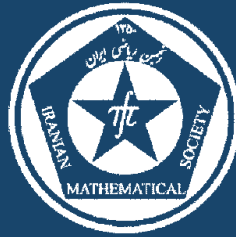


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A COMPREHENSIVE UNIFIED MODEL OF STRUCTURAL AND REDUCED FORM TYPE FOR DEFAULTABLE FIXED INCOME BONDS

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ABSTRACT. The aim of this paper is to generalize the comprehensive structural model for defaultable fixed income bonds (considered in R. Agliardi, A comprehensive structural model for defaultable fixed-income bonds, *Quant. Finance* 11 (2011), no. 5, 749–762.) into a comprehensive unified model of structural and reduced form models. In our model the bond holders receive the deterministic coupon at predetermined coupon dates and the face value (debt) and the coupon at the maturity as well as the effect of government taxes which are paid on the proceeds of an investment in bonds is considered. The expected default event occurs when the equity value is not enough to pay coupon or debt at the coupon dates or maturity and an unexpected default event can occur at any time interval with the probability of given default intensity. We consider the model and pricing formula for equity value and using it calculate expected default barrier. Then we provide pricing model and formula for defaultable corporate bonds with discrete coupons, and consider the duration and the effect of the government taxes.

Keywords: Defaultable corporate bond, discrete coupon, tax, default intensity, default barrier, duration.

MSC(2010): Primary: 35C15; Secondary: 91G80.

1. Introduction

The study on defaultable corporate bonds is recently one of the most interesting areas of cutting edge in financial mathematics.

As well known, there are two main approaches to modeling credit risk and pricing defaultable corporate bonds; one is the *structural approach* and the other one is the *reduced form approach*. In the structural method, we think that the default event occurs when the firm value is not enough to repay debt, that is,

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the firm value reaches a certain lower threshold (*default barrier*) from the above. Such a default is predictable and thus we call it *expected default*. In the reduced-form approach, the default is treated as an unpredictable event governed by a default intensity process. In this case, the default event can occur without any correlation with the firm value and such a default is called *unexpected default*. In the reduced-form approach, if the default probability in time interval $[t, t + \Delta t]$ is λ , then $\lambda \Delta t$ is called *default intensity* or *hazard rate* [4, 8, 10, 19, 20]. The *third approach* is to unify the structural and reduced form approaches [6, 7, 12, 13]. As for the history of the above three approaches and their advantages and shortcomings, readers can refer to [20] and the introductions of [6, 19]. Combining the elements of the structural approach and reduced-form approach is one of the recent trends [2, 3, 7].

On the other hand, many models related to coupon bonds approximate actual coupon bearing debts with a continuous coupon stream or even zero coupon contracts, but such approaches have restrictions [10].

There has been relatively little work on the most realistic payout structure providing fixed discrete coupons [1]. Geske [9] (1977) is the first who study this problem, where discrete interest payouts prior to maturity were modeled as determinants of default risk. The introduction and the conclusions of [1] include much useful information about corporate discrete coupon bonds. Recently, Agliardi [1] (2011) generalized the Geskes formula for defaultable coupon bonds, incorporated a stochastic risk free term structure and the effects of bankruptcy cost and government taxes on bond interest and studied the duration of defaultable bonds. Agliardi's approach in [1] to corporate coupon bonds is a kind of structural approaches as shown in its title. In [16, 17], the authors tried to generalize the result of [1] into a comprehensive unified model of structural and reduced form models. Unlike [1], the authors of [16] calculated the expected barrier from the bond price. In [17], the authors calculated the expected barrier from the equity price like [1]. The main different point of the approach of [16] and [17] is that they assumed that the discrete coupons are *discounted values of those at the maturity* in order to get analytical pricing formulae, which is different from [1] and seems not compatible with financial reality.

The aim of this paper is to generalize the comprehensive structural model for defaultable fixed income bonds (considered in [1]) into a comprehensive unified model of structural and reduced form models. In our model the bond holders receive the deterministic coupon at predetermined coupon dates and the face value (debt) and the coupon at the maturity like [1]. The effect of government taxes which are paid on the proceeds of an investment in bonds is considered under *constant short rate* while Agliardi [1] considered *stochastic model* of short rate such as Vasicek model. The aim of this change is to get analytical pricing formulae.

The expected default event occurs when the equity value is not enough to pay coupon or debt at the coupon dates or maturity, and unexpected default event can occur at any time interval with the probability of given default intensity (provided by a step function of time variable) multiplied by the length of the time interval. We consider the model and pricing formula for equity value and using it calculate expected default barrier. Then we provide pricing model and formula for defaultable corporate bonds with discrete coupons, and consider the duration and the effect of the government taxes.

In our case, the pricing model between every adjacent two coupon dates becomes an inhomogeneous Black-Scholes equation with discontinuous terminal value condition and can be solved by the method of higher order binaries with constant coefficients which is used to price zero-coupon bond in [18].

The remainder of the article is organized as follows. In Section 2 we consider the model and pricing formula for equity value and use it to calculate expected default barrier. Then we provide pricing model and formula for defaultable discrete coupon corporate bonds without consideration of taxes. In Section 3 we study the duration of our bond. In Section 4 we consider the effect of taxes. Sections 5 is the appendix where we give the sketch of the proof of pricing formulae for equity value and defaultable discrete coupon bond. The notions and the pricing formulae of higher order binaries with constant coefficients and some of their properties which are used in Sections 2, 3 and 4 will be referred to [15] or [18].

2. Mathematical model and pricing formulae for discrete coupon bond with both expected and unexpected defaults

2.1. **Assumptions.** (1) Short rate r is constant. Then the price of default free zero coupon bond with maturity T and face value 1 is $Z(t; T) = e^{-r(T-t)}$.

(2) The firm value $V(t)$ follows a geometric Brownian motion

$$dV(t) = (r - b)V(t)dt + s_V \cdot V(t)dW(t)$$

under the risk neutral martingale measure. Here the volatility s_V of the firm value is a constant and the firm continuously pays out dividend in rate b (constant) for a unit of firm value.

(3) Let $0 = T_0 < T_1 < \dots < T_{N_1} < T_N = T$ and let T be the maturity of our corporate bond with face value F (unit of currency). At time T_i ($i = 1, \dots, N - 1$) the bond holder receives the coupon of quantity C_i (unit of currency) from the firm and at time $T_N = T$ the bond holder receives the face value F and the last coupon C_N (unit of currency). (That is, the coupons are the same as in [1].)

(4) The expected default occurs only at time T_i when the equity of the firm is not enough to pay debt and coupon. If the expected default occurs, the bond

holder receives $\delta \cdot V$ as *default recovery*. Here δ ($0 \leq \delta \leq 1$) is called a *fractional recovery rate* of firm value at default.

(5) The unexpected default can occur at any time. For every $i = 0, \dots, N - 1$ the unexpected default probability in the interval $[t, t + \Delta t] \cap (T_i, T_{i+1})$ is $\lambda_i \Delta t$. Here the *default intensity* λ_i is a constant. If the unexpected default occurs at time $t \in (T_i, T_{i+1})$, the bond holder receives $\min\{\delta \cdot V, \sum_{k=i+1}^N C_k Z(t; T_K) + FZ(t; T_N)\}$ as default recovery. Here the *reason* why the expected default recovery and unexpected recovery are given in *different forms* is to avoid the possibility of *paying more than the price of a default free discrete coupon bond* with the face value F and coupons C_k (at time T_k) as a default recovery when the unexpected default event occurs. In what follows we call the unexpected default occurred at time $t \in (T_i, T_{i+1})$ with default recovery $\sum_{k=i+1}^N C_k Z(t; T_K) + FZ(t; T_N)$ as the *unexpected default without loss*.

(6) In the subinterval (T_i, T_{i+1}) , the price of our corporate bond and the equity of the firm are given by a sufficiently smooth function $B_i(V, t)$ and $E_i(V, t)$ ($i = 0, \dots, N - 1$), respectively.

2.2. Mathematical model for equity and expected default barriers.

According to [17], we can derive a PDE for the equity E when the firm has constant default intensity λ under the assumptions (1) and (2).

$$\frac{\partial E}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 E}{\partial V^2} + (r - b)V \frac{\partial E}{\partial V} - (r + \lambda)E = 0.$$

From the above PDE of the equity and the above assumption (5) and (6) the equity price E_i satisfies the following PDE in every subinterval (T_i, T_{i+1}) ($i = 0, \dots, N - 1$):

$$(2.1) \quad \frac{\partial E_i}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 E_i}{\partial V^2} + (r - b)V \frac{\partial E_i}{\partial V} - (r + \lambda_i)E_i = 0.$$

From assumption (3) we have:

$$(2.2) \quad \begin{aligned} E_N(V, T_N) &= (V - F - C_N) \cdot 1\{V > F + C_N\}, \\ E_i(V, T_{i+1}) &= [E_{i+1}(V, T_{i+1}) - C_{i+1}] \cdot 1\{E_i(V, T_{i+1}) > C_{i+1}\}, \quad i = 0, \dots, N - 2. \end{aligned}$$

We will use the following notation for simplicity.

$$(2.3) \quad \begin{aligned} K_N &= F + C_N; \quad \bar{c}_N = F + C_N; \\ \bar{c}_i &= C_i, \quad i = 1, \dots, N - 1; \\ \Delta T_i &= T_{i+1} - T_i, \quad i = 0, \dots, N - 1. \end{aligned}$$

Remark 2.1. \bar{c}_i is the payoff to bondholders at time T_i ($i = 1, \dots, N$) and K_N denotes the default barrier at time T_N .

