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MODULES WHOSE DIRECT SUMMANDS ARE FI-EXTENDING

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Abstract. A module M is called FI-extending if every fully invariant submodule of M is essential in a direct summand of M. It is not known whether a direct summand of an FI-extending module is also FI-extending. In this study, it is given some answers to the question that under what conditions a direct summand of an FI-extending module is an FI-extending module?
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MSC(2010): Primary: 16D50; Secondary: 16D80, 16D70.

1. Introduction

Throughout this paper all rings are associative with unity and R always denotes such a ring. Modules are unital and for an Abelian group M, we use M_R to denote a right R-module. Recall that a submodule K of an R-module M is called fully invariant if $\varphi(K) \leq K$ for every R-endomorphism φ of M. A module M is called FI-extending if every fully invariant submodule of M is essential in a direct summand of M. FI-extending modules were introduced in [1] and further studied in [2] and [7]. It is not known whether a direct summand of an FI-extending module is also FI-extending (see, [1]). In this paper, we supply certain conditions which guaranties that a direct summand of an FI-extending module is an FI-extending module.

2. Direct summands of FI-extending modules

Lemma 2.1 ([7, Lemma 2.1])**.** *A module M is FI-extending if and only if for any fully invariant submodule A of M, there exists a direct summand K of M such that $A \cap K = 0$ and $A \oplus K$ is essential in M.*

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Lemma 2.2 ([1, Theorem 1.3]). *Let $M = \bigoplus_{i\in I} X_i$. If each X_i is an FI-extending module then M is an FI-extending module.*

Lemma 2.3 ([5, Lemma 6]). *Let N be a submodule of a module M such that N has a unique closure K in M. Then K is the sum of all submodules L of M containing N such that N is essential in L.*

The next few results concern a left exact preradical r in the category of right modules over a ring R. For the definition and basic properties of left exact radicals, refer to [6]. In particular, we shall need the following properties of a left exact preradical r for a ring R:

(i) $r(M)$ is a submodule of M for every right R-module M;
(ii) $r(M_1 \oplus M_2) = r(M_1) \oplus r(M_2)$ for all right R-modules M_1, M_2;
(iii) $r(N) = N \cap r(M)$ for every submodule N of a right R-module M;
(iv) $\varphi(r(M)) \subseteq r(M^{'})$ for every homomorphism $\varphi : M \longrightarrow M^{'}$ for right R-modules M, $M^{'}$.

We first prove an easy lemma.

Lemma 2.4. *Let R be a ring, r a left exact preradical in the category of right R-modules, and M a right R-module which is FI-extending. Then $M = M_1 \oplus M_2$ such that $r(M_1)$ is essential in M_1 and $r(M_2) = 0$.*

Proof. By Lemma 2.1, there exists submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$, $r(M) \cap M_2 = 0$ and $r(M) \oplus M_2$ is essential in M. Since r is left exact, it follows that $r(M_2) = M_2 \cap r(M) = 0$. Let $\pi : M \longrightarrow M_1$ denote the canonical projection. Then $\pi(r(M)) \subseteq r(M_1)$. For any $0 \neq m \in M_1$, there exists $t \in R$ such that $0 \neq mt \in r(M) \oplus M_2$, and hence, $0 \neq mt = \pi(mt) \in \pi(r(M)) \subseteq r(M_1)$. It follows that $r(M_1)$ is essential in M_1. □

Theorem 2.5. *Let R be a ring, r a left exact preradical for the category of right R-modules, and M a right R-module such that $r(M)$ has a unique closure in M. Then M is an FI-extending module if and only if $M = M_1 \oplus M_2$ is a direct sum of FI-extending modules M_1 and M_2 such that $r(M_1)$ is essential in M_1 and $r(M_2) = 0$.*

Proof. The sufficiency follows from Lemma 2.2. Conversely, suppose M is an FI-extending module. By Lemma 2.4, $M = M_1 \oplus M_2$ such that $r(M_1)$ is essential in M_1 and $r(M_2) = 0$. Note that $r(M) = r(M_1) \oplus r(M_2) = r(M_1)$, so M_1 is the (unique) closure of $r(M)$ in M. Let $\pi_i : M \to M_i$ $(i = 1, 2)$ denote the canonical projections.

