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L^p BOUNDEDNESS OF THE BERGMAN PROJECTION ON SOME GENERALIZED HARTOGS TRIANGLES

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Abstract. In this paper we investigate two classes of domains in $\mathbb{C}^n$ generalizing the Hartogs triangle. We prove optimal estimates for the mapping properties of the Bergman projection on these domains.
Keywords: Hartogs triangle, Bergman projection, Bergman kernel.
MSC(2010): Primary: 32W05; Secondary: 32A25.

1. **Introduction**

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. The Bergman space $L^2_a(\Omega)$ is defined to be the intersection $L^2(\Omega) \cap \mathcal{O}(\Omega)$ of the space $L^2(\Omega)$ of square integrable functions on Ω (with respect to the Lebesgue measure of $\mathbb{C}^n$) with the space $\mathcal{O}(\Omega)$ of holomorphic functions on Ω. By the Bergman inequality, $L^2_a(\Omega)$ is a closed subspace of $L^2(\Omega)$. The orthogonal projection operator $\mathbf{P}\colon L^2(\Omega) \to L^2_a(\Omega)$ is the Bergman projection associated with the domain Ω. It follows from the Riesz representation theorem that the Bergman projection is an integral operator with the kernel $K_\Omega(z,w)$ on $\Omega \times \Omega$, i.e., $\mathbf{P}f(z) = \int_\Omega K_\Omega(z,w) f(w)\, dV(w)$ for all $f \in L^2(\Omega)$. The Bergman kernel depends on the choice of Ω and is also represented by

$$K_\Omega(z,w) = \sum_{j=0}^{\infty} \phi_j(z)\overline{\phi_j(w)}, \quad (z,w) \in \Omega \times \Omega,$$

where $\{\phi_j(\cdot)\colon j = 0,1,2,\dots\}$ is a complete orthonormal basis for $L^2_a(\Omega)$. If Ω is the Hermitian unit ball $\mathbb{B}_n$ defined by

$$\mathbb{B}_n = \{z \in \mathbb{C}^n\colon |z_1|^2 + \ldots + |z_n|^2 < 1\},$$

it is easy to see that z^α, $\alpha \in \mathbb{Z}^n_+$ form an orthogonal basis of $L^2_a(\mathbb{B}_n)$. A direct computation shows that $\|z^\alpha\| = \sqrt{\frac{\alpha!\pi^n}{(n+|\alpha|)!}}$. So the functions $\varphi_\alpha = \sqrt{\frac{(n+|\alpha|)!}{\alpha!\pi^n}} z^\alpha$,

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$\alpha \in \mathbb{Z}_+^n$, form an orthonormal basis of $L_a^2(\mathbb{B}_n)$. An easy computation gives:

$$K_{\mathbb{B}_n}(z,w) = \frac{n!}{\pi^n}\frac{1}{(1-\langle z,w\rangle)^{n+1}}, \tag{1.1}$$

where $\langle z,w\rangle := z_1\overline{w}_1 + \ldots + z_n\overline{w}_n$. Similarly as before, one can show that

$$K_{\mathbb{D}^n}(z,w) = \prod_{j=1}^n K_{\mathbb{D}}(z_j,w_j) = \frac{1}{\pi^n}\prod_{j=1}^n \frac{1}{(1-z_j\overline{w}_j)^2}, \tag{1.2}$$

where $\mathbb{D}^n$ is the Cartesian product of n copies of the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. One of the features that makes the Bergman kernel both important and useful is its invariance under biholomorphic mappings. This fact is useful in conformal mapping theory, and it also gives rise to the Bergman metric. The fundamental result is as follows.

Proposition 1.1. *Let Ω_1 and Ω_2 be domains in $\mathbb{C}^n$, and let $f\colon \Omega_1 \to \Omega_2$ be biholomorphic. Then*

$$\det J_{\mathbb{C}}f(z) K_{\Omega_2}(f(z),f(w)) \det \overline{J_{\mathbb{C}}f(w)} = K_{\Omega_1}(z,w).$$

Here $\det J_{\mathbb{C}}f$ is the complex Jacobian matrix of the mapping f (see [5], Section 1.1 for more on this topic).