Theorem 2.2. (Equity Price) The solutions of (2.1) and (2.2) are provided as follows:

(2.4)

$$E_i(V, t) = e^{-\lambda_i(T_{i+1}-t)} \left\{ e^{-\sum_{k=i+1}^{N-1} \lambda_k \Delta T_k} A_{K_{i+1} \dots K_N}^+ \dots^+ (V, t; T_{i+1}, \dots, T_N; r, b, sV) - \sum_{m=i}^{N-1} \bar{c}_{m+1} e^{-\sum_{k=i+1}^m \lambda_k \Delta T_k} B_{K_{i+1} \dots K_{m+1}}^+ \dots^+ (V, t; T_{i+1}, \dots, T_{m+1}; r, b, sV) \right\} \\ (T_i < t \leq T_{i+1}, V > 0, i = 0, \dots, N - 1).$$

Here $B_{K_1 \dots K_m}^+ \dots^+ (x, t; t_1, \dots, t_m; r, q, \sigma)$ and $A_{K_1 \dots K_m}^+ \dots^+ (x, t; t_1, \dots, t_m; r, q, \sigma)$ are respectively the prices of m -th order bond and asset binaries with risk free rate r , dividend rate q and volatility σ (see [15, Theorem 1]) and K_i ($i = 1, \dots, N - 1$) is the unique root of the equation $E_i(V, T_i) = C_i$. Using multi-variate normal distribution functions, (2.4) are represented in terms of the debt F , the coupons C_i and the firm value V as follows:

(2.5)

$$E_i(V, t) = V e^{-(\lambda_i+b)(T_{i+1}-t) - \sum_{k=i+1}^{N-1} (\lambda_k+b)\Delta T_k} N_{N-i}(d_{i+1}^+(t), \dots, d_N^+(t); A_{i+1,N}(t)) - e^{-\lambda_i(T_{i+1}-t)} \left\{ (F + C_N)Z(t; T_N) e^{-\sum_{k=i+1}^{N-1} \lambda_k \Delta T_k} N_{N-i}(d_{i+1}^-(t), \dots, d_N^-(t); A_{i+1,N}(t)) - \sum_{m=i}^{N-2} C_{m+1}Z(t; T_{m+1}) e^{-\sum_{k=i+1}^m \lambda_k \Delta T_k} N_{m+1-i}(d_{i+1}^-(t), \dots, d_N^-(t); A_{i+1,m+1}(t)) \right\}.$$

Here the cumulative distribution function $N_m(a_1, \dots, a_m; A)$ of m -variate normal distribution with zero mean vector and a covariance matrix A^{-1} , $d_i^\pm(t)$ and $A_{k,m}^{-1}(t) = (r_{ij}(t))_{i,j=k}^m$ are given by:

(2.6)

$$N_m(a_1, \dots, a_m; A) = \int_{-\infty}^{a_1} \dots \int_{-\infty}^{a_m} \frac{1}{(\sqrt{2\pi})^m} \sqrt{\det A} \exp\left(-\frac{1}{2}y^T A y\right) dy, \\ d_i^\pm(t) = \frac{\ln \frac{V}{K_i} + (r - b \pm \frac{sV}{2})(T_i - t)}{sV\sqrt{T_i - t}}, \quad i = 1, \dots, N - 1, \\ d_N^\pm(t) = \frac{\ln \frac{V}{F+C_N} + (r - b \pm \frac{sV}{2})(T_N - t)}{sV\sqrt{T_N - t}}, \\ r_{ij}(t) = \sqrt{\frac{T_i - t}{T_j - t}}, \quad r_{ji}(t) = r_{ij}(t), \quad i \leq j \quad (i, j = k, \dots, m).$$

Remark 2.3. Theorem 1 gives us the expected default barrier K_i at time T_i ($i = 1, \dots, N - 1$). That is, if $V < K_i$ at time T_i , then the expected default occurs. Note that the difference of (2.4) and (2.5) from [17, (2.13), (2.14), and (2.15)] comes from the coupon structures' difference. If $b = 0$ and $\lambda_k = 0$ ($k =$

$0, \dots, N-1$), then our pricing formula (2.5) has the same type with the formula (2) of [1] at page 751 but we should note that here short rate r is constant. If $b = 0$, $C_k = 0$ and $\lambda_k = 0$ ($k = 0, \dots, N-1$), then the formula (2.5) with (2.6) includes the formula (12) of Merton (1974) [14].

In what follows, we provide a numerical example for the equity calculated using Matlab. Here the basic data are as follows: $N = 2$, $T_1 = 3$, $T_2 = 6$ (annum), $r = 0.02$, $b = 0.05$, $s_V = 1.0$, $\lambda_0 = 0.002$, $\lambda_1 = 0.004$, $\delta = 0.5$, $F = 10$, $C_1 = C_2 = 1.0$, and in Figure 1 we give the graph (V - Equity price) when V changes from 0 to 30 and the default barrier K_1 .

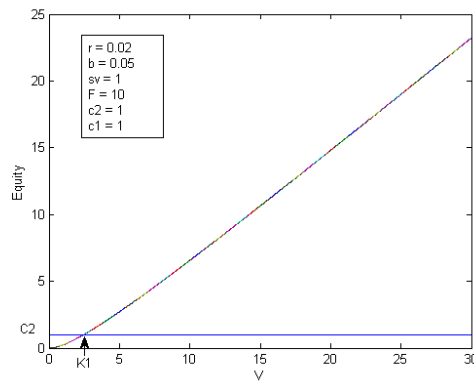


FIGURE 1. Plot (V - Equity) and the default barrier K_1

In Figure 2 we give the graphs (t - Equity price) when $V = 20$ and $C_1 = C_2 = 2, 1, 0.1$. The jumps at $T_1 = 3$ reflect the coupon payment.

2.3. Model and pricing formulae of the defaultable discrete coupon bond. In this subsection we derive the representation of the price $B_i(V, t)$ of the defaultable discrete coupon bond in the interval $(T_i, T_{i+1}]$ ($i = 0, \dots, N-1$). In this subsection we neglect the effect of the taxation. We use the notation of (2.3) and the following notation

$$(2.7) \quad \Phi_i(t) = \sum_{k=i+1}^N C_k Z(t; T_K) + FZ(t; T_N), \quad t \in (T_i, T_{i+1}].$$

That is, $\Phi_i(t)$ is time t -value of default free discrete coupon bond with the maturity T_N - face value F and coupons C_{i+1}, \dots, C_N at time T_{i+1}, \dots, T_N , respectively.

Now we consider the defaultable discrete coupon bond under the assumptions (1)–(6) in Subsection 2.1. From assumptions (5) and (6), using the

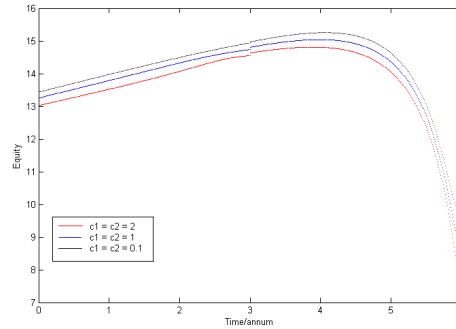


FIGURE 2. Plot $(t - Equity)$ when $V = 20$.

method of [20] we can know that our bond price $B_i(V, t)$ satisfies the following PDE in every subinterval (T_i, T_{i+1}) ($i = 0, \dots, N - 1$):

$$(2.8) \quad \frac{\partial B_i}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 B_i}{\partial V^2} + (r - b)V \frac{\partial B_i}{\partial V} - (r + \lambda_i)B_i + \lambda_i \min\{\delta V, \Phi_i(t)\} = 0, \\ T_i < t < T_{i+1}, V > 0.$$

In Theorem 2.2, we have calculated the expected default barrier K_i ($i = 1, \dots, N$). (See Remark 2.3.) Thus from assumptions (3) and (4) we have the following terminal value conditions:

$$(2.9) \quad B_{N-1}(V, T_N) = \bar{c}_N \cdot 1\{V > K_N\} + \delta V \cdot 1\{V \leq K_N\}, V > 0, \\ B_i(V, T_{i+1}) = [B_{i+1}(V, T_{i+1}) + \bar{c}_{i+1}] \cdot 1\{V > K_{i+1}\} + \delta V \cdot 1\{V \leq K_{i+1}\}, \\ V > 0 \text{ and } i = 0, \dots, N - 2.$$

The problem (2.8) and (2.9) is just the *pricing model of our defaultable discrete coupon bond*.

Remark 2.4. In our model (2.8) and (2.9) the consideration of *unexpected default risk* and *dividend* of firm value is added to the model on defaultable discrete coupon bond of [1]. Another difference from [1]'s approach is that risk free rate r is constant (but not stochastic process). The difference from the model [17, (2.18) and (2.19)] is the coupon structures. Our model (2.8) and (2.9) has some difference in default barriers and default recovery from the model [18, (21)] for defaultable zero coupon bond with discrete default information and endogenous default recovery but it is very similar with the fundamental problem [18, (32)] which is a terminal value problem for an inhomogeneous Black - Scholes equation with constant coefficients and binary type terminal value.

