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DETERMINATION OF A JUMP BY FOURIER AND FOURIER-CHEBYSHEV SERIES

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Abstract. By observing the equivalence of assertions on determining the jump of a function by its differentiated or integrated Fourier series, we generalize a previous result of Kvernadze, Hagstrom and Shapiro to the whole class of functions of harmonic bounded variation. This is achieved without the finiteness assumption on the number of discontinuities. Two results on determination of jump discontinuities by means of the tails of integrated Fourier-Chebyshev series are also derived.
Keywords: Fourier series, generalized bounded variation, jump discontinuities.
MSC(2010): Primary: 42A24; Secondary: 26A45.

1. Introduction

The problem of approximating the magnitudes of jumps of a function by means of its truncated Fourier series arises naturally from the attempt to overcome the Gibbs phenomenon which describes the characteristic oscillatory behaviour of the Fourier partial sums of a piecewise smooth function in the neighbourhood of a point of discontinuity. It has been known for a long time that the jumps of a function of bounded variation (BV) can be expressed through its differentiated Fourier series. Let $S_n^{'}(f,x)$ denote the nth partial sum of the differentiated Fourier series of a function f at a point x. The relation

$$\lim_{n\to\infty}\frac{S_n^{'}(f,x)}{n}=\frac{1}{\pi}\left[f\left(x+0\right)-f\left(x-0\right)\right] \tag{1.1}$$

was proved by L. Fejér [8] for f satisfying the so-called Dirichlet-condition, by P. Csillag [7] for functions of bounded variation and by B.I. Golubov [9, Theorem 1, p. 20] for functions in V_p, $1\leq p<\infty$, of Wiener's bounded variation. M. Avdispahić [3, Theorem 1, p. 268] has shown that equation (1.1)

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holds for any function $f \in HBV$ and that HBV is the limiting case in the scale of ΛBV spaces for validity of (1.1). The corresponding formula which involves the partial sums of the conjugate Fourier series of $f \in HBV$ is also derived there. A number of results from [1, 2, 3, 4] related to the classes V_ϕ, ΛBV and $V[\nu]$ were later rediscovered and differently proved in [10]. G. Kvernadze [10] extended [3, Theorem 1'(1)] to the setting of the generalized Fourier-Jacobi series.

G. Kvernadze, T. Hagstrom and H. Shapiro [11] proved that the jumps of a $2\pi-$periodic function from V_p, $1 \le p < 2$, can be also determined by means of the tails of its integrated Fourier series. This was established under the condition that the number of discontinuities of f is finite.

Our paper consists of two main parts. In the first part, we generalize a result of [11] to the whole class of functions of harmonic bounded variation. We remove the finiteness assumption on the number of discontinuities in the trigonometric case. New results that express jump discontinuities of functions from HBV or its subclass V_2 through their integrated Fourier-Chebyshev series are presented in the second part.

2. Jump of a HBV function and integrated Fourier series

2.1. **Generalized bounded variation.** A concept of bounded variation of a higher order was firstly introduced by N. Wiener [16]. A function f is said to be of *bounded $p-$variation* on $[0, 2\pi]$, $p \ge 1$, and belongs to the class V_p if

$$V_p(f) = \sup \left\{ \sum_i |f(I_i)|^p \right\}^{1/p} < \infty,$$

where the supremum is taken over all finite collections of nonoverlapping subintervals I_i of $[0, 2\pi]$. The quantity $V_p(f)$ is called the *$p-$variation of f* on $[0, 2\pi]$.

This concept has been generalized by L.C. Young [17]. Let ϕ be a continuous function defined on $[0, \infty)$ and strictly increasing from 0 to ∞. A function f is said to be of *bounded $\phi-$variation* on $[0, 2\pi]$ and belongs to the class V_ϕ if

$$V_\phi(f) = \sup \left\{ \sum_i \phi\left(|f(I_i)|\right) \right\} < \infty,$$

where the supremum is taken over all finite collections of nonoverlapping subintervals I_i of $[0, 2\pi]$. The quantity $V_\phi(f)$ is called the *$\phi-$variation of f* on $[0, 2\pi]$.

By taking $\phi(u) = u$ we get Jordan's class BV, while $\phi(u) = u^p$ gives Wiener's class V_p.

Another type of generalization of the class BV was introduced by D. Waterman in [15]. It was influenced by Waterman's joint work with C. Goffman on everywhere convergence of Fourier series. Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers tending to infinity, such that $\sum 1/\lambda_n$ diverges. A function f is said to be of *bounded* $\Lambda-$*variation* on $[0,2\pi]$ and belongs to the class ΛBV if

$$V_{\Lambda}(f)=\sup\left\{\sum_i |f(I_i)|/\lambda_i\right\}<\infty,$$

where the supremum is taken over all finite collections of nonoverlapping subintervals I_i of $[0,2\pi]$. The quantity $V_{\Lambda}(f)$ is called the $\Lambda-$*variation of* f on $[0,2\pi]$. In the case when $\Lambda=\{n\}$, the sequence of positive integers, the function f is said to be of *harmonic bounded variation* and the corresponding class is denoted by HBV.

