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LOCALIZATION AT PRIME IDEALS IN BOUNDED RINGS

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ABSTRACT. In this paper we investigate the sufficiency criteria which guarantee the classical localization of a bounded ring at its prime ideals.
Keywords: Localization, projective ideal, eventually idempotent ideal, bounded rings.
MSC(2010): Primary: 16A08; Secondary: 16P50.

1. Introduction

The question of localization in non-commutative rings was first considered by Ore in 1933 (see [10]). Since the process of localization is a powerful algebraic tool with which to study the structure of the ring, a great deal of work has been done by various authors since then (see for example [1, 3, 6, 8, 9, 11]). A particular contribution to the theory of localization at a prime ideal is due to Hajarnavis (see [8]). He proves that in a prime Noetherian PI ring a right invertible prime ideal is localizable. In fact, a right invertible ideal in a prime Noetherian PI ring is invertible. Thus it is non-eventually idempotent and projective on both sides. In this paper, we prove that in a prime Noetherian right bounded ring R, a non-eventually idempotent prime ideal P which is left projective is right localizable. This result both weakens the PI condition and generalizes the result of Hajarnavis to a larger set of prime ideals. To prove this result, the first observation we made is that in a result of Braun and Hajarnavis ([1, Proposition 3.7]), the PI condition is superfluous (see Proposition 3.1). We prove in Proposition 3.7 that if R is a bounded Noetherian prime ring, then any projective prime ideal P is (two-sided) localizable where localized ring R_P is a principal right and left ideal ring and P is a height-1 prime ideal of R. Moreover, we show that in a Noetherian prime ring R, if P is a projective prime ideal then $P^{(i)} = P^i$ is satisfied for all i, where $P^{(i)}$ denotes the i-th symbolic

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power of P. This proposition generalizes the result appeared in [3, Lemma 1.2] since an invertible ideal is projective.

2. **Preliminaries and notation**

Throughout this note R will denote a ring with nonzero identity.

Let R be an order in a simple Artinian ring Q. Let I be an ideal of R. Define $I^* = \{q \in Q : qI \subseteq R\}$ and $I^+ = \{q \in Q : Iq \subseteq R\}$. The ideal I is called *right invertible* if $II^+ = R$, *left invertible if* $I^*I = R$. If I is a non-zero ideal of R, then the dual basis Lemma [2, Proposition 3.1, p. 132] shows that I_R is projective if and only if $1 \in II^*$. Similarly $_RI$ is projective if and only if $1 \in I^+I$.

If A is an ideal of a ring R, then $\mathcal{C}(A)$ will denote the set

$$\{c \in R : [c + A] \text{ is a regular element of the ring } R/A\}.$$

Thus $\mathcal{C}(0)$ is the set of all regular elements of R.

A prime ideal P is said to have *height 1* if P does not properly contain a chain of two distinct prime ideals. Let P be a prime ideal of a Noetherian ring R. The symbolic powers $P^{(n)}$ of P are those described by Goldie, [6]. These have the property that

$$\mathcal{C}(P) = \mathcal{C}(P^{(n)}) \text{ for all } n \geq 1.$$

When R is a prime right Noetherian ring and a prime ideal P satisfies the right Ore condition with respect to $\mathcal{C}(P)$, we may form the right localization R_P which is a local ring with Jacobson radical PR_P. In this case we say that P is *right localizable*. Left localizability of a prime ideal is defined analogously. Note that, under two sided assumptions, the left localization coincides with the right localization. In this case we have $PR_P = R_PP$.

A ring R is called a *pri-(pli) ring* if every right (left) ideal of R is principal.

A ring R is called *right bounded* if each essential right ideal contains a nonzero two sided ideal which is essential as a right ideal. Left bounded rings can be defined analogously. A ring which is both right and left bounded is called a *bounded* ring.

An ideal I is called *eventually idempotent* if $I^n = I^{n+1}$ for some $n \geq 0$.