It is natural to consider the mapping properties of $\mathbf{P}$ on other spaces of functions on Ω, for example the L^p spaces. The L^p mapping properties of the Bergman projection have been determined for some classes of domains, we refer to the following articles and the references therein [1, 2, 6–12, 15]. In [1], Chakrabarti and Zeytuncu considered the classical Hartogs triangle and proved that the Bergman projection is a bounded operator from L^p to L^p_a if and only if $4/3 < p < 4$. In recent paper [3], L.D. Edholm and J.D. McNeal generalized the result obtained in [1] to the domains defined by

$$\{(z_1,z_2) \in \mathbb{C}^2 \colon |z_1|^k < |z_2| < 1\},$$

for every $k \in \mathbb{N} := \{1,2,3,\ldots\}$. Motivated by their work, in this paper we generalize their result to the following domains

$$\Omega_k := \left\{(z,w) \in \mathbb{C}^{1+n} \colon |w_1| < |z|^k < 1, \ldots, |w_n| < |z|^k < 1\right\}$$
$$\mathcal{H}_k := \left\{(z,w) \in \mathbb{C}^{1+n} \colon |w_1|^2 + \cdots + |w_n|^2 < |z|^{2k} < 1\right\},$$

where again $k \in \mathbb{N}$. See [14] for the group of holomorphic automorphisms of $\mathcal{H}_k$ and proper holomorphic mappings between generalized Hartogs triangles.

1.1. **The Bergman kernel for Ω_k and $\mathcal{H}_k$.** Now we discuss the Bergman kernel for Ω_k and $\mathcal{H}_k$.

Theorem 1.2. *The Bergman kernel for Ω_k is given by*

$$K_{\Omega_k}((z,w),(x,y)) = \frac{\eta^{nk}}{\pi^{n+1}(1-\eta)^2\prod_{j=1}^n(\eta^k - \nu_j)^2},$$

where $\nu_j = w_j\overline{y}_j$ *for every* $j = 1, \dots, n$ *and* $\eta = z\overline{x}$.

Proof. Using the biholomorphism $\phi\colon \Omega_k \to \mathbb{D}^* \times \mathbb{D}^n$ defined by $\phi(z,w) := (z, \frac{w_1}{z^k}, \dots, \frac{w_n}{z^k})$, well-known formula (1.2) and Proposition 1.1 we can easily prove the desired result. □

Using the same function $\phi\colon \mathcal{H}_k \to \mathbb{D}^* \times \mathbb{B}_n$, formula (1.1) and the behavior of the Bergman kernel under biholomorphic mappings we have

Theorem 1.3. *The Bergman kernel for* $\mathcal{H}_k$ *is given by*

$$K_{\mathcal{H}_k}((z,w),(x,y)) = \frac{\eta^k}{\pi^{n+1}(1-\eta)^2(\eta^k - \nu_1 - \dots - \nu_n)^{n+1}}. \tag{1.3}$$

2. L^p boundedness of the Bergman projection

Before we give the main result of this work we prove the following lemma that will be used to prove the main theorem.

Lemma 2.1. *For every* $\epsilon \in \left[\frac{1}{2}, \frac{kn+2}{2kn}\right)$, *we have*

$$\int_{\mathcal{H}_k} |K_{\mathcal{H}_k}((z,w),(x,y))|\, h(x,y)^{-\epsilon}\, dV(x,y) \le Ch(z,w)^{-\epsilon} \tag{2.1}$$

$$\int_{\Omega_k} |K_{\Omega_k}((z,w),(x,y))|\, g(x,y)^{-\epsilon}\, dV(x,y) \le C_1 g(z,w)^{-\epsilon} \tag{2.2}$$

for some constants C *and* C_1, *where*

$$h(x,y) = (1-|x|^2)|x|^{2k(n-1)}(|x|^{2k} - |y_1|^2 - \dots - |y_n|^2)$$
$$g(x,y) = (1-|x|^2)(|x|^{2k} - |y_1|^2) \cdots (|x|^{2k} - |y_n|^2).$$

