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FILTER THEORY IN MTL-ALGEBRAS BASED ON THE UNI-SOFT PROPERTY

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Abstract. The notion of (Boolean) uni-soft filters in MTL-algebras is introduced, and several properties of them are investigated. Characterizations of (Boolean) uni-soft filters are discussed, and some (necessary and sufficient) conditions for a uni-soft filter to be Boolean are provided. The condensational property for a Boolean uni-soft filter is established.
Keywords: MTL-algebras, (Boolean) filter, (Boolean) uni-soft filter.
MSC(2010): Primary: 03G25; Secondary: 06D72, 06E25.

1. **Introduction**

To formalize the many-valued logics induced by continuous t-norms on the real unit interval $[0,1]$, Hajek [4] introduced a very general many-valued logic, called Basic Logic. It is a well known fact that a t-norm has a residuum if and only if it is left-continuous, which shows that Basic Logic is not the most general t-norm based logic. In fact a logic weaker than Basic Logic, called Monoidal t-norm-based logic (MTL for short), was defined by Esteva and Godo in [3]. The MTL is indeed the logic of left-continuous t-norms, and MTL-algebras are the algebraic counterpart of this logic.

To solve complicated problems in economics, engineering, and environment, we can not successfully use the classical methods because of various uncertainties which are typical for those problems. Uncertainties can not be handled using the traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [7]. Maji et al. [6] and Molodtsov [7] suggested that one reason

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for these difficulties may be due to the inadequacy of the parametrization tool of the theory.

To overcome these difficulties, Molodtsov [7] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [6] described the application of the soft set theory to a decision making problem. Also, Maji et al. [5] studied several operations on the theory of soft sets. Chen et al. [2] presented a new definition of soft set parametrization reduction, and compared this definition with the related concept of attributes reduction in rough set theory.

In this paper, we introduce the notion of (Boolean) uni-soft filters in MTL-algebras, and investigate several properties of them. We discuss characterizations of (Boolean) uni-soft filters, and provide a necessary and sufficient condition for a uni-soft filter to be Boolean. We establish the condensational property for a Boolean uni-soft filter.

2. **Preliminaries**

By a *residuated lattice* we shall mean a lattice

$$L = (L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$$

containing the least element 0 and the largest element 1, and endowed with two binary operations $\odot$ (called *product*) and $\rightarrow$ (called *residuum*) such that

- $\odot$ is associative, commutative and isotone.
- $(\forall x \in L)\ (x \odot 1 = x)$.
- The Galois correspondence holds, that is,

$$(\forall x, y, z \in L)\,(x \odot y \leq z \iff x \leq y \rightarrow z).$$

In a residuated lattice, the following are true (see [8]):

$$x \leq y \ \Rightarrow\ x \rightarrow y = 1. \tag{2.1}$$

$$0 \rightarrow x = 1,\ 1 \rightarrow x = x,\ x \rightarrow (y \rightarrow x) = 1. \tag{2.2}$$

$$y \leq (y \rightarrow x) \rightarrow x. \tag{2.3}$$

$$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z). \tag{2.4}$$

$$x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y),\ x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z). \tag{2.5}$$

$$y \leq x \ \Rightarrow\ x \rightarrow z \leq y \rightarrow z,\ z \rightarrow y \leq z \rightarrow x. \tag{2.6}$$

$$\left(\bigvee_{i\in\Gamma} y_i\right) \rightarrow x = \bigwedge_{i\in\Gamma}(y_i \rightarrow x). \tag{2.7}$$

We define $x^* = \bigvee\{y \in L \mid x \odot y = 0\}$, equivalently, $x^* = x \to 0$. Then

$$0^* = 1,\ 1^* = 0,\ x \leq x^{**},\ x^* = x^{***}.$$

Based on the Hájek's results [4], and the axioms and provable formulas of MTL, Esteva and Godo [3] defined the so-called MTL-algebras corresponding to the MTL-logic in the following way.

Definition 2.1. An *MTL-algebra* is a residuated lattice $L = (L, \leq, \wedge, \vee, \odot, \to, 0, 1)$ satisfying the pre-linearity equation:

$$(x \to y) \vee (y \to x) = 1.$$

In an MTL-algebra, the following are true:

$$x \to (y \vee z) = (x \to y) \vee (x \to z). \tag{2.8}$$

$$x \odot y \leq x \wedge y. \tag{2.9}$$

Definition 2.2 ([3])**.** Let L be an MTL-algebra. A nonempty subset F of L is called a *filter* of L if it satisfies

(b1) $(\forall x, y \in F)\ (x \odot y \in F)$.
(b2) $(\forall x \in F)\ (\forall y \in L)\ (x \leq y \Rightarrow y \in F)$.