Theorem 2.5 (Discrete Coupon Bond Price). *The solution to the problem (2.8) and (2.9) is given as follows:*

(2.10)

$$\begin{aligned}
 & B_i(V, t) \\
 = & e^{-\lambda_i(T_{i+1}-t)} \left\{ \sum_{m=i}^{N-1} e^{-\sum_{k=i+1}^m \lambda_k \Delta T_k} \left[\bar{c}_{m+1} B_{K_{i+1} \dots K_{m+1}}^+ \dots^+ (V, t; T_{i+1}, \dots, T_{m+1}) \right. \right. \\
 & \left. \left. + \delta \cdot A_{K_{i+1} \dots K_m K_{m+1}}^+ \dots^+ (V, t; T_{i+1}, \dots, T_m, T_{m+1}) \right] \right. \\
 & + \sum_{m=i+1}^{N-1} \lambda_m e^{-\sum_{k=i+1}^{m-1} \lambda_k \Delta T_k} \\
 & \left. \cdot \int_{T_m}^{T_{m+1}} e^{-\lambda_m(\tau-T_m)} \left[\Phi_m(\tau) \cdot B_{K_{i+1} \dots K_m M_{m+1}(\tau)}^+ \dots^+ (V, t; T_{i+1}, \dots, T_m, \tau) \right. \right. \\
 & \left. \left. + \delta \cdot A_{K_{i+1} \dots K_m M_{m+1}(\tau)}^+ \dots^+ (V, t; T_{i+1}, \dots, T_m, \tau) \right] d\tau \right\} \\
 & + \lambda_i \int_t^{T_{i+1}} e^{-\lambda_i(\tau-t)} \left[\Phi_i(\tau) \cdot B_{M_{i+1}(\tau)}^+ (V, t; \tau) + \delta \cdot A_{M_{i+1}(\tau)}^- (V, t; \tau) \right] d\tau.
 \end{aligned}$$

Here $B_{K_1 \dots K_m}^+ \dots^+$ and $A_{K_1 \dots K_{m-1} K_m}^+ \dots^+$ are respectively the prices of m -th order bond and asset binaries with risk free rate r , dividend rate b and volatility s_V (cf. [15, Theorem 1]), \bar{c}_i and K_i ($i = 1, \dots, N$) are the same as in Theorem 2.2; $M_{i+1}(t) = \delta^{-1} \Phi_i(t)$. In particular the initial price of the bond is given by

(2.11)

$$\begin{aligned}
 B_0 &= B_0(V_0, 0) \\
 &= \sum_{m=0}^{N-1} e^{-\sum_{k=0}^{m-1} \lambda_k \Delta T_k} \left\{ e^{-\lambda_m \Delta T_m} \left[\bar{c}_{m+1} B_{K_1 \dots K_{m+1}}^+ \dots^+ (V_0, 0; T_1, \dots, T_{m+1}) \right. \right. \\
 & \left. \left. + \delta \cdot A_{K_1 \dots K_m K_{m+1}}^+ \dots^+ (V_0, 0; T_1, \dots, T_m, T_{m+1}) \right] \right. \\
 & + \lambda_m \int_{T_m}^{T_{m+1}} e^{-\lambda_m(\tau-T_m)} \left[\Phi_m(\tau) \cdot B_{K_1 \dots K_m M_{m+1}(\tau)}^+ \dots^+ (V_0, 0; T_1, \dots, T_m, \tau) \right. \\
 & \left. \left. + \delta \cdot A_{K_1 \dots K_m M_{m+1}(\tau)}^+ \dots^+ (V_0, 0; T_1, \dots, T_m, \tau) \right] d\tau \right\}, V > 0.
 \end{aligned}$$

Remark 2.6. (1) The proof of Theorem 2.5 is similar to the solution of [18, (32)] and we give a sketch of the proof in the Appendix. (2) The problem (2.8) and (2.9) is an inhomogeneous Black-Scholes equation with discontinuous terminal value. Thus using the results of [16], we can investigate such properties of

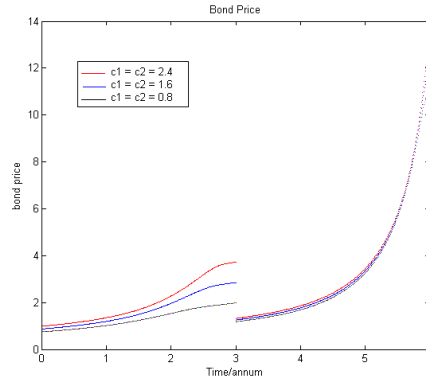


FIGURE 3. Plot (t -bond price). The jumps reflect coupon payment.

$B_i(V, t)$ as monotonicity, boundedness or gradient estimate and so on.

In what follows, we provide a numerical example for the bond price calculated using Matlab. Here the basic data are the same as in Subsection 2.2 but C_1 and C_2 are different.

Denote the *leverage ratio* by $L = F/V_0$ and the k -th *coupon rate* by $c_k = C_k/F$ ($k = 1, \dots, N$). Then we have the following *representation* of the initial price of the our defaultable discrete coupon bond in terms of leverage ratio, face value, coupon rates, default recovery rate and initial price of the default free zero coupon bonds with maturity T_k (coupon dates).

Corollary 2.7. *Under the assumption of Theorem 2.5, the initial price of the bond can be represented as follows:*

(2.12)

$$\begin{aligned}
 B_0 = & B_0(L, F, c_1, \dots, c_N; \delta, \lambda_0, \dots, \lambda_{N-1}; r, b) \\
 = & F \left\{ e^{-\sum_{k=0}^{N-1} \lambda_k \Delta T_k} Z(0; T_N) N_N(d_1^-, \dots, d_N^-; A_N) \right. \\
 & + \sum_{m=0}^{N-1} e^{-\sum_{k=0}^{m-1} \lambda_k \Delta T_k} \left[e^{-\lambda_m \Delta T_m} c_{m+1} Z(0; T_{m+1}) N_{m+1}(d_1^-, \dots, d_{m+1}^-; A_{m+1}) \right. \\
 & + \lambda_m \phi_m(0) \int_{T_m}^{T_{m+1}} e^{-\lambda_m(\tau - T_m)} N_{m+1}(d_1^-, \dots, d_m^-, \tilde{d}_{m+1}^-(\tau, \delta); \tilde{A}_{m+1}(\tau)) d\tau \left. \right] \\
 & + \frac{\delta}{L} \sum_{m=0}^{N-1} e^{-\sum_{k=0}^{m-1} (\lambda_k + b) \Delta T_k} \left[e^{-(\lambda_m + b) \Delta T_m} N_{m+1}(d_1^+, \dots, d_m^+, -d_{m+1}^+; A_{m+1}^-) \right. \\
 & \left. \left. + \lambda_m \int_{T_m}^{T_{m+1}} e^{-(\lambda_m + b)(\tau - T_m)} N_{m+1}(d_1^+, \dots, d_m^+, -\tilde{d}_{m+1}^+(\tau, \delta); \tilde{A}_{m+1}^-(\tau)) d\tau \right] \right\}.
 \end{aligned}$$

Here $\phi_m(0) = \Phi_m(0)/F$ and $N_m(a_1, \dots, a_m; A)$, $d_i^\pm(0)$ and $A_m^{-1} = A_{1,m}^{-1}(0) = (r_{ij})_{i,j=k}^m$ are given by (2.6),

$$(2.13) \quad \tilde{d}_i^\pm(\tau, \delta) = \frac{\ln \frac{V}{M_i(\tau)} + (r - b \pm \frac{s_V^2}{2})\tau}{s_V \sqrt{\tau}}, \quad T_{i-1} \leq \tau < T_i; \quad i = 1, \dots, N,$$

$(\tilde{A}_m(\tau))^{-1} = (\tilde{r}_{ij}(\tau))_{i,j=1}^m$ is the matrix whose m -th row and column are given by

$$(2.14) \quad \tilde{r}_{im}(\tau) = \sqrt{T_i/\tau}, \tilde{r}_{mi}(\tau) = \tilde{r}_{im}(\tau), \quad i < m \quad (i = 1, \dots, m-1),$$

and other elements coincide with those of $(A_m)^{-1}$. The matrices $(A_m^-)^{-1} = (r_{ij}^-)_{i,j=1}^m$ and $(\tilde{A}_m^-(\tau))^{-1} = (\tilde{r}_{ij}^-(\tau))_{i,j=1}^m$ have such m -th rows and columns that

$$(2.15) \quad r_{im}^- = -r_{im}, \quad r_{mi}^- = r_{im}^-; \quad \tilde{r}_{im}^-(\tau) = -\tilde{r}_{im}(\tau), \quad \tilde{r}_{mi}^-(\tau) = \tilde{r}_{im}(\tau), \\ i < m \quad (i = 1, \dots, m-1),$$

and other elements coincide with those of $(A_m)^{-1}$ and $(\tilde{A}_m(\tau))^{-1}$, respectively.

Remark 2.8. If $b = 0$ and $\lambda_k = 0$ ($k = 0, \dots, N-1$), then our pricing formula (2.12) nearly coincides with formula (5) of [1] at page 752 and the only difference comes from the assumption of short rate. If $b = 0, C_k = 0$, and $\lambda_k = 0$ ($k = 0, \dots, N-1$), then the formula (2.12) includes the formula (13) of Merton (1974) [14].

