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THE w-FF PROPERTY IN TRIVIAL EXTENSIONS

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Abstract. Let D be an integral domain with quotient field K and E a K-vector space. Let $R = D \propto E$ be the trivial extension of D by E, and w the so-called w-operation. In this paper, we show that R is a w-FF ring if and only if D is a w-FF domain; and in this case, each w-flat w-ideal of R is w-invertible.

Keywords: w-flat module, w-FF ring, trivial extension.
MSC(2010): Primary: 13A15; Secondary: 13C11, 13E15.

1. Introduction: motivations and results

Let R be a commutative ring with identity. Finitely generated flat ideals were first studied by Sally and Vasconcelos in [13]. This paper contains a number of interesting probings into what makes a flat ideal of the polynomial ring $R[X]$ finitely generated. In [5], El Baghdadi *et al.* called a ring in which every flat ideal is finitely generated an FF ring and continued to investigate the stability of FF rings under localization and homomorphic image, and their transfer to various contexts of constructions such as direct products, pullback rings, and trivial extensions. It is well known that a nonzero finitely generated ideal of an integral domain is flat if and only if it is invertible; so an FF-domain is an integral domain in which each flat ideal is invertible. For example, Krull domains are FF domains [20, Corollary 3]. In [9], the authors introduced the notion of a w-FF domain as follows: An integral domain R is a *w-FF domain* if every nonzero w-flat w-ideal of R is w-invertible. It is known that w-FF-domains are FF domains [9, Proposition 3(1)]. Also, it is known that a w-finite type w-ideal of an integral domain is w-flat if and only if it is w-invertible [9, Lemma 1]. Therefore, R is a w-FF domain if and only if every w-flat w-ideal of R is of w-finite type. Recall that D is a Krull domain if and

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only if each nonzero ideal of D is w-invertible; so every Krull domain is a w-FF domain.

Let D be an integral domain with quotient field K, E a K-vector space, and $R = D \propto E$ the trivial extension of D by E. In [5, Theorem 3.11], the authors showed that $D \propto E$ is an FF ring if and only if D is an FF-domain. They also proved that if $(A, \mathfrak{m})$ is a local FF ring and if E' is an A-module with $\mathfrak{m}E' = 0$, then $A \propto E'$ is an FF ring [5, Theorem 3.14]. In this paper, we extend w-FF property to a commutative ring with zero divisors. Recall that a w-ideal has w-FF property if it is w-flat and of w-finite type. Precisely, we show that $R = D \propto E$ is a w-FF ring if and only if D is a w-FF domain; and in this case, each w-flat w-ideal of R is w-invertible. To do this, we completely characterize the ideals of R that are not contained in $(0) \propto E$. We also prove that if $(A, \mathfrak{m})$ is a w-local FF ring and E' is an A-module with $\mathfrak{m}E' = 0$, then $A \propto E'$ is a w-FF ring.

2. The w-operation and the trivial extension

Let R be a commutative ring with identity, $T(R)$ the total quotient ring of R and M a unitary R-module. An element of R is said to be *regular* if it is not a zero divisor. An ideal is *regular* if it contains a regular element. For a nonzero fractional ideal I of R, define $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$ and $I_v = (I^{-1})^{-1}$. A finitely generated ideal J of R is a *Glaz-Vasconcelos ideal* (*GV-ideal*) if the natural homomorphism $R \to \operatorname{Hom}_R(J, R)$ is an isomorphism. It is known that if R is an integral domain, then a nonzero finitely generated ideal J of R is a GV-ideal if and only if $J^{-1} = R$.