Since $\wedge$ is not definable from $\odot$ and $\to$ in an MTL-algebra, one could consider that the further condition

(b3) $(\forall x, y \in F)\ (x \wedge y \in F)$

should be also required for a filter. However, the condition (b3) is indeed redundant because it is a consequence of (b1) and (b2). Namely, since $x \odot y \leq x \wedge y$, if $x, y \in F$ then $x \odot y \in F$ and thus $x \wedge y \in F$ as well.

Proposition 2.3. *A nonempty subset F of an MTL-algebra L is a filter of L if and only if it satisfies:*

(b4) $1 \in F$.
(b5) $(\forall x \in F)\ (\forall y \in L)\ (x \to y \in F \Rightarrow y \in F)$.

A soft set theory, introduced by Molodtsov [7], and Çağman et al. [1], provided new definitions and various results.

In what follows, let U be an initial universe set and L be a set of parameters. Let $P(U)$ denote the power set of U and $A, B, C, \cdots \subseteq L$.

Definition 2.4 ([1, 7])**.** A *soft set* F_A of L (over U) is defined to be the set of ordered pairs

$$F_A := \left\{ \left(x, \tilde{f}_A(x)\right) : x \in L,\ \tilde{f}_A(x) \in P(U) \right\},$$

where $\tilde{f}_A : L \to P(U)$ satisfies $\tilde{f}_A(x) = \emptyset$ for $x \notin A$.

For a soft set F_L of L (over U), the set $e_L\left(\tilde{f}_L; \tau\right) = \left\{x \in L \mid \tilde{f}_L(x) \subseteq \tau\right\}$ is called the *τ-exclusive set* of F_L.

3. Uni-soft filters

In what follows let L denote an MTL-algebra unless otherwise specified.

Definition 3.1. A soft set F_L of L is called a *uni-soft filter* of L if it satisfies:

$$(\forall x, y \in L)\left(\tilde{f}_L(x \odot y) \subseteq \tilde{f}_L(x) \cup \tilde{f}_L(y)\right), \text{ and} \tag{3.1}$$

$$(\forall x, y \in L)\left(x \leq y \;\Rightarrow\; \tilde{f}_L(x) \supseteq \tilde{f}_L(y)\right). \tag{3.2}$$

Example 3.2. Let $L = [0, 1]$ and define a product $\odot$ and a residuum $\rightarrow$ on L as follows:

$$x \odot y := \begin{cases} x \wedge y & \text{if } x + y > \frac{1}{2}, \\ 0 & \text{otherwise} \end{cases} \quad x \rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ (0.5 - x) \vee y & \text{if } x > y \end{cases}$$

for all $x, y \in L$. Then L is an MTL-algebra. Let F_L be a soft set of L in which

$$\tilde{f}_L(x) := \begin{cases} \tau_1 & \text{if } x \in (0.5, 1], \\ \tau_2 & \text{otherwise,} \end{cases}$$

where $\tau_1 \subsetneq \tau_2$ in $P(U)$. Then it is straightforward to verify that F_L is a uni-soft filter of L.

We prove the following characterizations for uni-soft filters.

Theorem 3.3. *A soft set F_L of L is a uni-soft filter of L if and only if it satisfies:*

$$(\forall x \in L)\left(\tilde{f}_L(1) \subseteq \tilde{f}_L(x)\right), \textit{and} \tag{3.3}$$

$$(\forall x, y \in L)\left(\tilde{f}_L(y) \subseteq \tilde{f}_L(x) \cup \tilde{f}_L(x \rightarrow y)\right). \tag{3.4}$$

Proof. Assume that F_L is a uni-soft filter of L. Since $x \leq 1$ for all $x \in L$, it follows from (3.2) that $\tilde{f}_L(1) \subseteq \tilde{f}_L(x)$ for all $x \in L$. Since $x \leq (x \rightarrow y) \rightarrow y$, we have $x \odot (x \rightarrow y) \leq y$ for all $x, y \in L$ by the Galois correspondence. It follows from (3.2) and (3.1) that

$$\tilde{f}_L(y) \subseteq \tilde{f}_L(x \odot (x \rightarrow y)) \subseteq \tilde{f}_L(x) \cup \tilde{f}_L(x \rightarrow y)$$

for all $x, y \in L$.