In what follows, we use the following notation for simplicity:

$$(2.16) \quad G_N^+ = G_N^+(\lambda_0, \dots, \lambda_{N-1}; b) = e^{-\sum_{k=0}^{N-1} (\lambda_k + b) \Delta T_k} N_N(d_1^+, \dots, d_N^+; A_N), \\ G_{m+1}^- = G_{m+1}^-(\lambda_0, \dots, \lambda_m) = e^{-\sum_{k=0}^m \lambda_k \Delta T_k} N_{m+1}(d_1^-, \dots, d_{m+1}^-; A_{m+1}), \\ g_{m+1}^-(\tau) = g_{m+1}^-(\tau, \delta, \lambda_0, \dots, \lambda_m) \\ = e^{-\lambda_m(\tau - T_m) - \sum_{k=0}^{m-1} \lambda_k \Delta T_k} N_{m+1}(d_1^-, \dots, d_m^-, \tilde{d}_{m+1}^-(\tau, \delta); \tilde{A}_{m+1}(\tau)), \\ \tilde{G}_{m+1} = \tilde{G}_{m+1}(\lambda_0, \dots, \lambda_m; b) \\ = e^{-\sum_{k=0}^m (\lambda_k + b) \Delta T_k} N_{m+1}(d_1^+, \dots, d_m^+, -d_{m+1}^+; A_{m+1}^-), \\ g_{m+1}(\tau) = \tilde{g}_{m+1}(\tau, \delta, \lambda_0, \dots, \lambda_m; b) \\ = e^{-(\lambda_m + b)(\tau - T_m) - \sum_{k=0}^{m-1} (\lambda_k + b) \Delta T_k} \\ \cdot N_{m+1}(d_1^+, \dots, d_m^+, -\tilde{d}_{m+1}^+(\tau, \delta); \tilde{A}_{m+1}^-(\tau)), \\ m = 0, \dots, N-1.$$

Then from (2.5) and (2.12) we can write as follows:

$$\begin{aligned}
 (2.17) \quad E_0(V_0, r, 0) &= V_0 G_N^+ - (F + C_N) Z(0, T_N) G_N^- - \sum_{m=1}^{N-1} C_m Z(0; T_m) G_m^-, \\
 B_0(V_0, r, 0) &= (F + C_N) Z(0, T_N) G_N^- + \sum_{m=1}^{N-1} C_m Z(0; T_m) G_m^- \\
 &\quad + \sum_{m=0}^{N-1} \lambda_m \Phi_m(0) \int_{T_m}^{T_{m+1}} g_{m+1}^-(\tau) d\tau \\
 &\quad + \delta \cdot V_0 \sum_{m=0}^{N-1} \left(\tilde{G}_{m+1} + \lambda_m \int_{T_m}^{T_{m+1}} \tilde{g}_{m+1}(\tau) d\tau \right).
 \end{aligned}$$

If we take the sum of the above expressions, we have

$$\begin{aligned}
 E_0 + B_0 &= V_0 G_N^+ + \delta \cdot V_0 \sum_{m=0}^{N-1} \left(\tilde{G}_{m+1} + \lambda_m \int_{T_m}^{T_{m+1}} \tilde{g}_{m+1}(\tau) d\tau \right) \\
 &\quad + \sum_{m=0}^{N-1} \lambda_m \Phi_m(0) \int_{T_m}^{T_{m+1}} g_{m+1}^-(\tau) d\tau.
 \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 V_0 = E_0 + B_0 + V_0 &\left[1 - G_N^+ - \delta \sum_{m=0}^{N-1} \left(\tilde{G}_{m+1} + \lambda_m \int_{T_m}^{T_{m+1}} \tilde{g}_{m+1}(\tau) d\tau \right) \right] \\
 &\quad - \sum_{m=0}^{N-1} \lambda_m \Phi_m(0) \int_{T_m}^{T_{m+1}} g_{m+1}^-(\tau) d\tau.
 \end{aligned}$$

This shows that the Modigliani-Miller theorem holds (that is, $V = Equity + Debt$) when $\delta = 1$ and $\lambda_k = b = 0$. Here we considered the following fact [1]:

$$1 - N_N(d_1^+, \dots, d_N^+; A_N) = \sum_{m=0}^{N-1} N_{m+1}(d_1^+, \dots, d_m^+, -d_{m+1}^+; A_{m+1}^-).$$

In the case with possibility of default, it is modified as follows [1]:

$$V = Equity + Debt + Default Costs (bankruptcy costs).$$

From this fact, we have the representation of bankruptcy costs.

Corollary 2.9. (Bankruptcy Cost) *The current value of bankruptcy cost is as follows:*

$$(2.18) \quad V_0 - V_0 G_N^+ - \sum_{m=0}^{N-1} \left\{ \delta V_0 \tilde{G}_{m+1} + \lambda_m \int_{T_m}^{T_{m+1}} [\delta V_0 \tilde{g}_{m+1}(\tau) + \Phi_m(0) g_{m+1}^-(\tau)] d\tau \right\}.$$

Remark 2.10. In the formula (2.18), let $\lambda_k = b = 0$, then we have the formula (6) of [1] at page 752.

3. Duration

In this section we study the problem of duration for defaultable discrete coupon bond under the united model of structural and reduced form approaches we developed in the previous section. According to [1], when $B(V, t; r)$ is bond price, we use the following definition for duration with respect to the short rate:

$$(3.1) \quad D(V, t) = -\frac{1}{B(V, t; r)} \partial_r B(V, t; r).$$

Now we calculate the *duration* of our *defaultable discrete coupon bond*. We use the notation of (2.3). In (2.17) the third term can be rewritten as follows:

$$\sum_{m=0}^{N-1} \lambda_m \Phi_m(0) \int_{T_m}^{T_{m+1}} g_{m+1}^-(\tau) d\tau = \sum_{n=1}^N \bar{c}_n Z(0, T_n) \sum_{m=0}^{n-1} \lambda_m \int_{T_m}^{T_{m+1}} g_{m+1}^-(\tau) d\tau.$$

Then we have another more intuitional initial price representation:

$$(3.2) \quad B_0(V_0, r, 0) = \sum_{n=1}^N \bar{c}_n Z(0, T_n) \left[G_n^- + \sum_{m=0}^{n-1} \lambda_m \int_{T_m}^{T_{m+1}} g_{m+1}^-(\tau) d\tau \right] + \delta \cdot V_0 \sum_{m=0}^{N-1} \left(\tilde{G}_{m+1} + \lambda_m \int_{T_m}^{T_{m+1}} \tilde{g}_{m+1}(\tau) d\tau \right).$$

Here we let

$$(3.3) \quad f_n(r) = G_n^- + \sum_{m=0}^{n-1} \lambda_m \int_{T_m}^{T_{m+1}} g_{m+1}^-(\tau) d\tau, \quad n = 1, \dots, N;$$

$$h(r) = \sum_{m=0}^{N-1} \left(\tilde{G}_{m+1} + \lambda_m \int_{T_m}^{T_{m+1}} \tilde{g}_{m+1}(\tau) d\tau \right).$$

Remark 3.1. $f_n(r)$ may be considered as the probability of no default (or unexpected default without loss) prior to or at T_n and $h(r)$ the probability of expected default or unexpected default with recovery δV_0 .

Then $f_n, h > 0$ and the initial price of our bond is written as follows:

$$(3.4) \quad B_0 = B_0(V_0, r, 0) = \sum_{n=1}^N \bar{c}_n Z(0, T_n) f_n(r) + \delta \cdot V_0 h(r).$$

Thus we have

$$(3.5) \quad -\partial_r B_0 = \sum_{n=1}^N \bar{c}_n Z(0, T_n) [T_n f_n(r) - \partial_r f_n(r)] - \delta \cdot V_0 \partial_r h(r).$$

We use the lemma on derivatives of multi-variate normal distribution functions ([17, Lemma 3, p. 331]), and

$$\frac{\partial}{\partial r} d_i^\pm(0) = (s_V \sqrt{T_i})^{-1} T_i; \quad \frac{\partial}{\partial r} \tilde{d}_i^\pm(\tau, \delta) = (s_V \sqrt{\tau})^{-1} \tau, i = 1, \dots, N$$