Let M be an R-module. Define the w-envelope of M as

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some GV-ideal } J \text{ of } R\},$$

where $E(M)$ denotes the injective envelope (or injective hull) of M (see [7]). Then M_w is independent of $E(M)$, up to isomorphism. We say that M is *GV-torsion* if for each $x \in M$, there exists a GV-ideal J of R such that $Jx = 0$; and M is *GV-torsion-free* (or *co-semi-divisorial*) if whenever $Jx = 0$ for any GV-ideal J of R and $x \in M$, we have $x = 0$. It is proved, [19, Theorem 1.4], that M is GV-torsion-free if and only if $\operatorname{Hom}_R(N, M) = 0$ for any GV-ideal J of R and any R/J-module N. Recall that M is a *w-module* if $M_w = M$. An ideal I of R is called a *w-ideal* if I is a w-module as an R-module and an ideal P of R is called a *maximal w-ideal* of R if P is maximal among the proper w-ideals of R. An ideal I of R is *w-invertible* if $(II^{-1})_w = R$. We call M a *w-flat module* if M_P is a flat R_P-module for every maximal w-ideal P of R. We call M a *w-faithfully flat module* if M is w-flat and $(M/PM)_w \neq 0$ for all maximal w-ideals P of R. It was shown in [8, Proposition 2.5] that M is w-faithfully flat if and only if M_P is faithfully flat for all maximal w-ideals P of R. Thus, faithfully flat modules are w-faithfully flat. A module homomorphism $g : A \to B$ is a

w-*epimorphism* if $g_P : A_P \to B_P$ is an epimorphism for all maximal w-ideals P of R; M is *of w-finite type* if there exist a finitely generated free module F and a w-epimorphism $g : F \to M$. It is clear that if M is of w-finite type, then M_P is finitely generated for all maximal w-ideals P of R. We say that R is a *w-FF ring* if each w-flat w-ideal of R is of w-finite type. Clearly, an integral domain that is a w-FF ring is a w-FF domain.

Example 2.1. (1) An FF ring whose maximal ideals are w-ideals is a w-FF ring. Indeed, this follows from the fact that a commutative ring whose maximal ideals are w-ideals is a DW ring, i.e., every ideal is a w-ideal [14, Theorem 3.8].

(2) In [17], Wang and McCasland introduced and studied strong Mori domains, that is, integral domains which satisfy the ascending chain condition on integral w-ideals. This concept was generalized to the semistar operation context [12] and the setup of (commutative) rings with zero-divisors [19, Definition 4.1]. In the latter case, it is called *w-Noetherian.* It is known that R is w-Noetherian if and only if each w-ideal of R is of w-finite type [19, Proposition 4.3]. Clearly, Noetherian rings are w-Noetherian rings, and w-Noetherian rings are w-FF rings.

The *trivial extension* of R by M (also called the *idealization* of M over R) is a commutative ring $R \propto M$ with identity (even an R-algebra) whose underlying group is $R \oplus M$, and the multiplication is defined by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. It is well known that the prime (respectively, maximal) ideals of $R \propto M$ have the form $I \propto M$, where I is a prime (respectively, maximal) ideal of R [6, Theorem 25.1] (or [2, Theorem 3.2]); if I is a finitely generated ideal of R and M is a finitely generated R-module, then $I \propto M$ is a finitely generated ideal of $R \propto M$ [1, Theorem 9(1)]. Denote by $T(R)$ the total ring of fractions of R. Then by [6, Theorem 25.10], for an ideal I of R, we have $(I \propto M)^{-1} = I^{-1} \propto M$ if M is a $T(R)$-module.

Now let D be an integral domain, and R the trivial extension of D with respect to a vector space E over the quotient field K of D. Then an element $(a, e) \in R = D \propto E$ is regular if and only if $a \neq 0$ [6, Theorem 25.3].

Proposition 2.2. *Let D be an integral domain with quotient field K, E a vector space over K, and $R = D \propto E$ the trivial extension of D by E. Let J be an ideal of R such that $J \nsubseteq (0) \propto E$, and let $I = \{a \in D \mid (a, e) \in J$ for some $e \in E\}$. Then the following assertions hold.*

(1) *I is a nonzero ideal of D.*
(2) *$J = I \propto E = \langle\{(a, 0) \mid a \in I\}\rangle$.*
(3) *J is finitely generated if and only if I is finitely generated.*
(4) *$J \in \mathrm{GV}(R)$ if and only if $I \in \mathrm{GV}(D)$.*
(5) *$J_w = I_w \propto E$.*
(6) *J is of w-finite type if and only if I is of w-finite type.*

(7) *J is invertible* (*respectively, w-invertible*) *if and only if I is invertible* (*respectively, w-invertible*).