By W we denote the class of *regulated functions*, i.e. functions possessing the one-sided limits at each point. W is the union of all ΛBV spaces [12].

Z. Chanturiya [6] gave another interesting generalization using the modulus of variation. The *modulus of variation* of a bounded function f is the function ν_f whose domain is the set of positive integers, given by

$$\nu_f(n)=\sup_{\Pi_n}\left\{\sum_{k=1}^{n}|f(I_k)|\right\},$$

where $\Pi_n=\{I_k : k=1,\ldots,n\}$ is an arbitrary finite collection of n nonoverlapping subintervals of $[0,2\pi]$. The modulus of variation of any bounded function is nondecreasing and concave. Given a function ν with such properties, then by $V[\nu]$ one denotes the class of functions f for which $\nu_f(n)=O(\nu(n))$ as $n\to\infty$.

We note that $V_\phi\subseteq V\left[n\phi^{-1}(1/n)\right]$ and $W=\{f:\nu_f(n)=o(n)\}$ [6].

There exist the following inclusion relations between Wiener's, Waterman's and Chanturiya's classes.

Theorem 2.A (cf. [2, Theorem 4.4.]).

$$\{n^\alpha\}BV\subset V_{\frac{1}{1-\alpha}}\subset V[n^\alpha]\subset\left\{n^\beta\right\}BV,$$

for $0<\alpha<\beta<1$.

2.2. **Cesàro summability and differentiated Fourier series.** As well known, a sequence $\{s_n\}$ is *Cesàro* or $(C,1)$ *summable* to s if the sequence $\{\sigma_n\}$ of its arithmetical means converges to s, i.e.

$$\sigma_n=\frac{s_0+s_1+\cdots+s_n}{n+1}\to s,\qquad n\to\infty.$$

Analogously, a sequence $\{s_n\}$ is (C,α), $\alpha>-1$, summable to s, if the sequence

$$\sigma_n^{(\alpha)}=\frac{1}{\binom{n+\alpha}{n}}\sum_{i=0}^{n}\binom{n-i+\alpha-1}{n-i}s_i$$

converges to s.

It is obvious that Fejér's identity (1.1) is equivalent to Cesàro summability of the sequence $\{kb_k \cos kx - ka_k \sin kx\}$, where $a_k = a_k(f)$ and $b_k = b_k(f)$ are the kth cosine and sine coefficient of the Fourier series of a function f, respectively. There exist numerous generalizations of Fejér's theorem to more general summability methods. We recall the relationship between the order of Cesàro summability of the sequence $\{kb_k \cos kx - ka_k \sin kx\}$ and the "order of variation" of a function f.

Theorem 2.B ([3, 4, 5]). *Let f be a function of generalized bounded variation. The sequence $\{kb_k \cos kx - ka_k \sin kx\}$ of the terms of its differentiated Fourier series is (C,α) summable to $\frac{1}{\pi}[f(x+0) - f(x-0)]$ at every point x for*

(1) $\alpha > 0$, *if* $f \in BV$,
(2) $\alpha > 1 - \frac{1}{p}$, *if* $f \in V_p$, $1 < p < \infty$,
(3) $\alpha > \beta$, *if* $f \in V[n^\beta]$, $0 < \beta < 1$,
(4) $\alpha = 1$, *if* $f \in HBV$,
(5) $\alpha > 1$, *if* $f \in W$.

2.3. **Jump of a function and integrated Fourier series.** A method of determining jumps of a function by means of the tails of its integrated Fourier series was introduced in [11]. Special formulae were derived to determine the jumps of a $2\pi-$periodic function from V_p, $1 \le p < 2$, with a finite number of discontinuities.

For any function f, integrable on $[-\pi, \pi]$, we define $f^{(-r)}$, $r \in \mathbb{N}_0$, as

$$f^{(-r-1)} \equiv \int f^{(-r)},$$

where $f^{(0)} \equiv f$ and the constants of integration are successively determined by the condition

$$\int_{-\pi}^{\pi} f^{(-r)}(t)dt = 0, \qquad r \in \mathbb{N}_0.$$

We generalize a result of Kvernadze, Hagstrom and Shapiro [11, Theorem 4, p. 32] to the whole class of HBV functions. In doing so, we also prove that the finiteness assumption on the number of discontinuities is redundant here. The result is presented in the following theorem.