(see [11]) to get

$$\begin{aligned} & \partial_r N_{m+1}(d_1^-, \dots, d_{m+1}^-; A_{m+1}) \\ &= \sum_{i=1}^{m+1} \check{N}_{m+1,i}(d_1^-, \dots, d_{m+1}^-; A_{m+1})(s_V \sqrt{T_i})^{-1} T_i \geq 0, \\ & \partial_r N_{m+1}(d_1^+, \dots, d_m^+, -d_{m+1}^+; A_{m+1}^-) \\ &= \sum_{i=1}^m \check{N}_{m+1,i}(d_1^+, \dots, d_m^+, -d_{m+1}^+; A_{m+1}^-)(s_V \sqrt{T_i})^{-1} T_i \\ &\quad - \check{N}_{m+1,m+1}(d_1^+, \dots, d_m^+, -d_{m+1}^+; A_{m+1}^-)(s_V \sqrt{T_{m+1}})^{-1} T_{m+1}, \\ & \partial_r N_{m+1}(d_1^-, \dots, d_m^-, \tilde{d}_{m+1}^-(\tau, \delta); \tilde{A}_{m+1}(\tau)) \\ &= \sum_{i=1}^m \check{N}_{m+1,i}(d_1^-, \dots, d_m^-, \tilde{d}_{m+1}^-(\tau, \delta); \tilde{A}_{m+1}(\tau))(s_V \sqrt{T_i})^{-1} T_i \\ &\quad + \check{N}_{m+1,m+1}(d_1^-, \dots, d_m^-, \tilde{d}_{m+1}^-(\tau, \delta); \tilde{A}_{m+1}(\tau))(s_V \sqrt{\tau})^{-1} \tau \geq 0, \\ & \partial_r N_{m+1}(d_1^+, \dots, d_m^+, -\tilde{d}_{m+1}^+(\tau, \delta); \tilde{A}_{m+1}^-(\tau)) \\ &= \sum_{i=1}^m \check{N}_{m+1,i}(d_1^+, \dots, d_m^+, -\tilde{d}_{m+1}^+(\tau, \delta); \tilde{A}_{m+1}^-(\tau))(s_V \sqrt{T_i})^{-1} T_i \\ &\quad - \check{N}_{m+1,m+1}(d_1^+, \dots, d_m^+, -\tilde{d}_{m+1}^+(\tau, \delta); \tilde{A}_{m+1}^-(\tau))(s_V \sqrt{\tau})^{-1} \tau. \end{aligned}$$

Here $\check{N}_{m+1,i}$ is given in (A.14) at page 331 of [17]. From (3.3) and (2.16), we have

$$\begin{aligned} (3.6) \quad \partial_r f_n(r) &= \partial_r G_n^- + \sum_{m=0}^{n-1} \lambda_m \int_{T_m}^{T_{m+1}} \partial_r g_{m+1}^-(\tau) d\tau \\ &= \sum_{i=1}^n T_i (D_{n,i}^- + \tilde{J}_{i-1}^-), \quad n = 1, \dots, N; \end{aligned}$$

$$\begin{aligned} (3.7) \quad \partial_r h(r) &= \sum_{m=0}^{N-1} \left(\partial_r \tilde{G}_{m+1} + \lambda_m \int_{T_m}^{T_{m+1}} \partial_r \tilde{g}_{m+1}(\tau) d\tau \right) \\ &= \sum_{i=1}^N T_i (D_{N,i}^+ - \tilde{J}_{i-1}^+). \end{aligned}$$

Here

$$\begin{aligned}
 (3.8) \quad D_{n,i}^- &= (s_V \sqrt{T_i})^{-1} \left[e^{-\sum_{k=0}^{n-1} \lambda_k \Delta T_k} \check{N}_{n,i}(d_1^-, \dots, d_n^-; A_n) \right. \\
 &\quad + \sum_{m=i}^{n-1} \lambda_m \int_{T_m}^{T_{m+1}} e^{-\lambda_m(\tau-T_m) - \sum_{k=0}^{m-1} \lambda_k \Delta T_k} \\
 &\quad \cdot \check{N}_{m+1,i}(d_1^-, \dots, d_m^-, \tilde{d}_{m+1}^-(\tau, \delta); \tilde{A}_{m+1}(\tau)) d\tau \Big] \geq 0, \\
 &\quad i = 1, \dots, n; \\
 \tilde{J}_m^- &= \lambda_m \int_{T_m}^{T_{m+1}} e^{-\lambda_m(\tau-T_m) - \sum_{k=0}^{m-1} \lambda_k \Delta T_k} \\
 &\quad \cdot \frac{\check{N}_{m+1,m+1}(d_1^-, \dots, d_m^-, \tilde{d}_{m+1}^-(\tau, \delta); \tilde{A}_{m+1}(\tau))}{s_V T_{m+1}} \sqrt{\tau} d\tau \geq 0 \\
 &\quad m = 0, \dots, n-1; \\
 D_{N,i}^+ &= (s_V \sqrt{T_i})^{-1} \left[\sum_{m=i}^{N-1} \left(e^{-\sum_{k=0}^m (\lambda_k + b) \Delta T_k} \right. \right. \\
 &\quad \cdot \check{N}_{m+1,i}(d_1^+, \dots, d_m^+, -d_{m+1}^+; A_{m+1}^-) \\
 &\quad + \lambda_m \int_{T_m}^{T_{m+1}} e^{-(\lambda_m + b)(\tau-T_m) - \sum_{k=0}^{m-1} (\lambda_k + b) \Delta T_k} \\
 &\quad \cdot \check{N}_{m+1,i}(d_1^+, \dots, d_m^+, -\tilde{d}_{m+1}^+(\tau, \delta); \tilde{A}_{m+1}^-(\tau)) d\tau \Big) \\
 &\quad \left. - e^{-\sum_{k=0}^{i-1} (\lambda_k + b) \Delta T_k} \check{N}_{i,i}(d_1^+, \dots, d_{i-1}^+, -d_i^+; A_i^-) \right], \\
 &\quad i = 1, \dots, N-1; \\
 D_{N,N}^+ &= - (s_V \sqrt{T_N})^{-1} e^{-\sum_{k=0}^{N-1} (\lambda_k + b) \Delta T_k} \check{N}_{N,N}(d_1^+, \dots, d_{N-1}^+, -d_N^+; A_N^-) \leq 0, \\
 \tilde{J}_m^+ &= \lambda_m \int_{T_m}^{T_{m+1}} e^{-(\lambda_m + b)(\tau-T_m) - \sum_{k=0}^{m-1} (\lambda_k + b) \Delta T_k} \\
 &\quad \cdot \frac{\check{N}_{m+1,m+1}(d_1^+, \dots, d_m^+, -\tilde{d}_{m+1}^+(\tau, \delta); \tilde{A}_{m+1}^-(\tau))}{(s_V \sqrt{T_{m+1}})^{-1}} \sqrt{\tau} d\tau \geq 0.
 \end{aligned}$$

Using the notations (3.3) and (3.8), if we substitute (3.6) and (3.7) into (3.5) we have the representation of the duration of our defaultable discrete coupon bond:

$$\begin{aligned}
 (3.9) \quad \tilde{D} &= \frac{-\partial_r B_0}{B_0} = \frac{1}{B_0} \left\{ \sum_{i=1}^N T_i \left[\bar{c}_i Z(0; T_i) (f_i - D_{i,i}^- - \tilde{J}_{i-1}^-) - \delta V_0 (D_{N,i}^+ - \tilde{J}_{i-1}^+) \right] \right. \\
 &\quad \left. - \sum_{i=1}^{N-1} T_i \sum_{n=i+1}^N \bar{c}_n Z(0; T_n) (D_{n,i}^- + \tilde{J}_{i-1}^-) \right\}.
 \end{aligned}$$

4. Taxes on the coupons

In this section we extend the result of Section 2 along the line of the study of [1] on the effect of government taxes that paid on the proceeds of an investment in corporate bonds.

State income taxes are only paid on the proceeds of an investment and not on the principal. In this case the payoff to the bond holders is reduced but the equity is not changed. Thus the expected default condition is not changed and default barrier at time T_i is still K_i ($i = 1, \dots, N$) as calculated in Theorem 2.2. It means that when the *tax rate* is Λ (> 0), the *payoff to bondholders* at coupon dates is as follows:

i) At the maturity date T_N , $F + (1 - \Lambda)C_N$ if $V_{T_N} \geq K_N$ ($= F + C_N$) (firm value is large enough to pay debt principal F and coupon C_N); $F + (1 - \Lambda)(\delta V_{T_N} - F)$ if $F/\delta \leq V_{T_N} < K_N$ (firm value is large enough to pay debt principal but not enough to pay coupon); δV_{T_N} if $V_{T_N} < F/\delta$ (firm value is not large enough to pay even the principal, let alone the coupon). Here we should note that this structure of the payoff comes from the *implicit assumption* that $F/\delta < F + C_N$ (equally $\delta > (1 + c_N)^{-1}$ or $c_N > \delta^{-1} - 1$; we call it *the case II* which is possible but generally unlikely because the recovery rate δ might not be able to be so large provided a coupon rate $c_N = C_N/F$ or the coupon rate c_N might not be able to be so large provided a recovery rate δ . For example, if $\delta = 1/2$, then we must have $c_N > 1$ which seems impossible. When $F/\delta \geq F + C_N$ (equally $\delta \leq (1 + c_N)^{-1}$ or $c_N \leq \delta^{-1} - 1$; we call it *the case I*, the payoff to bondholders at the maturity date T_N is $F + (1 - \Lambda)C_N$ if $V_{T_N} \geq K_N$ and δV_{T_N} if $V_{T_N} < K_N$. Here we only consider the case I as in [1].

ii) At the k -th coupon date T_i ($i = 1, \dots, N - 1$), $(1 - \Lambda)C_i$ if $V_{T_i} \geq K_i$; δV_{T_i} if $V_{T_i} < K_i$. (Note that it is possible to consider the case II as at time T_N but we do not consider it since it is generally unlikely.)