Proof. (1) This is clear.

(2) Let $(x, e) \in J$. Then $x \in I$. If $x \neq 0$, then $(x, e) = (x, 0) \cdot (1, \frac{1}{x}e) \in \langle\{(a, 0) \mid a \in I\}\rangle$. Next, if $x = 0$, then $(0, e) = (a', 0) \cdot (0, \frac{1}{a'}e) \in \langle\{(a, 0) \mid a \in I\}\rangle$ for every $0 \neq a' \in I$. Hence $J \subseteq \langle\{(a, 0) \mid a \in I\}\rangle$. Clearly, we have $\langle\{(a, 0) \mid a \in I\}\rangle \subseteq I \propto E$. For $I \propto E \subseteq J$, let $(b, \alpha) \in I \propto E$. If $b \neq 0$, then there is $(b, \beta) \in J$ for some $\beta \in E$. Hence $(b, \alpha) = (b, \beta) \cdot (1, \frac{1}{b}(\alpha - \beta)) \in J$. Assume $b = 0$. Then $(0, \alpha) = (c, \delta) \cdot (0, \frac{1}{c}\alpha) \in J$, where $(c, \delta) \in J$ with $c \neq 0$. Therefore $I \propto E \subseteq J$.

(3) This follows directly from (2).

(4) Note that if $0 \neq a \in D$, then (a, e) is regular in R for every $e \in E$. Thus by (2), J is regular. Hence $J \in \mathrm{GV}(R)$ if and only if $J_w = R$ and J is finitely generated; if and only if $J_v = R$ and J is finitely generated [19, Theorem 3.12]; if and only if $I_v = D$ and I is finitely generated (by (3) because $(I \propto E)_v = I_v \propto E$); if and only if $I \in \mathrm{GV}(D)$.

(5) Note that every GV-ideal J' of R is regular; so $J' \nsubseteq (0) \propto E$. Hence $(x, e) \in J_w$ if and only if $(x, e) \cdot (A \propto E) \subseteq I \propto E$ for some $A \in \mathrm{GV}(D)$ by (4); if and only if $xA \subseteq I$; if and only if $x \in I_w$; if and only if $(x, e) \in I_w \propto E$.

(6) This follows directly from (3) and (5).

(7) Note that $J^{-1} = (I \propto E)^{-1} = I^{-1} \propto E$; hence $JJ^{-1} = II^{-1} \propto E$. Thus, $JJ^{-1} = R$ (respectively, $(JJ^{-1})_w = R$) if and only if $II^{-1} = D$ (respectively, $(II^{-1})_w = D$). □

Let w-Spec(R) be the set of all prime w-ideals of R. The *w-dimension* of R, denoted by w-dim(R), is the supremum of the heights of maximal w-ideals of R. The next result is an immediate consequence of Proposition 2.2(5).

Corollary 2.3. *Let the notation be as in Proposition 2.2. Then there exists an order-preserving bijection from w-Spec(D) into w-Spec(R). Hence, w-dim$(D) = w$-dim(R).*

For more on the trivial extension, the readers can refer to Huckaba's book [6] and Anderson–Winders' interesting article [2].

3. The w-FF property in trivial extensions

Let R be a commutative ring with identity and let $R[X]$ be the polynomial ring over R. For $f \in R[X]$, let $c(f)$ be the ideal of R generated by the coefficients of f. Let $N_w = \{f \in R[X] \mid c(f)_w = R\}$. Let $f \in N_w$, and assume that $fg = 0$ for $g \in R[X]$. By [16, Theorem 1.7.16], there is an integer $m \geq 1$ such that $c(f)^{m+1}c(g) = c(f)^m c(fg)$. Hence, by [18, Lemma 2.9],

$$\begin{aligned} c(g) \subseteq c(g)_w &= ((c(f)^{m+1})_w c(g))_w = (c(f)^{m+1}c(g))_w \\ &= (c(f)^m c(fg))_w = ((c(f)^m)_w c(fg))_w \\ &= c(fg)_w = \{0\}, \end{aligned}$$

and thus $g = 0$. Therefore, every element of N_w is regular and N_w is a (saturated) multiplicative subset of $R[X]$. The quotient ring $R[X]_{N_w}$ is called the *w-Nagata ring* of R. Let M be an R-module. Then $M[X]_{N_w}$ is an $R[X]_{N_w}$-module and is called the *w-Nagata module* of M. In [15, Proposition 3.9(3)], it is shown that M is of w-finite type if and only if $M[X]_{N_w}$ is finitely generated.