Conversely, let F_L be a soft set of L which satisfy two conditions (3.3) and (3.4). Let $x, y \in L$ be such that $x \leq y$. Then $x \rightarrow y = 1$, and so

$$\tilde{f}_L(y) \subseteq \tilde{f}_L(x) \cup \tilde{f}_L(x \rightarrow y) = \tilde{f}_L(x) \cup \tilde{f}_L(1) = \tilde{f}_L(x),$$

for all $x \in L$. This proves (3.2). Using (2.4), we know that

$$x \rightarrow (y \rightarrow (x \odot y)) = (x \odot y) \rightarrow (x \odot y) = 1.$$

Using (3.3) and (3.4), we have

$$\begin{aligned}\tilde{f}_L(x \odot y) &\subseteq \tilde{f}_L(y) \cup \tilde{f}_L(y \to (x \odot y))\\ &\subseteq \tilde{f}_L(y) \cup \Big(\tilde{f}_L(x) \cup \tilde{f}_L(x \to (y \to (x \odot y)))\Big)\\ &= \tilde{f}_L(y) \cup \Big(\tilde{f}_L(x) \cup \tilde{f}_L(1)\Big)\\ &= \tilde{f}_L(x) \cup \tilde{f}_L(y)\end{aligned}$$

for all $x, y \in L$. Therefore, F_L is a uni-soft filter of L. □

Theorem 3.4. *A soft set F_L of L is a uni-soft filter of L if and only if it satisfies:*

$$(\forall a, b, c \in L)\left(a \le b \to c \;\Rightarrow\; \tilde{f}_L(c) \subseteq \tilde{f}_L(a) \cup \tilde{f}_L(b)\right). \tag{3.5}$$

Proof. Assume that F_L is a uni-soft filter of L. Let $a, b, c \in L$ be such that $a \le b \to c$. Then $\tilde{f}_L(a) \supseteq \tilde{f}_L(b \to c)$ by (3.2), and so

$$\tilde{f}_L(c) \subseteq \tilde{f}_L(b) \cup \tilde{f}_L(b \to c) \subseteq \tilde{f}_L(b) \cup \tilde{f}_L(a).$$

Conversely, let F_L be a soft set of L satisfying (3.5). Since $x \le x \to 1$ for all $x \in L$, it follows from (3.5) that

$$\tilde{f}_L(1) \subseteq \tilde{f}_L(x) \cup \tilde{f}_L(x) = \tilde{f}_L(x)$$

for all $x \in L$. Since $x \to y \le x \to y$ for all $x, y \in L$, we have

$$\tilde{f}_L(y) \subseteq \tilde{f}_L(x) \cup \tilde{f}_L(x \to y)$$

for all $x, y \in L$. Therefore F_L is a uni-soft filter of L by Theorem 3.3. □

Corollary 3.5. *A soft set F_L of L is a uni-soft filter of L if and only if it satisfies*

$$\tilde{f}_L(x) \subseteq \bigcup_{k=1}^{n} \tilde{f}_L(a_k) \tag{3.6}$$

whenever $a_n \to (\cdots \to (a_2 \to (a_1 \to x))\cdots) = 1$ for any $a_1, a_2, \cdots, a_n \in L$.

Proof. It can be easily checked by induction on n. □

Theorem 3.6. *For a filter F of L and $a \in L$, let F_L be a soft set of L defined by*

$$\tilde{f}_L(x) := \begin{cases} \tau_1 & \text{if } x \in \{z \in L \mid a \vee z \in F\},\\ \tau_2 & \text{otherwise,}\end{cases}$$

for all $x \in L$ where $\tau_1 \subsetneq \tau_2$ in $P(U)$. Then F_L is a uni-soft filter of L.

Proof. Since $a \vee 1 \in F$, we have $1 \in \{z \in L \mid a \vee z \in F\}$ and so $\tilde{f}_L(1) = \tau_1 \subseteq \tilde{f}_L(x)$ for all $x \in L$. Now if $y \in \{z \in L \mid a \vee z \in F\}$, then clearly

$$\tilde{f}_L(y) = \tau_1 \subseteq \tilde{f}_L(x) \cup \tilde{f}_L(x \to y).$$

Suppose that $y \notin \{z \in L \mid a \vee z \in F\}$. Then at least one of x and $x \to y$ does not belong to $\{z \in L \mid a \vee z \in F\}$. Hence

$$\tilde{f}_L(y) = \tau_2 = \tilde{f}_L(x) \cup \tilde{f}_L(x \to y),$$

and therefore F_L is a uni-soft filter of L by Theorem 3.3. □

Theorem 3.7. *A soft set F_L of L is a uni-soft filter of L if and only if the nonempty τ-exclusive set $e_L\left(\tilde{f}_L;\tau\right)$ is a filter of L for all $\tau \in P(U)$.*