Let modify our pricing model (2.8) and (2.9) under consideration of taxes on the coupons provided in the above. We introduce the following notation for simplicity of pricing formulae as the previous subsections.

$$(4.1) \quad \begin{aligned} \tilde{c}_N &= F + (1 - \Lambda)C_N; \\ \tilde{c}_i &= (1 - \Lambda)C_i, \quad \tilde{\Phi}_i(t) = \sum_{k=i+1}^N \tilde{c}_k Z(t; T_k), \quad i = 1, \dots, N - 1. \end{aligned}$$

That is, \tilde{c}_i is the time T_N -value of the payoff to bondholders at time T_i ($i = 1, \dots, N$) and $\tilde{\Phi}_i(t)$ is time t -value of *default free discrete coupon bond* with the maturity T_N -face value F and coupons C_{i+1}, \dots, C_N at time T_{i+1}, \dots, T_N under consideration of the tax rate Λ .

Under the above assumption and the notation (4.1), our bond price \tilde{B}_i satisfies the following PDE in every subinterval (T_i, T_{i+1}) ($i = 0, \dots, N - 1$):

$$(4.2) \quad \frac{\partial \tilde{B}_i}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 \tilde{B}_i}{\partial V^2} + (r - b)V \frac{\partial \tilde{B}_i}{\partial V} - (r + \lambda_i)\tilde{B}_i + \lambda_i \min\{\delta V, \tilde{\Phi}_i(t)\} = 0, \\ T_i < t < T_{i+1}, \quad V > 0.$$

If we consider the payoff to bondholders at coupon dates, we can derive the following terminal value conditions:

$$(4.3) \quad \begin{aligned} \tilde{B}_{N-1}(V, T_N) &= \tilde{c}_N \cdot 1\{V > K_N\} + \delta V \cdot 1\{V \leq K_N\}, \quad V > 0; \\ \tilde{B}_i(V, T_{i+1}) &= [\tilde{B}_{i+1}(V, T_{i+1}) + \tilde{c}_{i+1}] \cdot 1\{V > K_{i+1}\} + \delta V \cdot 1\{V \leq K_{i+1}\}, \\ V > 0, \quad i &= 0, \dots, N - 2. \end{aligned}$$

The problem (4.2) and (4.3) with the notation (4.1) is just the *pricing model* of our *defaultable discrete coupon bond* under consideration of *taxes on coupons* and it is the same problem with (2.8) and (2.9). Thus we have the solution representation of it just as in Theorem 2.5 or Corollary 2.7.

Theorem 4.1. *Unless the coupon rates are large relative to $1/\delta$, under State tax rate Λ , we have the following representation of the initial price of the our defaultable discrete coupon bond in terms of debt, coupon rates, default recovery rate, default intensity, and initial price of the default free zero coupon bond and initial firm value:*

(4.4)

$$\begin{aligned} \tilde{B}_0 &= \tilde{B}_0(V_0, F, c_1, \dots, c_N; \delta, \lambda_0, \dots, \lambda_{N-1}; r, b, \Lambda) \\ &= F \left\{ e^{-\sum_{k=0}^{N-1} \lambda_k \Delta T_k} Z(0; T_N) N_N(d_1^-, \dots, d_N^-; A_N) \right. \\ &\quad + \sum_{m=0}^{N-1} e^{-\sum_{k=0}^{m-1} \lambda_k \Delta T_k} \left[e^{-\lambda_m \Delta T_m} (1 - \Lambda) c_{m+1} Z(0; T_{m+1}) \right. \\ &\quad \quad \cdot N_{m+1}(d_1^-, \dots, d_{m+1}^-; A_{m+1}) \\ &\quad \quad \left. + \lambda_m \tilde{\phi}_m(0) \int_{T_m}^{T_{m+1}} e^{-\lambda_m(\tau - T_m)} N_{m+1}(d_1^-, \dots, d_m^-, \tilde{d}_{m+1}^-(\tau, \delta); \tilde{A}_{m+1}(\tau)) d\tau \right] \Big\} \\ &\quad + \delta V_0 \sum_{m=0}^{N-1} e^{-\sum_{k=0}^{m-1} (\lambda_k + b) \Delta T_k} \left[e^{-(\lambda_m + b) \Delta T_m} N_{m+1}(d_1^+, \dots, d_m^+, -d_{m+1}^+; A_{m+1}^-) \right. \\ &\quad \left. + \lambda_m \int_{T_m}^{T_{m+1}} e^{-(\lambda_m + b)(\tau - T_m)} N_{m+1}(d_1^+, \dots, d_m^+, -\tilde{d}_{m+1}^+(\tau, \delta); \tilde{A}_{m+1}(\tau)) d\tau \right]. \end{aligned}$$

Here

$$\tilde{\phi}_m(0) = \tilde{\Phi}_m(0)/F = Z(0; T_N) + \sum_{k=m+1}^N (1 - \Lambda) c_k Z(0; T_k),$$

and $N_m(a_1, \dots, a_m; A)$, $d_i^\pm(0)$, $A_m, A_m^-, \tilde{A}_m(\tau), \tilde{A}_m^-(\tau)$ and d_i^\pm are the same as in Theorem 2.5 and,

$$(4.5) \quad \begin{aligned} \tilde{d}_i^\pm(\tau, \delta, \Lambda) &= \frac{\ln \frac{V}{\tilde{M}_i(\tau)} + (r - b \pm \frac{s_V^2}{2})\tau}{s_V \sqrt{\tau}}, \quad T_{i-1} \leq \tau < T_i; \quad i = 1, \dots, N, \\ \tilde{M}_i(t) &= \delta^{-1} \tilde{\Phi}_i(t). \end{aligned}$$

Remark 4.2. If $b = 0$ and $\lambda_k = 0$ ($k = 0, \dots, N - 1$), then our pricing formula (4.4) nearly coincides with the formula (10) of [1] at page 756 and the only difference comes from the fact that the short rate is constant in our model.

Remark 4.3. As in the Section 2, the equation of the problem (4.2) and (4.3) is an inhomogeneous Black-Scholes equation with discontinuous terminal value. Thus using the results of [17], we can investigate such properties of $\tilde{B}_i(V, t)$ as monotonicity, boundedness or gradient estimate and so on.

Remark 4.4. In formulaes (2.5), (2.12) and (4.4), multivariate normal probability functions can easily be computed using standard functions on computers. (For example, we used the standard function “mvncdf” in Matlab to obtain our numerical results. Representations of covariance matrices including (2.6), (2.14) and (2.15) make convenient to use the standard function “mvncdf”.

Example 4.5 (Numerical example). Basic data for numerical experiments are the same as in Subsection 2.3 but coupons are given by $C_1 = C_2 = 2\frac{2}{3}$ when $\Lambda = 0.1$; $C_1 = C_2 = 2$ when $\Lambda = 0.2$ and $C_1 = C_2 = 1\frac{1}{7}$ when $\Lambda = 0.3$. Then we exactly get Figure 3.

5. Appendix: proofs of theorems

The proof of Theorem 1. Now we solve problems (2.1) and (2.2). Under the notation (2.3), when $i = N - 1$, we have in the interval (T_{N-1}, T_N)

$$(5.1) \quad \frac{\partial E_{N-1}}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 E_{N-1}}{\partial V^2} + (r - b)V \frac{\partial E_{N-1}}{\partial V} - (r + \lambda_{N-1})E_{N-1} = 0.$$

$$(5.2) \quad E_{N-1}(V, T_N) = (V - \bar{c}_N) \cdot 1\{V > K_N\}, \quad V > 0.$$

The equation (5.1) is the Black-Scholes equation with the short rate $r + \lambda_{N-1}$, the dividend rate $\lambda_{N-1} + b$ and the volatility s_V . The terminal value condition (5.2) can be written as

$$E_{N-1}(V, T_N) = V \cdot 1\{V > K_N\} - \bar{c}_N \cdot 1\{V > K_N\}.$$

This is the terminal value of binary option ([5, 15]) and thus we have the solution – representation in terms of binary options:

$$(5.3) \quad \begin{aligned} E_{N-1}(V, t) &= A_{K_N}^+(V, t; T_N; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V) \\ &\quad - \bar{c}_N B_{K_N}^+(V, t; T_N; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V) \\ &= e^{-\lambda_{N-1}(T_N - t)} [A_{K_N}^+(V, t; T_N; r, b, s_V) - \bar{c}_N B_{K_N}^+(V, t; T_N; r, b, s_V)], \\ &\quad T_{N-1} \leq t < T_N. \end{aligned}$$

Here $A_{K_N}^+(V, t; T_N; r, q, \sigma)$, and $B_{K_N}^+(V, t; T_N; r, q, \sigma)$ are the prices of the asset and bond binary options with risk free rate r , the dividend rate q and the volatility σ (given by [18, Lemma 1]), and we used [18, (11)]. In particular for the next step of study we rewrite $E_{N-1}(V, T_{N-1})$ as

$$(5.4) \quad \begin{aligned} E_{N-1}(V, T_{N-1}) &= e^{-(\lambda_{N-1} - \lambda_{N-2})(T_N - T_{N-1})} [A_{K_N}^+(V, T_{N-1}; T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\ &\quad - \bar{c}_N B_{K_N}^+(V, T_{N-1}; T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)]. \end{aligned}$$

By (5.2) and [15, Lemma 1, p. 253] we have

$$(5.5) \quad 0 < \frac{\partial E_{N-1}}{\partial V}(V, T_{N-1}) < e^{-(\lambda_{N-1}+b)\Delta T_{N-1}} \leq 1, \quad V > 0.$$