Theorem 2.1. (a) *Let $g \in HBV$ and $r = 0, 1, 2, \ldots$. Then, for any point x_0 one has*

$$\lim_{n\to\infty} n^{2r+1} R_n^{(-2r-1)}(g, x_0) = \frac{(-1)^{r+1}}{(2r+1)\pi}[g(x_0+0) - g(x_0-0)],$$

where $R_n(g,x)$ denotes the nth order tail of the Fourier series of g, i.e.

$$R_n(g,x) = \sum_{k=n}^{\infty} (a_k(g) \cos kx + b_k(g) \sin kx).$$

(b) *If* Λ *is such that* $\Lambda BV \supsetneqq HBV$, *the assertion* (a) *does not hold for* $\Lambda BV \setminus HBV$.

Proof. (a) Let $g \in HBV$ and $S_n^{'}(g, x_0) = \sum_{k=1}^{n} (-ka_k(g) \sin kx_0 + kb_k(g) \cos kx_0)$. For brevity, we denote by $c \equiv c(g, x_0) = \frac{1}{\pi}[g(x_0+0) - g(x_0-0)]$ the jump of the function g at x_0 and put $A_k \equiv A_k(g, x_0) = a_k(g) \sin kx_0 - b_k(g) \cos kx_0$. According to [3, Theorem 1, p. 268], one has

$$\lim_{n\to\infty} \frac{S_n^{'}(g, x_0)}{n} = \frac{1}{\pi}[g(x_0+0) - g(x_0-0)],$$

or equivalently

$$s_n \equiv s_n(g, x_0) \equiv c + \frac{1}{n}\sum_{k=1}^{n} kA_k = o(1), \ n \to \infty. \tag{2.1}$$

Multiplying (2.1) by n and rearranging the terms, we get

$$ns_n = \sum_{k=1}^{n} (kA_k + c) = o(n), \qquad n \to \infty. \tag{2.2}$$

Obviously,

$$ns_n - (n-1)s_{n-1} = nA_n + c \tag{2.3}$$

and

$$\begin{aligned} R_n^{(-2r-1)}(g, x_0) &= \sum_{k=n}^{\infty} \frac{(-1)^r (a_k(g) \sin kx_0 - b_k(g) \cos kx_0)}{k^{2r+1}} \\ &= (-1)^r \sum_{k=n}^{\infty} \frac{A_k}{k^{2r+1}}. \end{aligned}$$

Now, it is enough to prove that

$$n^{2r+1} \sum_{k=n}^{\infty} \frac{A_k}{k^{2r+1}} \to -\frac{c}{2r+1}, \qquad n \to \infty. \tag{2.4}$$

Using (2.1), (2.2), (2.3), Abel's partial summation formula and the fact that $\lim\limits_{n\to\infty} n^{2r+1}\sum_{k=n}^{\infty}\frac{1}{k^{2r+2}}=\frac{1}{2r+1}$, we get

$$\begin{aligned}
n^{2r+1}\sum_{k=n}^{\infty}\frac{A_k}{k^{2r+1}} &= n^{2r+1}\sum_{k=n}^{\infty}\frac{kA_k}{k^{2r+2}} = n^{2r+1}\sum_{k=n}^{\infty}\frac{ks_k-(k-1)\,s_{k-1}-c}{k^{2r+2}}\\
&= n^{2r+1}\sum_{k=n}^{\infty}\frac{ks_k-(k-1)\,s_{k-1}}{k^{2r+2}} - cn^{2r+1}\sum_{k=n}^{\infty}\frac{1}{k^{2r+2}}\\
&= n^{2r+1}\left\{-\frac{(n-1)\,s_{n-1}}{n^{2r+2}}+\sum_{k=n}^{\infty}\left[\frac{1}{k^{2r+2}}-\frac{1}{(k+1)^{2r+2}}\right]ks_k\right\}\\
&\qquad -cn^{2r+1}\sum_{k=n}^{\infty}\frac{1}{k^{2r+2}}.
\end{aligned}$$

Notice that $s_k=o(1)$ and

$$\left[\frac{1}{k^{2r+2}}-\frac{1}{(k+1)^{2r+2}}\right]k=\frac{(k+1)^{2r+2}-k^{2r+2}}{k^{2r+1}(k+1)^{2r+2}}=\frac{(2r+2)\xi_k^{2r+1}}{k^{2r+1}(k+1)^{2r+2}},$$

where $\xi_k\in(k,k+1)$. Thus,

$$\begin{aligned}
n^{2r+1}\sum_{k=n}^{\infty}\frac{A_k}{k^{2r+1}} &= -\frac{(n-1)}{n}s_{n-1}+o\left(n^{2r+1}\sum_{k=n}^{\infty}\frac{1}{k^{2r+2}}\right)\\
&- cn^{2r+1}\sum_{k=n}^{\infty}\frac{1}{k^{2r+2}}\to-\frac{c}{2r+1},\qquad n\to\infty.
\end{aligned}$$

The proof of (a) is complete.