Let N be any fully invariant submodule of M_1. By Lemma 2.1, there exist submodules K, $K^{'}$ of M such that $M = K \oplus K^{'}$, $(N \oplus M_2) \cap K = 0$ and $N \oplus M_2 \oplus K$ is essential in M. Since $K \cap M_2 = 0$, it follows that $K \cong \pi_1(K)$. Note that, because r is left exact, $r(\pi_1(K)) = \pi_1(K) \cap r(M_1)$ is essential in $\pi_1(K)$. Hence $r(K)$ is essential in K and, in addition, $r(M) = r(K) \oplus r(K^{'})$

is essential in $K \oplus r(K^{'})$. By Lemma 2.3, $K \oplus r(K^{'}) \subseteq M_1$ and, in particular, $K \subseteq M_1$. Now $M_1 = K \oplus (M_1 \cap K^{'})$, and $N \oplus K = (N \oplus M_2 \oplus K) \cap M_1$ is essential in M_1. By Lemma 2.1, M_1 is an FI-extending module.

Next, let H be any fully invariant submodule of M_2. By Lemma 2.1, there exist submodules L, $L^{'}$ of M such that $M = L \oplus L^{'}$, $(H \oplus M_1) \cap L = 0$, and $H \oplus M_1 \oplus L$ is essential in M. Note that $r(M) \subseteq M_1$ gives that $r(L) = L \cap r(M) \subseteq L \cap M_1 = 0$, and hence, $r(M) = r(L) \oplus r(L^{'}) = r(L^{'}) \subseteq L^{'}$. Let $L^{''}$ be a closure of $r(M)$ in $L^{'}$. Since $L^{'}$ is a direct summand of M, it follows that $L^{''}$ is a closure of $r(M)$ in M (see, [3, p. 6]), and hence, $M_1 = L^{''} \subseteq L^{'}$. Now $L^{'} = M_1 \oplus (L^{'} \cap M_2)$ and

$$M = L \oplus L^{'} = L \oplus M_1 \oplus (L^{'} \cap M_2) = \pi_2(L) \oplus M_1 \oplus (L^{'} \cap M_2).$$

We deduce that $\pi_2(L)$ is a direct summand of M_2 and

$$\pi_2(L) \oplus H = (\pi_2(L) \oplus M_1 \oplus H) \cap M_2 = (L \oplus M_1 \oplus H) \cap M_2$$

which is essential in M_2. By Lemma 2.1, M_2 is an FI-extending module. □

Before giving another case when a direct summand of an FI-extending module is an FI-extending module, we first prove the next lemma.

Lemma 2.6. *Let $M = M_1 \oplus M_2$. M_1 is FI-extending module if and only if for any fully invariant submodule N of M_1, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap N = 0$, and $K \oplus N$ is essential in M.*

Proof. Assume that M_1 is FI-extending module. Let N be a fully invariant submodule of M_1. By Lemma 2.1, there exists a direct summand L of M_1 such that $N \cap L = 0$ and $N \oplus L$ is essential in M_1. Then $(L \oplus M_2) \cap N = 0$ and $(L \oplus M_2) \oplus N$ is essential in M. Conversely, suppose M_1 has the stated propery. Let H be a fully invariant submodule of M_1. By hypothesis, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap H = 0$ and $K \oplus H$ is essential in M. Now, $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2$ so that $K \cap M_1$ is a direct summand of M, and hence also of M_1, $H \cap (K \cap M_1) = 0$, and $H \oplus (K \cap M_1) = M_1 \cap (H \oplus K)$ which is essential in M_1. By Lemma 2.1, M_1 is FI-extending. □

Theorem 2.7. *Let $M = M_1 \oplus M_2$ be an FI-extending module such that for every direct summand K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M. Then M_1 is an FI-extending module.*

Proof. Let N be any fully invariant submodule of M_1. By hypothesis, there exists a direct summand K of M such that $(N \oplus M_2) \cap K = 0$ and $N \oplus M_2 \oplus K$ is essential in M by Lemma 2.1. Moreover $M_2 \oplus K$ is a direct summand of M. The result follows by Lemma 2.6. □