Now consider the case when $i = N - 2$. In this case, in the interval (T_{N-2}, T_{N-1}) , the problem (2.1) and (2.2) becomes

$$(5.6) \quad \frac{\partial E_{N-2}}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 E_{N-2}}{\partial V^2} + (r - b)V \frac{\partial E_{N-2}}{\partial V} - (r + \lambda_{N-2})E_{N-2} = 0.$$

$$(5.7) \quad E_{N-2}(V, T_{N-1}) = (E_{N-1}(V, T_{N-1}) - \bar{c}_{N-1}) \cdot \mathbf{1}\{E_{N-1}(V, T_{N-1}) > \bar{c}_{N-1}\}.$$

The equation (5.6) is the Black-Scholes equation with the short rate $r + \lambda_{N-2}$, the dividend rate $\lambda_{N-2} + b$ and the volatility s_V . From (5.7) the equation $E_{N-1}(V, T_{N-1}) = \bar{c}_{N-1}$ has unique root K_{N-1} and $E_{N-1}(V, T_{N-1}) \geq \bar{c}_{N-1} \Leftrightarrow V \geq K_{N-1}$. (Note that $\bar{c}_{N-1} = 0 \Leftrightarrow K_{N-1} = 0$.) Thus by (5.4) the terminal value condition (5.7) can be written as follows:

$$\begin{aligned} & E_{N-2}(V, T_{N-1}) \\ &= E_{N-1}(V, T_{N-1}) \cdot \mathbf{1}\{V \geq K_{N-1}\} - \bar{c}_{N-1} \cdot \mathbf{1}\{V \geq K_{N-1}\} \\ &= e^{-(\lambda_{N-1} - \lambda_{N-2})(T_N - T_{N-1})} \\ & \quad \cdot [A_{K_N}^+(V, T_{N-1}; T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \cdot \mathbf{1}\{V \geq K_{N-1}\} \\ & \quad - \bar{c}_{N-1} B_{K_N}^+(V, T_{N-1}; T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \cdot \mathbf{1}\{V \geq K_{N-1}\}] \\ & \quad - \bar{c}_{N-1} \cdot \mathbf{1}\{V \geq K_{N-1}\}. \end{aligned}$$

This is the terminal value of combination of the second order binaries and bond binary ([5, 15]) with the short rate $r + \lambda_{N-2}$, the dividend rate $\lambda_{N-2} + b$ and the volatility s_V . Thus we have the following representation:

$$\begin{aligned} (5.8) \quad & E_{N-2}(V, t) \\ &= e^{-(\lambda_{N-1} - \lambda_{N-2})\Delta T_{N-1}} [A_{K_{N-1}K_N}^+ + (V, t; T_{N-1}, T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\ & \quad - \bar{c}_N B_{K_{N-1}K_N}^+ + (V, t; T_{N-1}, T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)] \\ & \quad - \bar{c}_{N-1} B_{K_{N-1}}^+ (V, t; T_{N-1}; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\ &= e^{-\lambda_{N-2}(T_{N-1}-t) - \lambda_{N-1}\Delta T_{N-1}} [A_{K_{N-1}K_N}^+ + (V, t; T_{N-1}, T_N; r, b, s_V) \\ & \quad - \bar{c}_N B_{K_{N-1}K_N}^+ + (V, t; T_{N-1}, T_N; r, b, s_V)] \\ & \quad - e^{-\lambda_{N-2}(T_{N-1}-t)} \bar{c}_{N-1} B_{K_{N-1}}^+ (V, t; T_{N-1}; r, b, s_V), \quad T_{N-2} \leq t < T_{N-1}. \end{aligned}$$

Here $A_{K_1K_2}^+ + (V, t; T_{N-1}, T_N; r, q, \sigma)$, and $B_{K_1K_2}^+ + (V, t; T_{N-1}, T_N; r, q, \sigma)$ are the prices of the second order asset and bond binary options with risk free rate r , the dividend rate q and the volatility σ (see [18, Lemma 1]), and we used [18, (11)]. In particular

for the next step of study we rewrite $E_{N-2}(V, T_{N-2})$ as

$$(5.9) \quad \begin{aligned} E_{N-2}(V, T_{N-2}) = & e^{-\lambda_{N-1}\Delta T_{N-1} - \lambda_{N-2}\Delta T_{N-2} + \lambda_{N-3}(T_N - T_{N-2})} \\ & \cdot [A_{K_{N-1}K_N}^{++}(V, T_{N-2}; T_{N-1}, T_N; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V) \\ & - \bar{c}_N B_{K_{N-1}K_N}^{++}(V, T_{N-2}; T_{N-1}, T_N; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V)] \\ & - e^{-(\lambda_{N-2} - \lambda_{N-3})\Delta T_{N-2}} \\ & \cdot \bar{c}_{N-1} B_{K_{N-1}}^+(V, T_{N-2}; T_{N-1}; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V). \end{aligned}$$

By (5.5), (5.7) and [15, Lemma 1], we have

$$(5.10) \quad 0 < \frac{\partial E_{N-2}}{\partial V}(V, T_{N-2}) < e^{-(\lambda_{N-2} + b)\Delta T_{N-2} - (\lambda_{N-1} + b)\Delta T_{N-1}} \leq 1, \quad V > 0.$$

Now consider the case when $i = N - 3$. In this case, in the interval (T_{N-3}, T_{N-2}) , (2.1) and (2.2) become

$$(5.11) \quad \frac{\partial E_{N-3}}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 E_{N-3}}{\partial V^2} + (r - b)V \frac{\partial E_{N-3}}{\partial V} - (r + \lambda_{N-3})E_{N-3} = 0.$$

$$(5.12) \quad E_{N-3}(V, T_{N-2}) = [E_{N-2}(V, T_{N-2}) - \bar{c}_{N-2}] \cdot 1\{E_{N-2}(V, T_{N-2}) > \bar{c}_{N-2}\}.$$

The equation (5.11) is the Black-Scholes equation with the short rate $r + \lambda_{N-3}$, the dividend rate $\lambda_{N-3} + b$ and the volatility s_V . From (5.10) the equation $E_{N-2}(V, T_{N-2}) = \bar{c}_{N-2}$ has unique root K_{N-2} and $E_{N-2}(V, T_{N-2}) \geq \bar{c}_{N-2} \iff V \geq K_{N-2}$. (Note that $\bar{c}_{N-2} = 0 \iff K_{N-2} = 0$.) Thus by (5.9) the terminal value condition (5.12) can be written as follows:

$$\begin{aligned} & E_{N-3}(V, T_{N-3}) \\ = & E_{N-2}(V, T_{N-2}) \cdot 1\{V \geq K_{N-2}\} - \bar{c}_{N-2} \cdot 1\{V \geq K_{N-2}\} \\ = & e^{-\lambda_{N-1}\Delta T_{N-1} - \lambda_{N-2}\Delta T_{N-2} + \lambda_{N-3}(T_N - T_{N-2})} \\ & \cdot [A_{K_{N-1}K_N}^{++}(V, T_{N-2}; T_{N-1}, T_N; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V) \cdot 1\{V \geq K_{N-2}\} \\ & - \bar{c}_N B_{K_{N-1}K_N}^{++}(V, T_{N-2}; T_{N-1}, T_N; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V) \cdot 1\{V \geq K_{N-2}\}] \\ & - e^{-(\lambda_{N-2} - \lambda_{N-3})\Delta T_{N-2}} \bar{c}_{N-1} B_{K_{N-1}}^+(V, T_{N-2}; T_{N-1}; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V) \\ & \cdot 1\{V \geq K_{N-2}\} - \bar{c}_{N-2} \cdot 1\{V \geq K_{N-2}\}. \end{aligned}$$

This is a linear combination of the terminal values of third or lower order binary options with the short rate $r + \lambda_{N-3}$, the dividend rate $\lambda_{N-3} + b$ and the volatility s_V in the meaning of [15] and the solution $E_{N-3}(V, t)$ is given by the third or lower

order binary options:

$$\begin{aligned}
 (5.13) \quad E_{N-3}(V, t) &= e^{-\lambda_{N-1}\Delta T_{N-1} - \lambda_{N-2}\Delta T_{N-2} + \lambda_{N-3}(T_N - T_{N-2})} \\
 &\quad \cdot [A_{K_{N-2}K_{N-1}K_N}^+(V, t; T_{N-2}, T_{N-1}, T_N; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V) \\
 &\quad - \bar{c}_N B_{K_{N-2}K_{N-1}K_N}^+(V, t; T_{N-2}, T_{N-1}, T_N; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V)] \\
 &\quad - e^{-(\lambda_{N-2} - \lambda_{N-3})\Delta T_{N-2}} \\
 &\quad \cdot \bar{c}_{N-1} B_{K_{N-2}K_{N-1}}^+(V, t; T_{N-2}, T_{N-1}; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V) \\
 &\quad - \bar{c}_{N-2} B_{K_{N-2}}^+(V, t; T_{N-2}; r + \lambda_{N-3}, \lambda_{N-3} + b, s_V) \\
 &= e^{-\lambda_{N-1}\Delta T_{N-1} - \lambda_{N-2}\Delta T_{N-2} - \lambda_{N-3}(T_N - 2 - t)} \\
 &\quad \cdot [A_{K_{N-2}K_{N-1}K_N}^+(V, t; T_{N-2}, T_{N-1}, T_N; r, b, s_V) \\
 &\quad - \bar{c}_N B_{K_{N-2}K_{N-1}K_N}^+(V, t; T_{N-2}, T_{N-1}, T_N; r, b, s_V)] \\
 &\quad - e^{-\lambda_{N-2}\Delta T_{N-2} - \lambda_{N-3}(T_N - 2 - t)} \bar{c}_{N-1} B_{K_{N-2}K_{N-1}}^+(V, t; T_{N-2}, T_{N-1}; r, b, s_V) \\
 &\quad - e^{-\lambda_{N-3}(T_N - 2 - t)} \bar{c}_{N-2} B_{K_{N-2}}^+(V, t; T_{N-2}; r, b, s_V), \quad T_{N-3} \leq t < T_{N-2}.
 \end{aligned}$$