Let $D \to R$ be a faithfully flat ring homomorphism and let M be a D-module. It is known that if the R-module $M \otimes_D R$ is finitely generated, then M is finitely generated [11, Exercise 7.3]. The following is the w-operation analogue of this result.

Lemma 3.1. *Let $D \to R$ be a w-faithfully flat ring homomorphism and let M be a D-module. If the R-module $M \otimes_D R$ is of w-finite type, then M is of w-finite type.*

Proof. Let X be an indeterminate and $N_w = \{f \in D[X] \mid c(f)_w = D\}$. Since the argument in the proof of [4, Theorem 1.7] works equally well for D being a commutative ring with identity, $R[X]_{N_w}$ is faithfully flat over $D[X]_{N_w}$. By [15, Proposition 3.9(3)], $M[X]_{N_w} \otimes_{D[X]_{N_w}} R[X]_{N_w}$ is a finitely generated $R[X]_{N_w}$-module. Thus, $M[X]_{N_w}$ is a finitely generated $D[X]_{N_w}$-module [11, Exercise 7.3]. Hence, again by [15, Proposition 3.9(3)], M is of w-finite type, □

Lemma 3.2. *Let $D \subseteq R$ be an extension of commutative rings such that R is a w-faithfully flat D-module. If R is a w-FF ring, then D is a w-FF ring*

Proof. Let I be a w-flat w-ideal of D. Then $I \otimes_D R \cong IR$ is a w-flat w-ideal of R [4, Lemma 1.5]. Since R is a w-FF ring, IR is of w-finite type. Since R is a w-faithfully flat D-module, I is of w-finite type by Lemma 3.1. Thus, D is a w-FF ring. □

Corollary 3.3. *Let $R[X]$ be the polynomial ring over R. If $R[X]$ is a w-FF ring, then R is a w-FF ring.*

Proof. This follows directly from Lemma 3.2 because $R[X]$ is a faithfully (hence w-faithfully) flat R-module. □

Let M be a unitary R-module, and let $\varphi_1 : R \to R \propto M$ and $\varphi_2 : R \propto M \to R$ be two ring homomorphisms defined by $\varphi_1(r) = (r, 0)$ and $\varphi_2(r, m) = r$, respectively. Then φ_1 and φ_2 induce functors ${}_{R\propto M}\mathcal{M} \to {}_R\mathcal{M}$ and ${}_R\mathcal{M} \to {}_{R\propto M}\mathcal{M}$, where ${}_{R\propto M}\mathcal{M}$ (respectively, ${}_R\mathcal{M}$) denotes the category of $(R \propto M)$-modules (respectively, R-modules) and the respective "scalar products" are

$ra := (r, 0)a$ and $(r, m)a := ra$. Note that the map ${}_R\mathcal{M} \to {}_{R\propto M}\mathcal{M} \to {}_R\mathcal{M}$ is the identity map [2, p. 24].

From now on, unless otherwise stated, D always denotes an integral domain with quotient field K, E stands for a K-vector space, $R = D \propto E$ is the trivial extension of D by E, and $T = K \propto E$ is the trivial extension of K by E. It is well-known (special case of [2, Corollary 3.4]) that the ideals of R are either of the form $I \propto E$ for an ideal I of D or $0 \propto F$ where F is a D-submodule of E. Note that the ring R may be defined as a pullback construction [5], as shown by the following pullback diagram:

$$\begin{array}{ccc} R = D \propto E & \longrightarrow & D = R/(0 \propto E) \\ {\scriptstyle i}\big\downarrow & & {\scriptstyle j}\big\downarrow \\ T = K \propto E & \xrightarrow{\varphi} & K = T/(0 \propto E). \end{array} \tag{3.1}$$

We next give the main result of this paper, which is the w-operation analogue of [5, Theorem 3.11] that $R = D \propto E$ is an FF ring if and only if D is an FF-domain.