Proof. We start with inequality (2.1), by Theorem 1.3

$$\int_{\mathcal{H}_k} |K_{\mathcal{H}_k}((z,w),(x,y))|\, h(x,y)^{-\epsilon}\, dV(x,y)$$
$$= \int_{\mathcal{H}_k} \frac{|x|^{-2kn\epsilon}(1-|x|^2)^{-\epsilon}\left(1 - \left|\frac{y_1}{x^k}\right|^2 - \dots - \left|\frac{y_n}{x^k}\right|^2\right)^{-\epsilon}}{\pi^{n+1}|z\overline{x}|^{kn}|1-z\overline{x}|^2\left|1 - \frac{w_1\overline{y}_1}{(z\overline{x})^k} - \dots - \frac{w_n\overline{y}_n}{(z\overline{x})^k}\right|^{n+1}}\, dV(x,y).$$

Make the substitution $y_j = x^k u_j$ for every $j = 1, \dots, n$. This transformation sends $\mathcal{H}_k$ to $\mathbb{D}^* \times \mathbb{B}_n$, so that the above expression becomes

$$\int_{\mathbb{D}^* \times \mathbb{B}_n} \frac{|x|^{kn-2kn\epsilon}(1-|x|^2)^{-\epsilon}\left(1 - |u_1|^2 - \dots - |u_n|^2\right)^{-\epsilon}}{\pi^{n+1}|z|^{kn}|1-z\overline{x}|^2\left|1 - \frac{w_1}{z^k}\overline{u}_1 - \dots - \frac{w_n}{z^k}\overline{u}_n\right|^{n+1}}\, dV(x,u).$$

Next by [13, Proposition 1.4.10] we have

$$\int_{\mathcal{H}_k} |K_{\mathcal{H}_k}((z,w),(x,y))|\, h(x,y)^{-\epsilon}\, dV(x,y)$$
$$\leq C \int_{\mathbb{D}} \frac{|x|^{kn-2kn\epsilon}(1-|x|^2)^{-\epsilon}}{\pi^{n+1}|z|^{kn}|1-z\overline{x}|^2} \left(1-\left|\frac{w_1}{z^k}\right|^2 - \cdots - \left|\frac{w_n}{z^k}\right|^2\right)^{-\epsilon} dV(x).$$

If $kn - 2kn\epsilon > -2$, then by [3, Lemma 3.2]

$$\int_{\mathbb{D}} \frac{|x|^{kn-2kn\epsilon}(1-|x|^2)^{-\epsilon}}{\pi^{n+1}|z|^{kn}|1-z\overline{x}|^2} \left(1-\left|\frac{w_1}{z^k}\right|^2 - \cdots - \left|\frac{w_n}{z^k}\right|^2\right)^{-\epsilon} dV(x)$$
$$\leq C' \frac{(1-|z|^2)^{-\epsilon}}{|z|^{kn}} \left(1-\left|\frac{w_1}{z^k}\right|^2 - \cdots - \left|\frac{w_n}{z^k}\right|^2\right)^{-\epsilon}.$$

Finally when $2kn\epsilon \geq kn$ we obtain the desired result. The estimation (2.2) can be obtained by the similar method and we omit the details. □

Now we are ready to formulate the main result

Theorem 2.2. *The Bergman projections*

$$\mathbf{P}_{\Omega_k} \colon L^p(\Omega_k) \to L^p_a(\Omega_k) \quad \textit{and} \quad \mathbf{P}_{\mathcal{H}_k} \colon L^p(\mathcal{H}_k) \to L^p_a(\mathcal{H}_k)$$

are bounded operators if and only if $p \in \left(\frac{2nk+2}{nk+2}, \frac{2nk+2}{nk}\right)$.