Proof. Suppose that F_L is a uni-soft filter of L. Let $\tau \in P(U)$ be such that $e_L\left(\tilde{f}_L;\tau\right) \neq \emptyset$. Then there exists an $a \in e_L\left(\tilde{f}_L;\tau\right)$, and so $\tilde{f}_L(a) \subseteq \tau$. It follows from (3.3) that $\tau \supseteq \tilde{f}_L(a) \supseteq \tilde{f}_L(1)$. Thus $1 \in e_L\left(\tilde{f}_L;\tau\right)$. Let $x, y \in L$ be such that $x \to y \in e_L\left(\tilde{f}_L;\tau\right)$ and $x \in e_L\left(\tilde{f}_L;\tau\right)$. Then $\tilde{f}_L(x \to y) \subseteq \tau$ and $\tilde{f}_L(x) \subseteq \tau$. It follows from (3.4) that

$$\tau \supseteq \tilde{f}_L(x \to y) \cup \tilde{f}_L(x) \supseteq \tilde{f}_L(y),$$

that is, $y \in e_L\left(\tilde{f}_L;\tau\right)$. Thus $e_L\left(\tilde{f}_L;\tau\right) (\neq \emptyset)$ is a filter of L by Proposition 2.3.

Conversely, assume that the nonempty τ-exclusive set $e_L\left(\tilde{f}_L;\tau\right)$ is a filter of L for all $\tau \in P(U)$. For any $x \in L$, let $\tilde{f}_L(x) = \tau$. Then $x \in e_L\left(\tilde{f}_L;\tau\right)$. Since $e_L\left(\tilde{f}_L;\tau\right)$ is a filter of L, we have $1 \in e_L\left(\tilde{f}_L;\tau\right)$ and so $\tilde{f}_L(x) = \tau \supseteq \tilde{f}_L(1)$. For any $x, y \in L$, let $\tilde{f}_L(x \to y) \cup \tilde{f}_L(x) = \tau$. Then $x \to y \in e_L\left(\tilde{f}_L;\tau\right)$ and $x \in e_L\left(\tilde{f}_L;\tau\right)$. It follows from (b5) that $y \in e_L\left(\tilde{f}_L;\tau\right)$. Hence, $\tilde{f}_L(y) \subseteq \tau = \tilde{f}_L(x \to y) \cup \tilde{f}_L(x)$. Therefore, F_L is a uni-soft filter of L by Theorem 3.3. □

Theorem 3.8. *If F_L is a uni-soft filter of L, then the set*

$$\Omega_a := \{x \in L \mid \tilde{f}_L(x) \subseteq \tilde{f}_L(a)\}$$

is a filter of L for every $a \in L$.

Proof. Since $\tilde{f}_L(1) \subseteq \tilde{f}_L(a)$ for all $a \in L$. we have $1 \in \Omega_a$. Let $x, y \in L$ be such that $x \in \Omega_a$ and $x \to y \in \Omega_a$. Then $\tilde{f}_L(a) \supseteq \tilde{f}_L(x)$ and $\tilde{f}_L(a) \supseteq \tilde{f}_L(x \to y)$. Since F_L is a uni-soft filter of L, it follows from (3.4) that

$$\tilde{f}_L(y) \subseteq \tilde{f}_L(x) \cup \tilde{f}_L(x \to y) \subseteq \tilde{f}_L(a)$$

so $y \in \Omega_a$. Hence Ω_a is a filter of L by Proposition 2.3. □

Theorem 3.9. *Let $a \in L$ and let F_L be a soft set of L. Then*

(1) *If Ω_a is a filter of L, then F_L satisfies the following implication for all $x, y \in L$:*

$$\tilde{f}_L(a) \supseteq \tilde{f}_L(x \to y) \cup \tilde{f}_L(x) \;\Rightarrow\; y \in \Omega_a, \text{ i.e.,} \tilde{f}_L(y) \subseteq \tilde{f}_L(a). \tag{3.7}$$

(2) *If F_L satisfies (3.3) and (3.7), then Ω_a is a filter of L.*

Proof. (1) Assume that Ω_a is a filter of L. Let $x, y \in L$ be such that

$$\tilde{f}_L(a) \supseteq \tilde{f}_L(x \to y) \cup \tilde{f}_L(x).$$

Then $x \to y \in \Omega_a$ and $x \in \Omega_a$. Using (b5), we have $y \in \Omega_a$, i.e., $\tilde{f}_L(y) \subseteq \tilde{f}_L(a)$.

(2) Suppose that $\tilde{f}_L$ satisfies (3.3) and (3.7). From (3.3) it follows that $1 \in \Omega_a$. Let $x, y \in L$ be such that $x \in \Omega_a$ and $x \to y \in \Omega_a$. Then

$$\tilde{f}_L(a) \supseteq \tilde{f}_L(x) \text{ and } \tilde{f}_L(a) \supseteq \tilde{f}_L(x \to y),$$

which imply that $\tilde{f}_L(a) \supseteq \tilde{f}_L(x) \cup \tilde{f}_L(x \to y)$. Thus $y \in \Omega_a$ by (3.7). Therefore, Ω_a is a filter of L by Proposition 2.3. □

Theorem 3.10. *Let F_L be a uni-soft filter of L. Then the following are equivalent for all x, y, z in L:*

(1) $\tilde{f}_L(x \to z) \subseteq \tilde{f}_L(x \to (y \to z)) \cup \tilde{f}_L(x \to y)$.
(2) $\tilde{f}_L(x \to y) \subseteq \tilde{f}_L(x \to (x \to y))$.
(3) $\tilde{f}_L((x \to y) \to (x \to z)) \subseteq \tilde{f}_L(x \to (y \to z))$.

