^*(R_1\cap K)=R_1\cap R_1^*K=R_1\cap P_1K$ is normal in G, and so $|R_1|=p$, which contradicts (3). Therefore $R_1\cap K\neq 1$, and thereby $R_1\leq K$. Then $R_1^*=R_1\cap P_1=R_1\cap S$. Since τ is regular, $|G:N_G(R_1^*)|$ is a power of p. It follows that R_1^* is normal in G, a contradiction. Thus (4) holds.

(5) $F(G)=P$ and P contains a unique minimal normal subgroup N of G.

Suppose that $1\neq Q$ is a Sylow q-subgroup of $F(G)$ for some $p\neq q$. Clearly, $Q\trianglelefteq G$. Let N be a minimal normal subgroup of G contained in $P\cap\Phi(G)$ by (4). By Lemma 2.7(5), $F^*(G/N)=F(G/N)E(G/N)$ and $[F(G/N),E(G/N)]=1$, where $E(G/N)$ is the layer of G/N. Let $E(G/N)=E/N$. Then $[Q,E]\leq N\cap Q=1$. Hence by (3), $EF(G)\leq C_G(Q)$. If $C_G(Q)\neq G$. Then by (2), $C_G(Q)$ is soluble. It follows that $F^*(G/N)$ is soluble and so $F^*(G/N)=F(G/N)=F(G)/N$. By Lemma 2.2(1), $(G/N,G/N)$ satisfies the hypothesis. The choice of (G,E) implies that G/N is soluble and thereby G is soluble. This contradiction shows $C_G(Q)=G$. Hence $Q\leq Z(G)$. With a similar proof as above (3), we have that G is soluble. The contradiction shows $F(G)=P$.

Let L be a minimal normal subgroup of G contained in P with $N\neq L$. Let $E/N=E(G/N)$ be the layer of G/N again. As above, $[F(G),E]\leq N$. It follows that $[L,E]\leq L\cap N=1$. Hence $F(G)E\leq C_G(L)$, and so one can obtain that $L\leq Z(G)$. Applying a same discussion as in (3), we can derive a contradiction. Thus N is the unique minimal normal subgroup of G contained in P.

(6) Final contradiction.

Let N_1 be a maximal subgroup of N such that N_1 is normal in some Sylow p-subgroup of G. Then by (3), $P_1 = N_1S$ is a maximal subgroup of P, where S is a complement of N in P. Obviously, $P = P_1N$ and $(P_1)_G = 1$ by (5). Therefore, G has an S-quasinormal subgroup T and a τ-subgroup S contained in P_1 such that P_1T is S-quasinormal in G and $P_1 \cap T \le S$. By (2), we derive that $P \cap T$ is normal in G. If $P \cap T = 1$, then $P_1 = P \cap P_1T$ is S-quasinormal in G. Hence $N_1 = N \cap P_1$ is S-quasinormal in G, and so N_1 is normal in G. It implies that $|N| = p$, which contradicts (3). Thus $P \cap T \ne 1$. Then $N \le P \cap T$ by (5). It follows that $N_1 = N \cap P_1 = N \cap S$. By using a similar discussion as in (4), we have that N_1 is normal in G. The final contradiction completes the proof. □

Remark 3.5. The Corollary 3.3 shows that if every maximal subgroup of every noncyclic Sylow subgroup of G is Φ-τ-quasinormal in G, then G is supersoluble. But, the following example shows that if every maximal subgroup of every noncyclic Sylow subgroup of $F^*(G)$ is Φ-τ-quasinormal in G, G may be not supersoluble.

Example 3.6. Let $A = S_3$ be a symmetric group of degree 3 and $B = C_3$ be a cyclic group of order 3. Let $G = A \wr B$ be the regular wreath product of A by B. Let $K = A_1 \times A_2 \times A_3$ be the base group of G, where $A_i = \langle \alpha_i, \beta_i \mid \alpha_i^3 = \beta_i^2 = 1, \alpha_i^{\beta_i} = \alpha_i^2 \rangle \cong S_3$, $i = 1, 2, 3$. It is easy to see that $F^*(G) = F(G) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$. Since $\langle \alpha_i \rangle$ is S-quasinormal in G by Lemma 2.1(5), $i = 1, 2, 3$, every maximal subgroup of $F^*(G)$ has an S-quasinormal complement $\langle \alpha_i \rangle$ in $F^*(G)$ for some i. Hence every maximal subgroup of $F^*(G)$ is Φ-τ-quasinormal in G. But, clearly, G is not supersoluble. □