(b) If Λ is such that $\Lambda BV \supsetneqq HBV$, by [3, Remark 4, p. 269] there exists a continuous function $g\in\Lambda BV$ with the property

$$\sum_{k=1}^{n}kA_k\neq O\left(n\right). \tag{2.5}$$

Suppose (2.4) holds true for g and some nonnegative integer r. Then, denoting $\sum_{k=n}^{\infty}\frac{A_k}{k^{2r+1}}$ by σ_n, we get

$$\begin{aligned}
\sum_{k=1}^{n}kA_k &= \sum_{k=1}^{n}k^{2r+2}\left(\sigma_k-\sigma_{k+1}\right)\\
&= \sigma_1+\sum_{k=2}^{n}\left(k^{2r+2}-(k-1)^{2r+2}\right)\sigma_k-n^{2r+2}\sigma_{n+1}\\
&= \sigma_1+\sum_{k=2}^{n}\left((2r+2)k^{2r+1}+O\left(k^{2r}\right)\right)\sigma_k-n^{2r+2}\sigma_{n+1}.
\end{aligned}$$

Hence,

$$\frac{1}{n}\sum_{k=1}^{n} kA_k = \frac{\sigma_1}{n} + \frac{1}{n}\sum_{k=2}^{n}(2r+2)k^{2r+1}\sigma_k + \frac{1}{n}\sum_{k=2}^{n} O\left(k^{2r}\right)\sigma_k - n^{2r+1}\sigma_{n+1}.$$

Letting $n\to\infty$ and having in mind that

$$\frac{\sigma_1}{n}\to 0,\ \frac{1}{n}\sum_{k=2}^{n}(2r+2)k^{2r+1}\sigma_k \sim (2r+2)n^{2r+1}\sigma_n \to -\frac{2r+2}{2r+1}c,$$

$$\frac{1}{n}\sum_{k=2}^{n} O\left(k^{2r}\right)\sigma_k \sim \frac{1}{n}O\left(n^{2r+1}\right)\sigma_n \to 0 \ \text{ and } \ n^{2r+1}\sigma_{n+1}\to -\frac{1}{2r+1}c,$$

we get

$$\frac{1}{n}\sum_{k=1}^{n} kA_k \to -c.$$

This obviously contradicts (2.5). □

Making use of [3, Theorem 1'(2)] and following the same line of argumentation as in the proof of Theorem 2.1, one obtains

Theorem 2.2. (a) *Let $g\in HBV$ and $r=1,\,2,\,\ldots$. Then, for any point x_0 we have*

$$\lim_{n\to\infty} n^{2r}\tilde{R}_n^{(-2r)}\left(g,x_0\right) = \frac{(-1)^{r+1}}{2r\pi}\left[g\left(x_0+0\right)-g\left(x_0-0\right)\right],$$

where $\tilde{R}_n\left(g,x\right)=\sum_{k=n}^{\infty}\left(a_k\left(g\right)\sin kx - b_k\left(g\right)\cos kx\right)$ is the tail of the conjugate Fourier series of g.

(b) *If Λ is such that $\Lambda BV \supsetneqq HBV$, the assertion* (a) *does not hold for $\Lambda BV\setminus HBV$.*

3. Generalized Fourier-Jacobi and Fourier-Chebyshev series

3.1. **Notation.** By $C^p[-1,1]$, $p\in\mathbb{N}_0$, we denote the space of $p-$times continuously differentiable functions on $[-1,1]$, where $C^0[-1,1]\equiv C[-1,1]$ is the space of continuous functions. Let $C^{-1}[-1,1]$ be the space of functions defined on $[-1,1]$ which may have discontinuities only of the first kind. We normalize these functions by imposing the condition $f(x)=\left(f(x+0)+f\left(x-0\right)\right)/2$. If $f\in C^{-1}[-1,1]$ has finitely many discontinuities, say $M\equiv M(f)$, let $x_m\equiv x_m(f)$ and $[f]_m\equiv f(x_m+0)-f(x_m-0)$, $m=1,\ldots,M$, denote these points of discontinuity and the associated jumps of the function f. The rth derivative of a function f which piecewise belongs to $C^p[-1,1]$, $p\geq r$, or which belongs to $C^{r-1}[-1,1]$, is defined as $f^{(r)}(x)=\left(f^{(r)}(x+0)+f^{(r)}\left(x-0\right)\right)/2$, whenever $f^{(r)}(x\pm 0)$ exist.