Proof. If D is a Reinhardt domain, $f \in L^2_a(D) := \mathcal{O}(D) \cap L^2(D)$, $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha$, then $\{z^\alpha \colon \alpha \in \sum(f)\} \subset L^2_a(D)$, where $\sum(f) := \{\alpha \in \mathbb{Z}^n \colon a_\alpha \neq 0\}$ (for the proof see [4, p. 67]). Therefore it is easy to check, that the set $\{z^\alpha w_1^{\beta_1} \cdots w_n^{\beta_n} \colon \beta_j \geq 0, \alpha \geq -k(|\beta| + n)\}$ is a complete orthogonal set for $L^2_a(\Omega_k)$ and $L^2_a(\mathcal{H}_k)$. Now it can be shown that

$$\mathbf{P}_{\Omega_k}(\overline{z}^{kn})(x,y) = \frac{C}{x^{nk}} \quad \text{and} \quad \mathbf{P}_{\mathcal{H}_k}(\overline{z}^{kn})(x,y) = \frac{C_1}{x^{nk}},$$

for some constants C and C_1. Thus,

$$\|\mathbf{P}_{\Omega_k}(\overline{z})\|_p^p \approx \int_{\Omega_k} \frac{1}{|x|^{pnk}}\, dV(x,y)$$
$$= (2\pi)^{n+1} \int_0^1 \int_0^{r^k} \cdots \int_0^{r^k} r^{1-pnk} s_1 \cdots s_n\, ds_1 \cdots ds_n dr$$
$$= 2\pi^{n+1} \int_0^1 r^{1-pnk+2nk}\, dr,$$

and

$$\|\mathbf{P}_{\mathcal{H}_k}(\overline{z})\|_p^p \approx \int_{\mathcal{H}_k} \frac{1}{|x|^{pnk}}\, dV(x,y)$$
$$= (2\pi)^{n+1} \int_0^1 \int_0^{r^k} \int_{S_+^{n-1}} r^{1-pnk}\rho^{2n-1}\omega d\sigma(\omega) d\rho dr$$
$$= \frac{2\pi^{n+1}}{n!} \int_0^1 r^{1-pnk+2nk}\, dr.$$

These integrals diverge when $p \geq \frac{2+2nk}{nk}$, so $\mathbf{P}_{\Omega_k}(\overline{z}) \notin L^p(\Omega_k)$ and $\mathbf{P}_{\mathcal{H}_k}(\overline{z}) \notin L^p(\mathcal{H}_k)$ for this range of p. The fact that the Bergman projection is selfadjoint, together with Hölder's inequality, shows that the $\mathbf{P}_{\mathcal{H}_k}$ and $\mathbf{P}_{\Omega_k}$ also fail to be a bounded operator on $L^p(\Omega_k)$ and $L^p(\mathcal{H}_k)$, respectively when $p \in (1, \frac{2kn+2}{kn+2}]$. In order to prove boundedness it is enough to combine Lemma 2.1 and Schur's lemma ([3, Lemma 2.4]). These yield that the operators $\mathbf{P}_{\Omega_k}$ and $\mathbf{P}_{\mathcal{H}_k}$ are bounded for $p \in \left(\frac{2nk+2}{nk+2}, \frac{2nk+2}{nk}\right)$. □

2.1. **Concluding remarks.**

(1) Putting together Theorem 2.2 and [3, Theorem 3.1], we have three different domains Ω_k, $\mathcal{H}_k$ and

$$\mathbb{H}_{nk} := \left\{(z,w) \in \mathbb{C}^2 \colon |w|^{nk} < |z|\right\}$$

for which Bergman projection is bounded with the same range of p.

(2) For Ω_k and $\mathcal{H}_k$ we can also prove similar results to those obtained in [1] related to weighted L^p mapping behavior.

(3) It would be interesting to investigate a class of domains considered by Zapałowski [14] which generalizes the classical Hartogs triangle, that is, domains of the form

$$\left\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m \colon \sum_{j=1}^n |z_j|^{2p_j} < \sum_{j=1}^m |w_j|^{2q_j} < 1\right\}.$$

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