Conditions will be assumed to hold on both right and left, unless otherwise stated. For any unexplained terminology we refer the reader to [4] or [7].

3. **Localization in bounded rings**

Let R be a ring and let I be a (two-sided) ideal of R. If I_R is a projective module then we say that I is a *right projective ideal* of R. Left projectivity of an ideal is defined analogously. If I is an ideal of R such that both $_RI$ and I_R are projective modules, then we say that I is a *projective ideal* of R.

Proposition 3.1. *Let R be a prime Noetherian ring, and let I_R be a non-zero right projective ideal. Then exactly one of the following holds:*

(1) *I is eventually idempotent.*
(2) $\bigcap_i I^i = \{0\}$.

Proof. Suppose that $\bigcap_i I^i \neq \{0\}$. Let c be a regular element in this intersection. Since $c \in I^i$ we have $(I^i)^*c \subseteq (I^i)^*I^i \subseteq R$ and so $(I^i)^* \subseteq Rc^{-1}$ for all i. It is easily seen that $(I^*)^i \subseteq (I^i)^*$ and $\{(I^*)^i\}$ is an ascending chain of left R-modules. Since R is Noetherian, we have $(I^*)^k = (I^*)^{k+1}$ for some k. Therefore $(I^*)^{k+1}I^k \subseteq R$ and so $I^{k+1}(I^*)^{k+1}I^k \subseteq I^{k+1}$. Since I_R is projective we have $R \subseteq (I^{k+1})(I^*)^{k+1}$. Hence we have $I^k \subseteq I^{k+1}$ and consequently $I^k = I^{k+1}$. □

Lemma 3.2. *Let R be an order in a simple Artinian ring, and let I be a right projective ideal of R. Then the following statements hold:*

(1) *I^*I is idempotent.*
(2) *If J is another right projective ideal of R, then IJ is also a right projective ideal of R. In particular, I^k is a right projective ideal of R for all $k \geq 0$.*

Proof. (1) is given in [5, Lemma 1.1].

For (2) let J be a right projective ideal of R. Then $1 \in II^*$ and $1 \in JJ^*$. Also it is clear that $J^*I^* \subseteq (IJ)^*$. It follows that

$$1 \in II^* \subseteq I(JJ^*)I^* \subseteq IJ(IJ)^*,$$

i.e., IJ is a right projective ideal of R. □

Note that similar statements as in the above lemma can be given for left projective ideals. The only difference occurs in part (1) in which we need to consider the ideal II^+ instead of I^*I.

Proposition 3.3. *Let R be an order in a simple Artinian ring, and let P be a right projective prime ideal of R. If P is eventually idempotent, then either P is idempotent or $P = PP^+$.*

Furthermore if P is also left projective, then P is idempotent.

Proof. Assume that $P^k = P^{k+1}$ for some $k \geq 1$. Assume also that P is not idempotent. Then $k \geq 2$. Choose k as the smallest positive integer such that $P^k = P^{k+1}$. Since $\left(P^k\right)^* P^k \subseteq R$, we have $\left(P^k\right)^* P^k = \left(P^k\right)^* P^{k+2} \subseteq P^2$. Then

$$\left(P^{k-1}\right)^* P^{k-1} PP^+ \subseteq \left(P^k\right)^* P^k P^+ \subseteq P^2 P^+ \subseteq P.$$

Since $\left(P^{k-1}\right)^* P^{k-1} \subseteq R$, $PP^+ \subseteq R$, and P is prime, we have either $\left(P^{k-1}\right)^* P^{k-1} \subseteq P$ or $PP^+ \subseteq P$.

Suppose that $\left(P^{k-1}\right)^* P^{k-1} \subseteq P$. Since P is right projective, P^{k-1} is also right projective by Lemma 3.2, and so $\left(P^{k-1}\right)^* P^{k-1}$ is an idempotent ideal again by Lemma 3.2. This gives that $\left(P^{k-1}\right)^* P^{k-1} \subseteq P^{k-1}$, hence $\left(P^{k-1}\right)^* P^{k-1} = P^{k-1}$, which implies that P^{k-1} is idempotent, by Lemma 3.2. But since $k \geq 2$, $2k-2 \geq k$, and so $P^{k-1} = P^{2k-2} \subseteq P^k \subseteq P^{k-1}$. It follows that $P^{k-1} = P^k$, which contradicts with the choice of k.