We say that $\mathbf{w}$ is a *generalized Jacobi weight*, i.e., $\mathbf{w} \in GJ$, if

$$\mathbf{w}(t) = h(t)\,(1-t)^{\alpha}\,(1+t)^{\beta}\,|t-\tilde{x}_1|^{\delta_1}\cdots|t-\tilde{x}_N|^{\delta_N},$$
$$h \in C\,[-1,1]\,,\ h(t) > 0\ \ (|t| \le 1)\,,\ \omega\,(h;t;[-1,1])\,t^{-1} \in L^1\,[0,1]\,,$$
$$-1 < \tilde{x}_1 < \cdots < \tilde{x}_N < 1,\ \alpha,\beta,\delta_1,\ldots,\delta_N > -1,$$

where $L^1\,[0,1]$ is the space of Lebesgue integrable functions on $[0,1]$ and

$$\omega\,(f;t;[-1,1]) = \max\{|f(x)-f(y)| : x,y \in [-1,1] \wedge |x-y| \le t\}$$

is the modulus of continuity of $f \in C\,[-1,1]$ on $[-1,1]$. It is always assumed that $\tilde{x}_0 = -1$, and $\tilde{x}_{N+1} = 1$. In addition, for a fixed $\varepsilon \in (0,(\tilde{x}_{\nu+1}-\tilde{x}_{\nu})\,/2)$, $\nu = 0,1,\ldots,N$, we set $\Delta\,(\nu;\varepsilon) = [\tilde{x}_{\nu}+\varepsilon,\tilde{x}_{\nu+1}-\varepsilon]$.

Let $\sigma\,(\mathbf{w}) = (P_n\,(\mathbf{w};x))_{n=0}^{\infty}$ be the system of algebraic polynomials

$$P_n\,(\mathbf{w};x) = \gamma_n(\mathbf{w})x^n + \textit{lower degree terms}$$

with positive leading coefficients $\gamma_n(\mathbf{w})$, which are orthonormal on $[-1,1]$ with respect to the weight $\mathbf{w} \in GJ$, i.e.,

$$\int_{-1}^{1} P_n\,(\mathbf{w};t)\,P_m\,(\mathbf{w};t)\,\mathbf{w}(t)dt = \delta_{nm}.$$

Such polynomials are called the *generalized Jacobi polynomials*.

If $f\mathbf{w} \in L[-1,1]$, $\mathbf{w} \in GJ$, then f has the Fourier series with respect to the system $\sigma\,(\mathbf{w})$. This series is the *generalized Fourier-Jacobi series* of f. Let $S_n(\mathbf{w};f;x)$ and $R_n(\mathbf{w};f;x)$ denote its nth partial sum and nth order tail, respectively, i.e.,

$$S_n(\mathbf{w};f;x) = \sum_{k=0}^{n-1} a_k\,(\mathbf{w};f)\,P_k\,(\mathbf{w};x) = \int_{-1}^{1} f\,(t)\,K_n\,(\mathbf{w};x;t)\,\mathbf{w}(t)dt,$$
$$R_n(\mathbf{w};f;x) = \sum_{k=n}^{\infty} a_k\,(\mathbf{w};f)\,P_k\,(\mathbf{w};x)\,,$$

where

$$a_k\,(\mathbf{w};f) = \int_{-1}^{1} f\,(t)\,P_k\,(\mathbf{w};t)\,\mathbf{w}(t)dt$$

is the kth Fourier coefficient of the function f, and

$$K_n\,(\mathbf{w};x;t) = \sum_{k=0}^{n-1} P_k\,(\mathbf{w};x)\,P_k\,(\mathbf{w};t)$$

is the Dirichlet kernel of the system $\sigma\,(\mathbf{w})$.

When $h(t) \equiv 1$, $|t| \le 1$, and $N = 0$ (i.e., a weight does not have singularities strictly inside the interval $(-1,1)$), $\mathbf{w} \in GJ$ is called a *Jacobi weight*. In this case, we use the commonly accepted notation "(α,β)" instead of "$\mathbf{w}$" throughout. We write $S_n^{(\alpha,\beta)}(f;x)$ for $S_n(\mathbf{w};f;x)$. The corresponding series is called the *Fourier-Jacobi series*. If $\alpha = \beta = -\frac{1}{2}$, Fourier-Jacobi series become *Fourier-Chebyshev series*.

3.2. **Equiconvergence.** We shall start with a simple proposition on convergence of generalized Fourier-Jacobi series for functions of harmonic bounded variation.

Proposition 3.1. *Let $f \in HBV$, $f\mathbf{w} \in L[-1,1]$, $\mathbf{w} \in GJ$. Then*

$$\lim_{n\to\infty} S_n(\mathbf{w}; f; x) = \frac{f(x+0) + f(x-0)}{2}$$

for every $x \in (-1,1)$, $x \neq \tilde{x}_1, \dots, \tilde{x}_N$.