Proof. (1) ⇒ (2). Suppose that F_L satisfies the condition (1). Taking $z = y$ and $y = x$ in (1) and using (3.3), we have

$$\begin{aligned}
\tilde{f}_L(x \to y) &\subseteq \tilde{f}_L(x \to (x \to y)) \cup \tilde{f}_L(x \to x) \\
&= \tilde{f}_L(x \to (x \to y)) \cup \tilde{f}_L(1) \\
&= \tilde{f}_L(x \to (x \to y))
\end{aligned}$$

(2) ⇒ (3). Suppose that F_L satisfies the condition (2) and let $x, y, z \in L$. Since

$$x \to (y \to z) \le x \to ((x \to y) \to (x \to z)),$$

it follows from (2.4), (2) and (3.2) that

$$\begin{aligned}
\tilde{f}_L((x \to y) \to (x \to z)) &= \tilde{f}_L(x \to ((x \to y) \to z)) \\
&\subseteq \tilde{f}_L(x \to (x \to ((x \to y) \to z))) \\
&= \tilde{f}_L(x \to ((x \to y) \to (x \to z))) \\
&\subseteq \tilde{f}_L(x \to (y \to z)).
\end{aligned}$$

(3) $\Rightarrow$ (1). If F_L satisfies the condition (3), then

$$\begin{aligned}\tilde{f}_L(x \to z) &\subseteq \tilde{f}_L((x \to y) \to (x \to z)) \cup \tilde{f}_L(x \to y)\\ &\subseteq \tilde{f}_L(x \to (y \to z)) \cup \tilde{f}_L(x \to y)\end{aligned}$$

This completes the proof. □

Theorem 3.11. *For a fixed element $a \in L$, let F_L^a be a soft set of L defined by*

$$\tilde{f}_L^a(x) := \begin{cases} \tau_1 & \text{if } a \leq x, \\ \tau_2 & \text{otherwise,} \end{cases}$$

where $\tau_1 \subsetneq \tau_2$ in $P(U)$. Then F_L^a is a uni-soft filter of L if and only if it satisfies the following implication:

$$(\forall x, y \in L)\,(a \leq y \to x,\ a \leq y \ \Rightarrow\ a \leq x). \tag{3.8}$$

Proof. Assume that F_L^a is a uni-soft filter of L and let $x, y \in L$ be such that $a \leq y \to x$ and $a \leq y$. Then $\tilde{f}_L^a(y \to x) = \tau_1 = \tilde{f}_L^a(y)$, and thus

$$\tilde{f}_L^a(x) \subseteq \tilde{f}_L^a(y \to x) \cup \tilde{f}_L^a(y) = \tau_1$$

which implies that $\tilde{f}_L^a(x) = \tau_1$ and so $a \leq x$.

Conversely, suppose that (3.8) holds. Note that $e_L\left(\tilde{f}_L^a; \tau_2\right) = L$ and

$$e_L\left(\tilde{f}_L^a; \tau_1\right) = \{x \in L \mid a \leq x\}.$$

Obviously, $1 \in e_L\left(\tilde{f}_L^a; \tau_1\right)$. Let $x, y \in L$ be such that $x \in e_L\left(\tilde{f}_L^a; \tau_1\right)$ and $x \to y \in e_L\left(\tilde{f}_L^a; \tau_1\right)$. Then $a \leq x$ and $a \leq x \to y$, which imply from (3.8) that $a \leq y$, that is, $y \in e_L\left(\tilde{f}_L^a; \tau_1\right)$. Hence, $e_L\left(\tilde{f}_L^a; \tau_1\right)$ is a filter of L. By Theorem 3.7, F_L^a is a uni-soft filter of L. □

Definition 3.12. A uni-soft filter F_L of L is said to be *Boolean* if it satisfies the following identity

$$(\forall x \in L)\left(\tilde{f}_L(x \vee x^*) = \tilde{f}_L(1)\right). \tag{3.9}$$

Theorem 3.13. *Every Boolean uni-soft filter F_L of L satisfies the following inclusion for all $x, y, z \in L$:*

$$\tilde{f}_L(x \to z) \subseteq \tilde{f}_L(x \to (z^* \to y)) \cup \tilde{f}_L(y \to z). \tag{3.10}$$