Theorem 3.7. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E such that $(|E|, p-1) = 1$. Suppose that τ is a Φ-regular hereditary inductive subgroup functor of G contained in P. If every cyclic subgroup of P of prime order or order 4 (when P is a non-abelian 2-group) is Φ-τ-quasinormal in G, then E is p-nilpotent.*

Proof. Suppose that it is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. We prove theorem via the following steps.

First, we show that $E = G$ and G is a minimal non-nilpotent group. Assume that $E < G$. Then by Lemma 2.2(2), (E, E) satisfies the hypothesis. The choice of the (G, E) implies that E is p-nilpotent, a contradiciton. Thus $E = G$. Let M be any maximal subgroup of G. Similarly, M is p-nilpotent and so G is a minimal non-p-nilpotent group. In view of [13, Chapter IV, Satz 5.4] and [6, Chapter VII, Theorem 6.18], G is a minimal non-nilpotent group; $G = P \rtimes Q$, where Q is a Sylow q-subgroup of G with $q \neq p$; $P/\Phi(P)$ is a chief factor of G; the exponent of P is p or 4 (when P is a non-abelian 2-group).

Let $x \in P\backslash\Phi(P)$, $H = \langle x \rangle$. Then $|H| = p$ or 4 (when P is a non-abelian 2-group) and $H < P$. Since $P/\Phi(P)$ is a chief factor of G, $H_G \leq \Phi(P)$ and $H \neq H_G$. By the hypothesis, there exists an S-quasinormal subgroup T/H_G in G/H_G and some τ-subgroup S/H_G of G/H_G contained in H/H_G such that HT/H_G is S-quasinormal in G/H_G and $H/H_G \cap T/H_G \leq (S/H_G)\Phi(H/H_G)$. By Lemma 2.1(3)(5), $(P \cap T)\Phi(P)/\Phi(P)$ and $(P \cap HT)\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. It implies that $P \leq T$ and so $H = S$. Clearly, $P/\Phi(P) \nleq \Phi(G/\Phi(P))$. Therefore there exists a maximal $M/\Phi(P)$ subgroup of $G/\Phi(P)$ such that $G/\Phi(P) = P/\Phi(P) \rtimes M/\Phi(P)$. It is easy to see that $P \cap M_G = \Phi(P)$. It follows that PM_G/M_G is a minimal normal subgroup of G/M_G. Since τ is a Φ-regular inductive subgroup functor, $|G : N_G(HM_G)|$ is a power of p, and thereby $|G/\Phi(P) : N_{G/\Phi(P)}(HM_G/\Phi(P))|$ is a power of p. It implies that $HM_G/\Phi(P)$ is normal in $G/\Phi(P)$. Hence HM_G is normal in G and so $H\Phi(P) = P \cap HM_G$ is normal in G, which is impossible. The proof of the theorem is completes. □

Theorem 3.8. *Let E be a normal subgroup of G and τ be a Φ-regular hereditary inductive subgroup functor of G contained in E. Suppose that for every prime p dividing $|E|$ and every non-cyclic Sylow p-subgroup P of E, every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) is Φ-τ-quasinormal in G, then E is supersoluble.*

Proof. Suppose that it is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Applying a same discussion as in the proof of Theorem 3.7, we have that $E = G$ and G is a minimal non-supersoluble group. In view of [2, Theorem 12] and [6, Chapter VII, Theorem 6.18], G is a soluble group that has a unique normal Sylow p-subgroup, say G_p; $G_p = G^{\mathfrak{U}}$; $G_p/\Phi(G_p)$ is a chief factor of G; the exponent of G_p is p or 4 (when G_p is a non-abelian 2-group). Let $x \in G_p\backslash\Phi(G_p)$ and $H = \langle x \rangle$. Then $|H| = p$ or 4 (when G_p is a non-abelian 2-group) and $H < G_p$. Since $(G/\Phi(G_p))/(G_p/\Phi(G_p)) \cong G/G_p$ is supersoluble, $G_p/\Phi(G_p) \nleq \Phi(G/\Phi(G_p))$. By using a similar argument as in the proof of Theorem 3.7, we can get a contradiction. □