Corollary 2.8. *Let M be an FI-extending module and K is a direct summand of M such that M/K is K-injective. Then K is an FI-extending module.*

Proof. There exists a submodule $K^{'}$ of M such that $M = K \oplus K^{'}$ and, by hypothesis $K^{'}$ is K-injective. Let L be a direct summand of M such that $L \cap K^{'} = 0$. By [3, Lemma 7.5], there exists a submodule of H of M such that $H \cap K^{'} = 0$, $M = H \oplus K^{'}$, and $L \subseteq H$. Now L is a direct summand of H, and hence, $L \oplus K^{'}$ is a direct summand of $M = H \oplus K^{'}$. By Theorem 2.7, K is FI-extending. □

Corollary 2.9. *Let $M = M_1 \oplus M_2$ be a direct sum of a submodule M_1 and an injective submodule M_2. Then M is FI-extending module if and only if M_1 is FI-extending module.*

Proof. If M is FI-extending, then M_1 is FI-extending by Corollary 2.8. Conversely, If M_1 is FI-extending, then M is FI-extending by Lemma 2.2. □

Recall that, a module M satisfies property (C_3) if for any direct summands A and B with $A \cap B = 0$ then $A \oplus B$ is a direct summand of M.

Corollary 2.10. *Every direct summmand of an FI-extending module with (C_3) is again FI-extending.*

Proof. Assume that M is an FI-extending module and satisfies property (C_3). Let M_1 be a direct summand of M. There exists a submodule M_2 of M such that $M = M_1 \oplus M_2$. Let $\pi : M \to M_1$ denote the canonical projection. Let K be any fully invariant submodule of M_1. There exists a direct summand L of M such that $(K \oplus M_2) \cap L = 0$ and $K \oplus M_2 \oplus L$ is essential in M. Because M satisfies property (C_3), $M_2 \oplus L$ is a direct summand of M. Note that $M_2 \oplus L = M_2 \oplus \pi(L)$, and, hence $\pi(L)$ is a direct summand of M_1. Morever, $K \oplus M_2 \oplus L = K \oplus \pi(L) \oplus M_2$ being essential in M implies $K \oplus \pi(L)$ is essential in M_1. It follows that M_1 is an FI-extending module. □

Recall that, a module M is said to be SIP-extending if the intersection of every pair of direct summands is essential in a direct summand of M. It is known that, SIP-extending modules are proper generalization both SIP-modules and extending modules (see, [4]).

Wang and Chen proved in [7, Theorem 3.1] that every direct summand of an FI-extending module with SIP is also FI-extending. Next result generalizes [7, Theorem 3.1].

Proposition 2.11. *Let M be an FI-extending module. If M is SIP-extending then every direct summand of M is FI-extending.*

Proof. Assume that M is an FI-extending and SIP-extending module. Let M_1 be a direct summand of M. There exists a submodule M_2 of M such that $M = M_1 \oplus M_2$. Let N_1 be any fully invariant submodule of M_1. By proof of [7, Theorem 3.1], there exists a fully invariant submodule N_2 of M_2 such that $N_1 \oplus N_2$ is a fully invariant submodule of M. Since M is an FI-extending module, there exists a direct summand N of M such that $N_1 \oplus N_2$ is essential in N. Also $N_1 \oplus N_2$ is essential in $(M_1 \cap N) \oplus (M_2 \cap N)$. This implies that N_1 is essential in $M_1 \cap N$. Since M is SIP-extending, there exists a direct summand T of M such that $M_1 \cap N$ is essential in T. Since $M_1 \cap N$ is essential in T and $M_1 \cap N$ is a submodule of M_2, it is easy to see that T is a submodule of M_1. Thus, T is a direct summand of M_1. So, M_1 is an FI-extending module. □

Proposition 2.12. *Let $M = U \oplus V$ be a direct sum of uniform modules U and V. Then every direct summand of M is FI-extending.*

Proof. Let K be a non-zero direct summand of M. If $K = M$ then K is FI-extending by Lemma 2.2. If $K \neq M$ then K is uniform and hence K is FI-extending. □

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