Therefore we must have $PP^+ \subseteq P$, or equivalently $PP^+ = P$.

For the last statement of the proposition, suppose that P is also a left projective ideal of R. Then by (1) above, we see that either P is idempotent or $P = PP^+$. However PP^+ is an idempotent ideal by the left-handed version of Lemma 3.2. This completes the proof. □

Lemma 3.4. *Let R be a prime Noetherian ring, and let P be a right projective prime ideal of R. Then*

(1) *For every $n \geq 1$, $xc \in P^n$ for some $c \in \mathcal{C}(P)$ and $x \in R$ implies $x \in P^n$.*
(2) *If, further, P is not eventually idempotent, then $\mathcal{C}(P) \subseteq \mathcal{C}(0)$.*

Proof. To prove (1) we use induction on n. If $n = 1$, then clearly there is nothing to prove. Let $n > 1$ and assume that the first part of the lemma has been established for $n-1$. Let $xc \in P^n$ for some $x \in R$ and $c \in \mathcal{C}(P)$. Then clearly $x \in P$. Since $P^*x \subseteq R$ and $P^*xc \subseteq P^{n-1}$, we have $P^*x \subseteq P^{n-1}$ by our inductive hypothesis. Thus $x \in PP^*x \subseteq P^n$ because P_R is projective.

For (2), let $c \in \mathcal{C}(P)$. To see that $c \in \mathcal{C}(0)$, it is enough to show that the left annihilator of c is zero. Thus let $xc = 0$ for some $x \in R$. Then $x \in P^n$ for every n, by (1) above. Since P is not eventually idempotent, by Proposition 3.1, $\bigcap_{n\geq 1} P^n = 0$. Hence $x = 0$. This completes the proof. □

Note that the above lemma can also be given analogously for left projective prime ideals of a prime Noetherian ring.

Theorem 3.5. *Let R be a right bounded Noetherian prime ring. Let P be a non-eventually idempotent prime ideal such that ${}_RP$ is projective. Then P is right localizable. Furthermore, every non-idempotent projective prime ideal of a bounded Noetherian prime ring is localizable.*

Proof. We need to prove that the right Ore condition is satisfied with respect to $\mathcal{C}(P)$. Let $r, c \in R$ with $c \in \mathcal{C}(P)$. By Lemma 3.4, $c \in \mathcal{C}(0)$. Therefore cR is an essential right ideal of R. Since R is a right bounded ring, cR contains a nonzero ideal. Let B be the ideal of R which is maximal among all nonzero ideals contained in cR. In particular, $0 \neq B \subseteq cR$. Assume that $B \subseteq P$. Then $B \subseteq cP$ since $c \in \mathcal{C}(P)$. Thus $B \subseteq BP^+ \subseteq cR$, and hence $B = BP^+$ by maximality of B. Since P is left projective we have

$$B \subseteq BP^+P = BP \subseteq B,$$

which gives that $B = BP = BP^2 = \ldots$. Thus $B \subseteq \bigcap_n P^n$ which is a contradiction since $\bigcap_n P^n = \{0\}$ by Proposition 3.1. Therefore $B \nsubseteq P$, and so $B \cap \mathcal{C}(P) \neq \emptyset$, i.e. there exists $c' \in \mathcal{C}(P)$ such that $c' \in B$. Since $rB \subseteq B \subseteq cR$

we have $rc' = cr'$ for some $r' \in R$. It follows that the right Ore condition is satisfied with respect to $\mathcal{C}(P)$ and R is right localizable at P.