Proof. Using (2.5), we have

$$y \to z \leq (z^* \to y) \to (z^* \to z) \leq (x \to (z^* \to y)) \to (x \to (z^* \to z)).$$

It follows from (3.2) that

$$\tilde{f}_L(y \to z) \supseteq \tilde{f}_L((x \to (z^* \to y)) \to (x \to (z^* \to z)))$$

so from (3.4) that

$$\begin{aligned}\tilde{f}_L(x \to (z^* \to z)) &\subseteq \tilde{f}_L(x \to (z^* \to y)) \cup \tilde{f}_L((x \to (z^* \to y)) \\ &\quad \to (x \to (z^* \to z))) \\ &\subseteq \tilde{f}_L(x \to (z^* \to y)) \cup \tilde{f}_L(y \to z).\end{aligned}$$

Since

$$z^* \vee z = ((z^* \to z) \to z) \wedge ((z \to z^*) \to z^*) \leq (z^* \to z) \to z,$$

we have $\tilde{f}_L((z^* \to z) \to z) \subseteq \tilde{f}_L(z^* \vee z) = \tilde{f}_L(1)$. Since

$$x \to (z^* \to z) \leq ((z^* \to z) \to z) \to (x \to z),$$

it follows from (3.2) that

$$\tilde{f}_L(x \to (z^* \to z)) \supseteq \tilde{f}_L(((z^* \to z) \to z) \to (x \to z)).$$

Thus

$$\begin{aligned}\tilde{f}_L(x \to z) &\subseteq \tilde{f}_L((z^* \to z) \to z) \cup \tilde{f}_L(((z^* \to z) \to z) \to (x \to z)) \\ &\subseteq \tilde{f}_L(1) \cup \tilde{f}_L(x \to (z^* \to z)) \\ &= \tilde{f}_L(x \to (z^* \to z)) \\ &\subseteq \tilde{f}_L(x \to (z^* \to y)) \cup \tilde{f}_L(y \to z),\end{aligned}$$

for all $x, y, z \in L$. This completes the proof. □

We provide a necessary and sufficient condition for a uni-soft filter to be Boolean.

Theorem 3.14. *If a uni-soft filter F_L of L satisfies the following inclusion*

$$(3.11) \qquad (\forall x, y \in L) \left(\tilde{f}_L(x) \subseteq \tilde{f}_L((x \to y) \to x)\right),$$

then it is Boolean.

Proof. Using (2.2), (2.4) and (2.5), we have

$$\begin{aligned}1 &= x \to ((x^* \to x) \to x) \\ &\leq ((x^* \to x) \to x)^* \to x^* \\ &\leq (x^* \to x) \to (((x^* \to x) \to x)^* \to x) \\ &= ((x^* \to x) \to x)^* \to ((x^* \to x) \to x) \\ &= (((x^* \to x) \to x) \to 0) \to ((x^* \to x) \to x).\end{aligned}$$

It follows from (3.2), (3.3) and (3.11) that

$$\begin{aligned}\tilde{f}_L((x^* \to x) \to x) &\subseteq \tilde{f}_L((((x^* \to x) \to x) \to 0) \to ((x^* \to x) \to x)) \\ &\subseteq \tilde{f}_L(1).\end{aligned}$$

so that $\tilde{f}_L((x^* \to x) \to x) = \tilde{f}_L(1)$. Using (2.7) and (2.8), since

$$\begin{aligned}(x^* \to x) \to x &\leq ((x^* \to x) \to x) \vee ((x^* \to x) \to x^*)\\ &= (x^* \to x) \to (x \vee x^*)\\ &= (1 \wedge (x^* \to x)) \to (x \vee x^*)\\ &= ((x \to x) \wedge (x^* \to x)) \to (x \vee x^*)\\ &= ((x \vee x^*) \to x) \to (x \vee x^*),\end{aligned}$$

we get

$$\begin{aligned}\tilde{f}_L(1) &= \tilde{f}_L((x^* \to x) \to x)\\ &\supseteq \tilde{f}_L(((x \vee x^*) \to x) \to (x \vee x^*))\\ &\supseteq \tilde{f}_L(x \vee x^*),\end{aligned}$$

and so $\tilde{f}_L(x \vee x^*) = \tilde{f}_L(1)$. Therefore, F_L is Boolean. □

Theorem 3.15. *If a uni-soft filter F_L of L satisfies the condition (3.10), then it satisfies the condition (3.11).*

Proof. Since $(x \to y) \to x \leq x^* \to x$, it follows from (3.2) and (3.10) and (3.3) that

$$\begin{aligned}\tilde{f}_L(x) &= \tilde{f}_L(1 \to x)\\ &\subseteq \tilde{f}_L(1 \to (x^* \to x^*)) \cup \tilde{f}_L(x^* \to x)\\ &\subseteq \tilde{f}_L(1) \cup \tilde{f}_L((x \to y) \to x)\\ &= \tilde{f}_L((x \to y) \to x)\end{aligned}$$

for all $x, y \in L$. Hence, F_L satisfies the condition (3.11). □

Corollary 3.16. *Every uni-soft filter F_L of L satisfying the condition (3.10) is Boolean.*