Proof. Let $S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)$ be the nth partial sum of the Fourier-Chebyshev series of f. By [10, proof of Theorem 7, p. 185] we have the uniform equiconvergence of the Fourier-Chebyshev and generalized Fourier-Jacobi series for an arbitrary function $f \in HBV$ and a fixed $\varepsilon \in \left(0, \frac{\tilde{x}_{\nu+1} - \tilde{x}_\nu}{2}\right)$, $\nu = 0, 1, 2, \dots, N$, that is

$$\|S_n(\mathbf{w}; f; x) - S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\|_{C[\Delta(\nu;\varepsilon)]} = o(1). \tag{3.1}$$

Putting $x = \cos\theta$, $\theta \in (0,\pi)$, and $g(\theta) = f(\cos\theta)$, and taking into account that $g(\theta \mp 0) = f(x \pm 0)$, we get

$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) = S_n(g,\theta) \to \frac{g(\theta+0) + g(\theta-0)}{2} = \frac{f(x+0) + f(x-0)}{2}$$

as $n \to \infty$, according to Waterman [15, Theorem 2, p. 112]. For $x \neq \tilde{x}_1, \dots, \tilde{x}_N$, there exist ν_0 and ε such that $x \in [\tilde{x}_{\nu_0} + \varepsilon, \tilde{x}_{\nu_0+1} - \varepsilon]$. Now, we have

$$\begin{aligned} &\left| S_n(\mathbf{w}; f; x) - \frac{f(x+0) + f(x-0)}{2} \right| \\ &\leq \left| S_n(\mathbf{w}; f; x) - S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) \right| + \left| S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) - \frac{f(x+0) + f(x-0)}{2} \right| \\ &\leq \left\| S_n(\mathbf{w}; f; x) - S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) \right\|_{C[\Delta(\nu_0;\varepsilon)]} \\ &\quad + \left| S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) - \frac{f(x+0) + f(x-0)}{2} \right| = o(1). \end{aligned}$$

□

Corollary 3.2. *Let $f \in HBV$ and $\Delta(\nu;\varepsilon)$ be as above. Then,*

$$\|R_n(\mathbf{w}; f; x) - R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\|_{C[\Delta(\nu;\varepsilon)]} = o(1).$$

Proof. For $x \in (-1,1)$, $x \neq \tilde{x}_1, \dots, \tilde{x}_N$, Proposition 3.1 gives us

$$\begin{aligned} S_n(\mathbf{w}; f; x) &= \frac{f(x+0)+f(x-0)}{2} - R_n(\mathbf{w}; f; x), \\ S_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x) &= \frac{f(x+0)+f(x-0)}{2} - R_n^{(-\frac{1}{2},-\frac{1}{2})}(f; x). \end{aligned}$$

This and (3.1) yield the assertion. □

3.3. **Determination of a jump.** In order to prove an unconditional result on determination of a jump discontinuity of a function $f \in V_2$ by the tails of its integrated Fourier-Chebyshev series, we shall need the following lemma (cf. [2, Remark, p. 236]). For the sake of completeness of the argument, we include also the proof of the Lemma.

Lemma 3.3. *Let $f \in V_2$ be a $2\pi-$periodic function. Then, $n\sum_{k=n}^{\infty}\rho_k^2(f) = O(1)$, where $\rho_k^2(f) = a_k^2(f) + b_k^2(f)$ is the magnitude of the kth Fourier coefficient.*

Proof. If the Fourier series of f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx),$$

then the Fourier series of $f(\cdot + t)$ reads

$$f(x+t) \quad \sim \quad \frac{a_0}{2} + \sum_{m=1}^{\infty}(A_m(t)\cos mx + B_m(t)\sin mx),$$

where $A_m(t) = a_m \cos mt + b_m \sin mt$ and $B_m(t) = b_m \cos mt - a_m \sin mt$. Thus,

$$f(x+t) - f(x) \sim \sum_{m=1}^{\infty}\big((A_m(t) - a_m)\cos mx + (B_m(t) - b_m)\sin mx\big).$$

Simple calculations yield

$$A_m(t) - a_m = 2B_m\left(\frac{t}{2}\right)\sin\frac{mt}{2} \text{ and } B_m(t) - b_m = -2A_m\left(\frac{t}{2}\right)\sin\frac{mt}{2}.$$

Hence,

$$f(x+\frac{\pi}{n}) - f(x) \sim 2\sum_{m=1}^{\infty}\left[B_m\left(\frac{\pi}{2n}\right)\cos mx - A_m\left(\frac{\pi}{2n}\right)\sin mx\right]\sin\frac{m\pi}{2n}.$$

Parseval's identity gives us

$$\frac{1}{\pi}\int_0^{2\pi}\left[f\left(x+\frac{\pi}{n}\right) - f(x)\right]^2 dx = 4\sum_{m=1}^{\infty}\left[A_m^2\left(\frac{\pi}{2n}\right) + B_m^2\left(\frac{\pi}{2n}\right)\right]\sin^2\frac{m\pi}{2n}.$$