Theorem 3.4. *Given a pullback diagram (3.1), R is a w-FF ring if and only if D is a w-FF domain. In this case, each w-flat w-ideal of R is w-invertible.*

To prove this theorem, we first need several lemmas. Let $R \subseteq T$ be an extension of commutative rings. Following [18, Definition 3.1], we say that T is *w-linked over R* (or $R \subseteq T$ is a *w-linked extension*) if T as an R-module is a w-module. Clearly, flat $\Rightarrow$ w-flat $\Rightarrow$ w-linked.

Lemma 3.5. *With the same notation as in (3.1), the following assertions hold.*

(1) *T is a flat R-module, and hence $R \subseteq T$ is a w-linked extension.*
(2) *If J is w-flat over R, then $J \otimes_R T$ is w-flat over T.*
(3) *Every ideal of T is a w-ideal.*

Proof. (1) This is an immediate consequence of the fact that $T = R_{D\setminus\{0\}}$ (see the proof of [5, Theorem 3.11]).

(2) This follows from the fact that the argument in the proof of [4, Lemma 1.5] works equally well for D being a commutative ring with identity.

(3) Note that T is zero-dimensional. Thus, every ideal of T is a w-ideal [14, Theorem 3.8]. ☐

Our next lemma shows that the GV-torsion property descends under a certain w-linked extension. Note that $D \subseteq R$ is a w-linked extension (Proposition 2.2(5)).

Lemma 3.6. *If an R-module M is GV-torsion, then M as a D-module is GV-torsion.*

Proof. By [21, Lemma 1], it suffices to show that if a D-module N is GV-torsion-free, then $\mathrm{Hom}_D(R, N)$ is GV-torsion-free as an R-module. Let J be a GV-ideal of R. Then J is finitely generated, say

$$J = \langle \{(d_i, e_i) \mid d_i \in D, e_i \in E \text{ and } i = 1, \ldots, n\} \rangle.$$

Let $I = \langle d_1, \ldots, d_n \rangle$. Then I is a GV-ideal of D by Proposition 2.2(4). If $Jf = 0$ for $f \in \mathrm{Hom}_D(R, N)$, then for any $r \in R$, $If(r) \subseteq Jf(r) = 0$. Since N is GV-torsion-free as a D-module, we have $f(r) = 0$. Hence f is identically zero. Thus $\mathrm{Hom}_D(R, N)$ is GV-torsion-free as an R-module. □

The following lemma appears in [10, Theorem 3.3].

Lemma 3.7. *Let R be any commutative ring with identity and let M be an R-module. Then M is w-flat if and only if $\mathrm{Tor}_1^R(N, M)$ is GV-torsion for any R-module N.*

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. ($\Rightarrow$) Since $D \subseteq R$ is a faithfully flat extension [5], this follows from Lemma 3.2.

($\Leftarrow$) Assume that D is a w-FF domain, and let J be a nonzero w-flat w-ideal of R. Then $JT(= J \otimes_R T)$ is a nonzero flat ideal of T by Lemma 3.5 and Example 2.1(1), and hence $JT = T = K \propto E$ [5, Lemma 3.12]. Thus, there exists an $(a, e) \in J$ with $a \neq 0$, and hence $J = I \propto E$ for some nonzero w-ideal I of D by Proposition 2.2(2) and (5).

Claim: I is w-flat over D. Recall that for any D-module N,

$$\mathrm{Tor}_1^D(I, N \otimes_D R) \cong \mathrm{Tor}_1^R(I \otimes_D R, N \otimes_D R)$$

as abelian groups [3, Proposition 4.1.1, Section VI]; so $\mathrm{Tor}_1^R(I \otimes_D R, N \otimes_D R)$ is GV-torsion by Lemma 3.7, because $I \otimes_D R$ is w-flat over R. On the other hand, N is a direct summand of $N \otimes_D R$ since D is a direct summand of R. Therefore by Lemma 3.6, $\mathrm{Tor}_1^D(I, N)$ is GV-torsion. Thus again by Lemma 3.7, I is w-flat over D.