Theorem 3.17. *If a uni-soft filter F_L of L satisfies (3.11), then it satisfies the following inclusion for all $x, y, z \in L$:*

$$\tilde{f}_L(x \to z) \subseteq \tilde{f}_L(x \to (y \to z)) \cup \tilde{f}_L(x \to y). \tag{3.12}$$

Proof. Since $x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z))$, it follows from (3.2) that

$$\tilde{f}_L(x \to (y \to z)) \supseteq \tilde{f}_L((x \to y) \to (x \to (x \to z)))$$

so from (3.4) that

$$\begin{aligned}\tilde{f}_L(x \to (x \to z)) &\subseteq \tilde{f}_L(x \to y) \cup \tilde{f}_L((x \to y) \to (x \to (x \to z)))\\ &\subseteq \tilde{f}_L(x \to y) \cup \tilde{f}_L(x \to (y \to z)).\end{aligned}$$

Since

$$x \to (x \to z) \leq x \to (((x \to z) \to z) \to z) = ((x \to z) \to z) \to (x \to z),$$

we have

$$\begin{aligned}\tilde{f}_L(x \to z) &\subseteq \tilde{f}_L(((x \to z) \to z) \to (x \to z))\\ &\subseteq \tilde{f}_L(x \to (x \to z))\\ &\subseteq \tilde{f}_L(x \to y) \cup \tilde{f}_L(x \to (y \to z))\end{aligned}$$

by using (3.2) and (3.11). This completes the proof. □

Corollary 3.18. *Every uni-soft filter F_L of L satisfying the condition (3.10) satisfies the condition (3.12).*

Theorem 3.19. *Every Boolean uni-soft filter F_L of L satisfies the following inclusion:*

$$(\forall x, y, z \in L)\left(\tilde{f}_L(x \to z) \subseteq \tilde{f}_L(x \to (y \to z)) \cup \tilde{f}_L(x \to y)\right). \tag{3.13}$$

Proof. Note that $x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z))$ and

$$x \to (x \to z) \leq x \to (((x \to z) \to z) \to z) = ((x \to z) \to z) \to (x \to z),$$

for all $x, y, z \in L$. It follows from (3.2), (3.4), and Propositions 3.13 and 3.15 that

$$\begin{aligned}\tilde{f}_L(x \to z) &\subseteq \tilde{f}_L(((x \to z) \to z) \to (x \to z))\\ &\subseteq \tilde{f}_L(x \to (x \to z))\\ &\subseteq \tilde{f}_L(x \to y) \cup \tilde{f}_L((x \to y) \to (x \to (x \to z)))\\ &\subseteq \tilde{f}_L(x \to y) \cup \tilde{f}_L(x \to (y \to z)),\end{aligned}$$

for all $x, y, z \in L$. This completes the proof. □

Combining Theorems 3.13, 3.14 and 3.15, we have a characterization of a Boolean uni-soft filter.

Theorem 3.20. *Let F_L be a uni-soft filter of L. Then the following assertions are equivalent:*

(1) *F_L is Boolean.*
(2) *F_L satisfies the condition (3.10).*
(3) *F_L satisfies the condition (3.11).*

Theorem 3.21. *Every Boolean uni-soft filter F_L of L satisfies:*

$$(\forall x, y \in L)\left(\tilde{f}_L(x \to y) \supseteq \tilde{f}_L(((y \to x) \to x) \to y)\right). \tag{3.14}$$

Proof. Let F_L be a Boolean uni-soft filter of L.
Since $y \leq ((y \rightarrow x) \rightarrow x) \rightarrow y$, we have

$$(((y \rightarrow x) \rightarrow x) \rightarrow y) \rightarrow x \leq y \rightarrow x \tag{3.15}$$

by (2.6). Using (2.4), (2.5), (2.6) and (3.15), we get

$$\begin{aligned} x \rightarrow y &\leq ((y \rightarrow x) \rightarrow x) \rightarrow ((y \rightarrow x) \rightarrow y) \\ &= (y \rightarrow x) \rightarrow (((y \rightarrow x) \rightarrow x) \rightarrow y) \\ &\leq ((((y \rightarrow x) \rightarrow x) \rightarrow y) \rightarrow x) \rightarrow (((y \rightarrow x) \rightarrow x) \rightarrow y) \end{aligned}$$