Since $A_m^2(t) + B_m^2(t) = a_m^2 + b_m^2 = \rho_m^2$, the last equation becomes

$$\frac{1}{\pi}\int_0^{2\pi}\left[f\left(x+\frac{\pi}{n}\right)-f(x)\right]^2 dx = 4\sum_{m=1}^{\infty}\rho_m^2\sin^2\frac{m\pi}{2n}.$$

Due to periodicity of f, we have

$$\frac{1}{\pi}\int_0^{2\pi}\left[f\left(x+k\frac{\pi}{n}\right)-f\left(x+(k-1)\frac{\pi}{n}\right)\right]^2 dx = 4\sum_{m=1}^{\infty}\rho_m^2\sin^2\frac{m\pi}{2n}$$

for every positive integer k. Therefore,

$$\sum_{k=1}^{2n}\frac{1}{\pi}\int_0^{2\pi}\left[f\left(x+k\frac{\pi}{n}\right)-f\left(x+(k-1)\frac{\pi}{n}\right)\right]^2 dx = 8n\sum_{m=1}^{\infty}\rho_m^2\sin^2\frac{m\pi}{2n}.$$

Changing the order of summation and integration on the left-hand side in the above equation and taking into account that $f \in V_2$, we get

$$n\sum_{m=1}^{\infty}\rho_m^2\sin^2\frac{m\pi}{2n} = O(1).$$

Now,

$$n\sum_{k=1}^{\infty}\rho_k^2\sin^2\frac{k\pi}{2n} \geq n\sum_{k=1}^{n}\rho_k^2\sin^2\frac{k\pi}{2n} \geq n\sum_{k=1}^{n}\rho_k^2\left(\frac{2}{\pi}\cdot\frac{k\pi}{2n}\right)^2 = \frac{1}{n}\sum_{k=1}^{n}k^2\rho_k^2.$$

Thus,

$$\frac{1}{n}\sum_{k=1}^{n}k^2\rho_k^2 = O(1).$$

Using Abel's partial summation formula, we get

$$\begin{aligned}
\sum_{k=n}^{m}\rho_k^2 &= \sum_{k=n}^{m}\frac{1}{k^2}\left(k^2\rho_k^2\right) = \frac{1}{m^2}\sum_{i=n}^{m}i^2\rho_i^2 + \sum_{k=n}^{m-1}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)\sum_{i=n}^{k}i^2\rho_i^2 \\
&= O(1)\left[\frac{1}{m}\cdot\frac{1}{m}\sum_{i=n}^{m}i^2\rho_i^2 + \sum_{k=n}^{m-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\frac{1}{k}\sum_{i=n}^{k}i^2\rho_i^2\right] \\
&= O(1)\left[\frac{1}{m} + \sum_{k=n}^{m-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\right] = O\left(\frac{1}{n}\right)
\end{aligned}$$

for arbitrary positive integer $m > n$. Hence,

$$n\sum_{k=n}^{\infty}\rho_k^2 = O(1).$$

□

Now, we turn our attention to determination of jump discontinuities by means of the tails of integrated Fourier-Chebyshev series.

Theorem 3.4. (a) *If $f \in HBV$ has finitely many discontinuities, then*

$$\lim_{n\to\infty} n\left[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\right]^{(-1)} = -\frac{(1-x^2)^{\frac{1}{2}}}{\pi}(f(x+0)-f(x-0)) \tag{3.2}$$

is valid for each fixed $x \in (-1,1)$, where

$$\left[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\right]^{(-1)} = \int_{-1}^{x} R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;y)dy.$$

(b) *If $f \in V_2$, then the relation (3.2) holds true without restriction on the number of discontinuities.*

Proof. Integrating $R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;y)$ on $[-1,x]$ and using the identity

$$R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;y) = R_n(g,\theta),$$

where $y = \cos\theta$, we get

$$\begin{aligned}
[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} &= \int_{\arccos x}^{\pi} R_n(g,\theta)\sin\theta\ d\theta \\
&= \left[\sin\theta R_n^{(-1)}(g;\theta)\right]\Big|_{\arccos x}^{\pi} - \int_{\arccos x}^{\pi} R_n^{(-1)}(g;\theta)\cos\theta\ d\theta \\
&= -\sin\eta\ R_n^{(-1)}(g;\eta) - \int_{\eta}^{\pi} R_n^{(-1)}(g;\theta)\cos\theta\ d\theta \\
&= -(1-x^2)^{\frac{1}{2}} R_n^{(-1)}(g;\eta) - \int_{\eta}^{\pi} R_n^{(-1)}(g;\theta)\cos\theta\ d\theta.
\end{aligned} \tag{3.3}$$

Here we put $\eta = \arccos x$.