Theorem 3.9. *Let E be a normal subgroup of G and τ be a quasiregular hereditary inductive subgroup functor of G contained in E. Suppose that for every prime p dividing $|F^*(E)|$ and every non-cyclic Sylow p-subgroup P of $F^*(E)$, every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) is Φ-τ-quasinormal in G, then E is soluble.*

Proof. Suppose that it is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. (1) $E = G$ and $F^*(G) = F(G)$.

With a similar argument as in steps (1) of the proof of Theorem 3.4, we have that $E = G$. By Theorem 3.8, $F^*(G)$ is soluble. Then by Lemma 2.7(2), $F^*(G) = F(G)$.

(2) $G = O^q(G)$, where q is any prime divisor of $|G|$, the every S-quasinormal subgroup of G is normal in G and $Z_{\mathfrak{U}}(G) = Z(G)$.

By Lemma 2.7(1), $F^*(G') \leq F^*(G)$. Therefore, by Lemma 2.2(2), (G, G') satisfies the hypothesis of the theorem. If $G' \neq G$, the choice of (G, G) implies that G' is soluble, and so G is soluble, a contradiction. Thus $G = G'$. It implies that $G = O^q(G)$, where q is any prime divisor of $|G|$. Let T be an S-quasinormal subgroup of G and T_p be a Sylow p-subgroup of T, where p is any prime divisor of $|T|$. Without loss of generality, we may assume that $T_G = 1$. Then by Lemma 2.1(6)(7), T is nilpotent and T_p is S-quasinormal in G. Hence by Lemma 2.1(5), T_p is normal in G, and so T is normal in G.

Since $G = G'$, we have that $G/C_G(Z_{\mathfrak{U}}(G)) = (G/C_G(Z_{\mathfrak{U}}(G)))'$ and thereby $G/C_G(Z_{\mathfrak{U}}(G))$ is not soluble. But by [6, Chapter IV, Theorem 6.10], $G/C_G(Z_{\mathfrak{U}}(G))$ is supersoluble. It follows that $G = C_G(Z_{\mathfrak{U}}(G))$ and so $Z_{\mathfrak{U}}(G) \leq Z(G)$. Therefore $Z_{\mathfrak{U}}(G) = Z(G)$.

(3) $p > 2$.

Assume that $p = 2$. Let Q be an arbitrary Sylow q-subgroup of G, where $q \neq 2$ is a prime divisor of G. By Theorem 3.7, PQ is 2-nilpotent and so $Q \leq C_G(P)$. It follows that $O^2(G) \leq C_G(P)$, and thereby $P \leq Z(G)$ by (2). By Lemma 2.7(3), $F^*(G/P) = F^*(G)/P$. Therefore by Lemma 2.2(1), $(G/P, G/P)$ satisfies the hypothesis of the theorem, and thus G/P is soluble, a contradiction. Hence (3) holds.

(4) D contained a minimal normal subgroup N of prime order of G, where D is a Thompson critical subgroup of P of exponent p.

By (3) and [7, Chapter 5, Theorem 3.13], P contains a Thompson critical subgroup D of exponent p. Let N be a minimal normal subgroup of G contained in D. And let $H \leq N$ with $|H| = p$ and H be normal in some Sylow p-subgroup of G. By hypothesis and (2), G/H_G has a normal subgroup T/H_G and a τ-subgroup S/H_G contained in H/H_G such that HT/H_G is normal in G/H_G and $H/H_G \cap T/H_G \leq S/H_G$. First assume that $H_G = 1$. If $S = 1$, then $H \cap T = 1$ and so $N \cap T = 1$. It follows that $H = N \cap HT$ is normal in G. Now assume that $S \neq 1$. Since τ is a quasiregular subgroup functor, $|G : N_G(H)|$ is a power of p and so H is normal in G. If $H_G \neq 1$, then clearly, $H = H_G$. Hence for every case, we always have that H is normal in G, and thereby $H = N$.