and so

$$\begin{aligned} \tilde{f}_L(((y \rightarrow x) \rightarrow x) \rightarrow y) &\subseteq \tilde{f}_L(((((y \rightarrow x) \rightarrow x) \rightarrow y) \rightarrow x) \\ &\quad \rightarrow (((y \rightarrow x) \rightarrow x) \rightarrow y)) \\ &\subseteq \tilde{f}_L(x \rightarrow y) \end{aligned}$$

for all $x, y \in L$ by Theorem 3.20(3) and (3.2). □

Theorem 3.22. *A uni-soft filter F_L of L is Boolean if and only if it satisfies* (3.14) *and*

$$(\forall x, y \in L) \left(\tilde{f}_L((x \odot x) \rightarrow y) \supseteq \tilde{f}_L(x \rightarrow y) \right). \tag{3.16}$$

Proof. Assume that F_L is a Boolean uni-soft filter of L. Then $\tilde{f}_L$ satisfies the condition (3.14) (see Theorem 3.21). The condition (3.16) follows from Theorems 3.19 and 3.10.

Conversely, let F_L be a uni-soft filter of L that satisfies two conditions (3.14) and (3.16). Since

$$(x \rightarrow y) \rightarrow x \leq ((x \rightarrow y) \odot (x \rightarrow y)) \rightarrow y$$

for all $x, y \in L$, it follows from (3.2) and (3.16) that

$$\tilde{f}_L((x \rightarrow y) \rightarrow x) \supseteq \tilde{f}_L(((x \rightarrow y) \odot (x \rightarrow y)) \rightarrow y) \supseteq \tilde{f}_L((x \rightarrow y) \rightarrow y) \tag{3.17}$$

for all $x, y \in L$. Note that

$$((x \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow y)) \rightarrow x \leq ((x \rightarrow y) \rightarrow y) \rightarrow x$$

for all $x, y \in L$. Hence

$$\begin{aligned} \tilde{f}_L((x \rightarrow y) \rightarrow x) &\supseteq \tilde{f}_L(((x \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow y)) \rightarrow x) \\ &\supseteq \tilde{f}_L(((x \rightarrow y) \rightarrow y) \rightarrow x) \end{aligned} \tag{3.18}$$

for all $x, y \in L$ by (3.14) and (3.2). Since F_L is a uni-soft filter of L, it follows from (3.17), (3.18) and (3.4) that

$$\tilde{f}_L((x \rightarrow y) \rightarrow x) \supseteq \tilde{f}_L(((x \rightarrow y) \rightarrow y) \rightarrow x) \cup \tilde{f}_L((x \rightarrow y) \rightarrow y) \supseteq \tilde{f}_L(x)$$

for all $x, y \in L$. We conclude from Theorem 3.14 that F_L is a Boolean uni-soft filter of L. □

Theorem 3.23 (Condensational property for Boolean uni-soft filter)**.** *Let F_L and G_L be two uni-soft filters of L such that $\tilde{f}_L(1) = \tilde{g}_L(1)$ and $\tilde{f}_L(x) \supseteq \tilde{g}_L(x)$ for all $x \in L$. If F_L is Boolean, then so is G_L.*

Proof. Assume that F_L is a Boolean uni-soft filter of L. Then $\tilde{f}_L(x \vee x^*) = \tilde{f}_L(1)$ for all $x \in L$. It follows from the hypothesis that

$$\tilde{g}_L(x \vee x^*) \subseteq \tilde{f}_L(x \vee x^*) = \tilde{f}_L(1) = \tilde{g}_L(1). \tag{3.19}$$

Combining (3.19) and (3.3), we have $\tilde{g}_L(x \vee x^*) = \tilde{g}_L(1)$ for all $x \in L$. Hence G_L is a Boolean uni-soft filter of L. □

4. **Applications**

Soft set theory, introduced by Molodtsov [7], is an important mathematical tool to deal with uncertainties, fuzzy or vague objects; and has vast applications in real life situations. Several possible applications of soft set theory in various directions are given in [7].

In this paper, we presented an application of soft set theory in an algebraic structure, called an MTL-algebra. In fact, using the notion of uni-soft property, we introduced the notion of (Boolean) uni-soft filters in MTL-algebras, and investigated on several properties of them. Moreover, we discussed on characterizations of (Boolean) uni-soft filters, and provided a necessary and sufficient condition for a uni-soft filter to be Boolean. We also established the condensational property for a Boolean uni-soft filter.

We hope that this work will provide a deep impact on the upcoming research in this field and other soft algebraic studies to open up new horizons of interest and innovations. Indeed, this work may serve as a foundation for further study of soft MTL-algebras. To extend these results, one can further study the union soft substructures of different algebras such as hemirings, R0-algerbas, hyperalgebras and other mathematical branches. One may also apply this concept to study some applications in many fields like decision making, knowledge base systems, data analysis, etc.

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