For the second statement of the theorem, assume that R is a bounded Noetherian prime ring and P is a non-idempotent projective prime ideal of R. By Proposition 3.3, P is not eventually idempotent. It follows from the proof of the first part that P is right localizable. Symmetric arguments show that P is also left localizable. This completes the proof. $\square$

The following example shows that the condition that P is not eventually idempotent in the above theorem is not superfluous.

Example 3.6. Let

$$R = \begin{pmatrix} k[x,y] & (x,y) \\ k[x,y] & k[x,y] \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} (x,y) & (x,y) \\ k[x,y] & k[x,y] \end{pmatrix},$$

where k is a field and x, y are variables. Note that R is a bounded Noetherian prime ring. It is not difficult to see that P^+ is equal to the ring of 2×2 matrices over $k[x,y]$ and that $1 \in P^+ = P^+P$. It follows that ${}_RP$ is projective. Also since $R/P \cong k$, P is a maximal (and hence a prime) ideal of R. However P is not right localizable. Indeed, if we take

$$\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then $\beta \in \mathcal{C}(P)$ and one cannot find elements $\beta' \in \mathcal{C}(P)$ and $\alpha' \in R$ such that $\alpha\beta' = \beta\alpha'$.

Theorem 3.7. *Let R be a bounded Noetherian prime ring. Then for a non-idempotent projective prime ideal P, the localized ring R_P is a pri-pli ring and P has height 1. Moreover, $P^{(i)} = P^i$ for all i.*

Proof. Note that R_P is a prime Noetherian ring and $J = PR_P = R_PP$ is its Jacobson radical. Since P is right projective, we have $R \subseteq PP^*$ and so $R_P \subseteq PP^*R_P \subseteq PR_PP^*R_P \subseteq PR_P(PR_P)^*$. Therefore $J = PR_P$ is a right projective ideal of R_P. Similarly it can be shown that J is also a left projective ideal of R_P. Now we prove that J is not idempotent. Suppose on the contrary that $J = J^2$. Then we have $PR_P = P^2R_P$ and so $PR_P \cap R = P^2R_P \cap R$.

Claim: $P^nR_P \cap R = P^n$ for all n.

Proof of the claim: Clearly $P^n \subseteq P^nR_P \cap R$ is satisfied. For the converse let $x \in P^nR_P \cap R$. Then $xc \in P^n$ for some $c \in \mathcal{C}(P)$. By Lemma 3.4(1), we have $x \in P^n$.

Now since $PR_P \cap R = P^2R_P \cap R$, we have $P = P^2$, which contradicts with the assumption that P is not idempotent. Thus J is not idempotent. Clearly $JJ^+ \subseteq R_P$ is satisfied. Since J is left projective, if $JJ^+ \subseteq J$ we get $J = J^2$, which is a contradiction since J is non-idempotent. Thus $JJ^+ = R_P$, which means that J is right invertible. Similarly it can be shown that J is also left invertible and so J is invertible. By [9, Proposition 1.3] it follows that R_P is a pri-pli ring.

Now since J is an invertible prime ideal, J has height 1 in R_P by the invertible ideal theorem [4, Theorem 3.4], and thus P has height 1 in R.

Now we will prove that $P^{(i)} = P^i$ for all i. In fact, this result is true for all projective prime ideals in a prime Noetherian ring. Note that we use the same notation and definition of symbolic powers as in [6]. For $i = 1$, $P^{(1)} = P$ by definition.

Let $i > 1$ and assume that $P^{(i-1)} = P^{i-1}$. Now take any $x \in P^{(i)}$. Then there exist $c, d \in \mathcal{C}(P)$ such that $cxd \in PP^{(i-1)} = P^i$ by assumption. Thus $x \in P^i$ by Lemma 3.4(1) and its left analogue. Conversely take any $x \in P^i$ and choose $G = F = R$. Then $GxF \in PP^{i-1} \subseteq PP^{(i-1)}$, and therefore $x \in \kappa(PP^{(i-1)}) = P^{(i)}$. □

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