Under the present hypotheses, by Claim, we have that I is of w-finite type because D is a w-FF domain. Therefore by Proposition 2.2(6), J is of w-finite type over R.

For "In this case" part of the statement, recall from [9] that a w-flat w-ideal of D is of w-finite type if and only if it is w-invertible. Thus, a nonzero w-flat w-ideal of R is w-invertible by Proposition 2.2(7). □

Corollary 3.8. *Let K be a field, $K[X]$ the polynomial ring over K, $K[X]/X^2K[X]$ a quotient ring, x the image of X under the canonical map $K[X] \to K[X]/X^2K[X]$, and $R = D + xK[x]$. Then R is a w-FF ring if and only if D is a w-FF domain.*

Proof. Clearly, R is a subring of $K[x]$ and $xK[x]$ is a K-vector space. Also, $R \cong D \propto xK[x]$ since $x^2 = 0$. Thus, the result follows directly from Theorem 3.4. $\square$

It is clear that a nonzero finitely generated ideal is of w-finite type; hence Noetherian rings are w-FF rings. We next give examples of w-FF rings that are not Noetherian.

Example 3.9. (1) Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ the field of rational numbers, n a positive integer, and $R = \mathbb{Z} \propto (\mathbb{Q}[X]/X^n\mathbb{Q}[X])$. Then R is a w-FF ring by Theorem 3.4, but since $\mathbb{Q}[X]/X^n\mathbb{Q}[X]$ is not a finitely generated $\mathbb{Z}$-module, R is not Noetherian [2, Theorem 4.8]

(2) Let D be a Krull domain of dimension ≥ 2, and $R = D + xK[x]$ as in Corollary 3.8. Then R is a w-FF ring by Corollary 3.8. However, note that $D \neq K$; so K is not a finitely generated D-module. Thus, R is not a Noetherian ring.

Recall that a commutative ring is *w-local* if it is local and the unique maximal ideal is a w-ideal. The following result is the w-operation analogue of [5, Theorem 3.14].

Theorem 3.10. *Let $(A, \mathfrak{m})$ be a w-local FF ring and $R = A \propto E$ the trivial extension of A by an A-module E such that $\mathfrak{m}E = 0$. Then R is a w-FF ring.*

Proof. Since $(A, \mathfrak{m})$ is w-local, by Corollary 2.3, $(R, \mathfrak{m} \propto E)$ is also w-local. Since A is a w-local FF ring, A is a local FF ring. Then by [5, Theorem 3.14], R is an FF ring. By Example 2.1(1), R is a w-FF ring. $\square$

Corollary 3.11. *Let $(A, \mathfrak{m})$ be a w-local Noetherian ring and $R = A \propto E$ the trivial extension of A by an A-module E such that $\mathfrak{m}E = 0$. Then R is a w-FF ring.*

Proof. This follows from Theorem 3.10 because Notherian rings are w-FF rings. $\square$

Recall that if a commutative ring R is zero-dimensional, then R is a DW ring [16, Corollary 6.3.13].

Example 3.12. Let D be a one-dimensional local domain with maximal ideal M such that $M^2 \neq M$, $A := D/M^2$, $\mathfrak{m} := M/M^2$, and $E := \mathfrak{m}$. Then $\mathfrak{m}$ is a w-ideal of A, and hence A is a w-local FF ring (cf. [13, Lemma 2.1]). Note that $\mathfrak{m}E = 0$. Thus, $R = A \propto E$ is a w-FF ring by Theorem 3.10. For a concrete example, let $D = \mathbb{Q} + X\mathbb{C}[[X]]$ (respectively, $D = \mathbb{R} + X\mathbb{C}[[X]]$) and $M = X\mathbb{C}[[X]]$, where $\mathbb{Q}$ (respectively, $\mathbb{R}, \mathbb{C}$) is the field of rational numbers (respectively, real numbers, complex numbers). Then R is a non-Noetherian (respectively, Noetherian) w-FF ring.

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