(a) Any $g \in HBV$ with M points of discontinuity can be represented in the following form

$$g \equiv g_c + \frac{1}{\pi}\sum_{m=1}^{M}[g]_m G(\theta_m;\cdot), \tag{3.4}$$

where $G(\theta) = \dfrac{\pi-\theta}{2}$, $\theta \in (0,2\pi)$, is a $2\pi-$periodic sawtooth function, θ_m and $[g]_m$, $m = 1,2,\dots,M$, are the points of discontinuity and the associated jumps of the function g, respectively, and $G(\theta_m;\theta) = G(\theta-\theta_m)$. The function g_c is a $2\pi-$periodic continuous function, which is piecewise smooth on $[-\pi,\pi]$.

From $G(\theta) = \sum_{n=1}^{\infty}\frac{\sin n\theta}{n}$, we obviously have $R_n^{(-1)}(G;\theta) = O\left(\frac{1}{n}\right)$ and

$$nR_n^{(-1)}\left(G\left(\theta_m;\cdot\right);\theta\right) = O\left(1\right) \text{ uniformly, } m = 1,\dots,M. \tag{3.5}$$

Now, $g_c \in C \cap HBV$. Fourier series of g_c converges uniformly by a theorem of Waterman [15, Theorem 2, p. 112]. Since

$$R_n^{(-1)}(g_c;\theta) = \int R_n(g_c;\theta)d\theta$$

and $R_n(g_c;\theta)$ converges uniformly on $[-\pi,\pi]$, then

$$nR_n^{(-1)}(g_c;\theta) = o\,(1) \text{ uniformly} \tag{3.6}$$

by a theorem of Tong [14, Theorem, p. 252]. Combining (3.4), (3.5) and (3.6), we get

$$nR_n^{(-1)}(g;\theta) = O\,(1) \text{ uniformly.}$$

If θ is a point of continuity of g, Theorem 2.1 implies $nR_n^{(-1)}(g;\theta) \to 0$ as $n\to\infty$. Therefore,

$$\lim_{n\to\infty} nR_n^{(-1)}(g;\theta)\cos\theta = 0$$

everywhere except on a finite set of discontinuities of g. Applying the Lebesgue dominated convergence theorem [13, p. 267], we obtain

$$\lim_{n\to\infty}\int_{\arccos x}^{\pi} nR_n^{(-1)}(g;\theta)\cos\theta\; d\theta = 0. \tag{3.7}$$

Multiplying (3.3) by n, letting $n\to\infty$, using (3.7) and Theorem 2.1 with $r=0$ and taking into account that $f\,(x\pm 0) = g\,(\theta\mp 0)$, we get

$$\lim_{n\to\infty} n\left[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)\right]^{(-1)} = -\frac{(1-x^2)^{\frac{1}{2}}}{\pi}(f(x+0)-f(x-0)).$$

(b) For $g\in V_2$, applying the Cauchy-Schwartz inequality and Lemma 3.3, we get

$$\begin{aligned} n\left|R_n^{(-1)}(g;\theta)\right| &\leq n\sum_{k=n}^{\infty}\frac{|a_k(g)|+|b_k(g)|}{k} \\ &\leq \sqrt{2}n\left(\sum_{k=n}^{\infty}\left(a_k^2(g)+b_k^2(g)\right)\right)^{1/2}\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^{1/2} \\ &= \sqrt{2}nO\left(\frac{1}{\sqrt{n}}\right)O\left(\frac{1}{\sqrt{n}}\right) = O(1), \end{aligned}$$

i.e., $nR_n^{(-1)}(g;\theta) = O\,(1)$ uniformly. The Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty}\int_{\arccos x}^{\pi} nR_n^{(-1)}(g;\theta)\cos\theta d\theta = \int_{\arccos x}^{\pi}\lim_{n\to\infty} nR_n^{(-1)}(g;\theta)\cos\theta d\theta.$$

As already noticed, if θ is a point of continuity of the function g, Theorem 2.1 implies $nR_n^{(-1)}(g;\theta)\to 0$ as $n\to\infty$. Therefore, $\lim\limits_{n\to\infty} nR_n^{(-1)}(g;\theta)\cos\theta = 0$ everywhere except on a denumerable set of discontinuities of g. Thus, (3.7) and consequently (3.2) hold true for $g\in V_2$ without finiteness restriction on the number of discontinuities of g. □

Remark 3.5. Theorem 3.4 transfers a corresponding result by Kvernadze, Hagstrom and Shapiro [11] from the trigonometric case to the setting of Fourier-Chebyshev series. At the same time, it generalizes their result in two directions. If the finiteness assumption on the number of discontinuities of a function is kept, then we can deal with the whole class HBV, as demonstrated in part (a) of the proof. On the other hand, if the attention is restricted to the subclass V_2, then part (b) shows that the finiteness assumption can be removed.

Remark 3.6. In view of Theorem 2.A above, the part (b) of Theorem 3.4 is obviously valid for the Watermann class $\left\{n^{\frac{1}{2}}\right\}BV$ and Chanturiya's classes $V\left[n^{\alpha}\right]$, $0<\alpha<\frac{1}{2}$.

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