(5) Final contradiction

We claim that every prime order subgroup of D/N is Φ-τ-quasinormal in G/N. Suppose that it is false and let H/N be a subgroup of D/N such that $|H/N| = p$ but H/N is not Φ-τ-quasinormal in G/N. If H/N is normal in G/N, then obviously, H/N is Φ-τ-quasinormal in G/N, a contradiction. Thus $(H/N)_{G/N} = 1$. By (4), there exists an element $x \in H \backslash N$ such that $H = \langle x \rangle N$, $|\langle x \rangle| = p$ and $\langle x \rangle_G = 1$. By hypothesis and (2), G has a normal subgroup T and a τ-subgroup S contained in $\langle x \rangle$ such that $\langle x \rangle T$ is normal in G and $\langle x \rangle \cap T \leq S$. If $S = \langle x \rangle$ or $\langle x \rangle N/N \cap TN/N = 1$, it is easy to see that H/N is

Φ-τ-quasinormal in G/N, a contradiction too. Now assume that $\langle x\rangle \cap T = S = 1$ and $\langle x\rangle N/N \leq TN/N$. Obviously, $[\langle x\rangle, G]$ is normal in G. It is easy to see from (2) that $\langle x\rangle T/T \leq Z(G/T)$. Hence $[\langle x\rangle, G] \leq T$. Since $[\langle x\rangle[\langle x\rangle, G], G] \leq [\langle x\rangle, G]$, $\langle x\rangle[\langle x\rangle, G]/[\langle x\rangle, G] \leq Z(G/[\langle x\rangle, G])$. Therefore $\langle x\rangle[\langle x\rangle, G]$ is normal in G and $\langle x\rangle \cap [\langle x\rangle, G] \leq \langle x\rangle \cap T = 1$. Hence we may let $T = [\langle x\rangle, G]$. Since $\langle x\rangle N/N \leq TN/N$, there exists an element $t \in T\backslash N$ such that $\langle x\rangle N = \langle t\rangle N$ and $|\langle t\rangle| = p$. By using a similar discussion as above, G has a normal subgroup $T_1 = [\langle t\rangle, G]$ and a τ-subgroup $S_1 = 1$ contained in $\langle t\rangle$ such that $\langle t\rangle T_1$ is normal in G and $\langle t\rangle \cap T_1 \leq S_1$. By (2) and (4), we see that $N \leq Z(G)$. Hence $T_1 = [\langle t\rangle, G] = [\langle t\rangle N, G] = [\langle x\rangle N, G] = [\langle x\rangle, G] = T$, which contradicts $\langle t\rangle \cap T_1 = 1$. Therefore every prime order subgroup of D/N is Φ-τ-quasinormal in G/N.

By using a same argument as in (4), we can derive that D/N contained a minimal normal subgroup N_1/N of prime order of G/N. Then by analogy, we can find a chief series of G below D such that every G-chief factor of the series is cyclic. It implies that $D \leq Z_{\mathfrak{U}}(G)$. By Lemma 2.5, $P \leq Z_{\mathfrak{U}}(G)$, and thereby $F^*(G) \leq Z_{\mathfrak{U}}(G)$. It follows from Lemma 2.6 that $G \leq Z_{\mathfrak{U}}(G)$ which contradicts the assumption. This completes the proof. □

Remark 3.10. Similar to Theorem 3.4, if G satisfies the hypothesis of Theorem 3.9, G may be not supersoluble.

Example 3.11. Let G be the same group as in Example 3.6. It is easy to see that every minimal subgroup of $F^*(G)$ has an S-quasinormal complement in $F(G)$. Hence every minimal subgroup of $F^*(G)$ is Φ-τ-quasinormal in G. But, clearly, G is not supersoluble.

4. **Further applications**

Many known results are corollaries of our Theorems. For example, in view of [11, Example 1.5], Theorem 3.4 and Theorem 3.9 covers [10, Theorem C], Theorem 3.1 covers [19, Lemma 3.1], Theorem 3.4 covers [19, Lemma 3.6]; in view of [11, Example 1.7], Theorem 3.8 covers [16, Corollary 3.3]; in view of [5, Example 4.6], Theorem 3.1 covers [1, Corollary 4.4], Theorem 3.9 covers [1, Theorem 3.7 and Theorem 1.7].

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