Here $A_{K_{N-2}K_{N-1}K_N}^+$, and $B_{K_{N-2}K_{N-1}K_N}^+$ are the prices of the third order asset and bond binary options (see [15, Theorem 1] or [18, Lemma 1]) and we used [18, (11)]. By induction the formulae (2.4) are proved. \square

The proof of Theorem 2. Now we solve problem (2.8) and (2.9). The equation (2.8) is an inhomogeneous Black-Scholes equation with the short rate $r + \lambda_i$, the dividend rate $b + \lambda_i$, the volatility s_V , and the inhomogeneous term

$$\begin{aligned}
 (5.14) \quad g_i(V, t) &= \lambda_i \min\{\delta V, \Phi_i(t)\} = \lambda_i [\Phi_i(t) \cdot 1\{V \geq M_{i+1}(t) + \delta V\} \cdot 1\{V < M_{i+1}(t)\}], \\
 &\quad i = 0, \dots, N - 1.
 \end{aligned}$$

Here

$$(5.15) \quad M_{i+1}(t) = \delta^{-1} \Phi_i(t) = \delta^{-1} \left[\sum_{k=i+1}^N C_k e^{-r(T_k - t)} + F e^{-r(T_N - t)} \right].$$

When $i = N - 1$, we have

$$\begin{aligned}
 (5.16) \quad \frac{\partial B_{N-1}}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 B_{N-1}}{\partial V^2} + (r - b)V \frac{\partial B_{N-1}}{\partial V} - (r + \lambda_{N-1})B_{N-1} \\
 + \lambda_{N-1} \min\{\delta V, \Phi_{N-1}(t)\} = 0, \quad T_{N-1} < t < T_N, \quad V > 0,
 \end{aligned}$$

$$(5.17) \quad B_{N-1}(V, T_N) = \bar{c}_N \cdot 1\{V \geq K_N\} + \delta V \cdot 1\{V < K_N\}, \quad V > 0.$$

The solution of (5.16) and (5.17) is given by the sum of the following two problems:

$$\begin{aligned}
 (5.18) \quad \frac{\partial X}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 X}{\partial V^2} + (r - b)V \frac{\partial X}{\partial V} - (r + \lambda_{N-1})X = 0, \\
 T_{N-1} < t < T_N, \quad V > 0,
 \end{aligned}$$

$$(5.19) \quad X(V, T_N) = \bar{c}_N \cdot 1\{V \geq K_N\} + \delta V \cdot 1\{V < K_N\}, \quad V > 0.$$

$$(5.20) \quad \frac{\partial Y}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 Y}{\partial V^2} + (r - b)V \frac{\partial Y}{\partial V} - (r + \lambda_{N-1})Y + g_{N-1}(t) = 0, \\ T_{N-1} < t < T_N, \quad V > 0,$$

$$(5.21) \quad Y(V, T_N) = 0, \quad V > 0.$$

The problem (5.18) and (5.19) is a binary option pricing problem with the short rate $r + \lambda_{N-1}$, the dividend rate $b + \lambda_{N-1}$ and the volatility s_V . Thus using the notation and binary option pricing formulae of [5, 15] we have

$$X = \bar{c}_N B_{K_N}^+(V, t; T_N; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V) \\ + \delta A_{K_N}^-(V, t; T_N; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V), \quad T_{N-1} \leq t < T_N, \quad V > 0.$$

The problem (5.20) and (5.21) is a 0-terminal value problem of an inhomogeneous equation and thus we use the *Duhamel's principle* to solve it. (See [18]). Fix $\tau \in (T_{N-1}, T_N]$ and let $W(V, t; \tau)$ be the solution to the following terminal value problem:

$$\frac{\partial W}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 W}{\partial V^2} + (r - b)V \frac{\partial W}{\partial V} - (r + \lambda_{N-1})W = 0, \quad T_{N-1} < t < \tau, \quad V > 0, \\ W(V, \tau; \tau) = g_{N-1}(V, \tau), \quad V > 0.$$

Consider (5.14) with $i = N - 1$ and use again the notation and binary option pricing formulae of [5, 15]. We have

$$W(V, t; \tau) = \lambda_{N-1} [\Phi_{N-1}(\tau) B_{M_N(\tau)}^+(V, t; \tau; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V) \\ + \delta A_{M_N(\tau)}^-(V, t; \tau; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V), \quad T_{N-1} \leq t < \tau, \quad V > 0.$$

Then the solution Y to the problem (5.20) and (5.21) is given as follows:

$$Y(V, t) = \lambda_{N-1} \int_t^{T_N} [\Phi_{N-1}(\tau) B_{M_N(\tau)}^+(V, t; \tau; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V) \\ + \delta A_{M_N(\tau)}^-(V, t; \tau; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V)] d\tau, \\ T_{N-1} \leq t < T_N, \quad V > 0.$$

Therefore $B_{N-1}(V, t)$, $T_{N-1} \leq t < T_N$ is provided as follows:

(5.22)

$$B_{N-1}(V, t) = \bar{c}_N B_{K_N}^+(V, t; T_N; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V) \\ + \delta A_{K_N}^-(V, t; T_N; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V) \\ + \lambda_{N-1} \int_t^{T_N} [\Phi_{N-1}(\tau) B_{M_N(\tau)}^+(V, t; \tau; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V) \\ + \delta A_{M_N(\tau)}^-(V, t; \tau; r + \lambda_{N-1}, \lambda_{N-1} + b, s_V)] d\tau \\ = e^{-\lambda_{N-1}(T_N - t)} [\bar{c}_N B_{K_N}^+(V, t; T_N; r, b, s_V) + \delta A_{K_N}^-(V, t; T_N; r, b, s_V)] \\ + \lambda_{N-1} \int_t^{T_N} e^{-\lambda_{N-1}(\tau - t)} [\Phi_{N-1}(\tau) B_{M_N(\tau)}^+(V, t; \tau; r, b, s_V) \\ + \delta A_{M_N(\tau)}^-(V, t; \tau; r, b, s_V)] d\tau, \quad T_{N-1} \leq t < T_N, \quad V > 0.$$

Here the last equality comes from (11) of [18].

Now we consider the case when $i = N - 2$. Then the problem (2.8) and (2.9) becomes

$$(5.23) \quad \frac{\partial B_{N-2}}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 B_{N-2}}{\partial V^2} + (r - b)V \frac{\partial B_{N-2}}{\partial V} - (r + \lambda_{N-2})B_{N-2} + g_{N-2}(t) = 0, \\ T_{N-2} < t < T_{N-1}, \quad V > 0,$$

$$(5.24) \quad B_{N-2}(V, T_{N-1}) = [B_{N-1}(V, T_{N-1}) + \bar{c}_{N-1}] \cdot 1\{V \geq K_{N-1}\} + \delta V \cdot 1\{V < K_{N-1}\}, \\ V > 0.$$

The solution to (5.23) and (5.24) is provided by the sum of the following two problems:

$$(5.25) \quad \frac{\partial X}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 X}{\partial V^2} + (r - b)V \frac{\partial X}{\partial V} - (r + \lambda_{N-2})X = 0, \\ T_{N-2} < t < T_{N-1}, \quad V > 0,$$

$$(5.26) \quad X(V, T_{N-1}) = [B_{N-1}(V, T_{N-1}) + \bar{c}_{N-1}] \cdot 1\{V \geq K_{N-1}\} + \delta V \cdot 1\{V < K_{N-1}\}, \\ V > 0.$$

$$(5.27) \quad \frac{\partial Y}{\partial t} + \frac{s_V^2}{2} V^2 \frac{\partial^2 Y}{\partial V^2} + (r - b)V \frac{\partial Y}{\partial V} - (r + \lambda_{N-2})Y + g_{N-2}(t) = 0, \\ T_{N-2} < t < T_{N-1}, \quad V > 0,$$

$$(5.28) \quad Y(V, T_{N-1}) = 0, \quad V > 0.$$

The problem (5.27) and (5.28) is the same type as the problem (5.20) and (5.21) and thus the solution to (5.27) and (5.28) is provided as follows:

$$Y(V, t) = \lambda_{N-2} \int_t^{T_{N-1}} [\Phi_{N-2}(\tau) B_{M_{N-1}(\tau)}^+(V, t; \tau; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\ + \delta A_{M_{N-1}(\tau)}^-(V, t; \tau; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)] d\tau, \quad T_{N-2} \leq t < T_{N-1}, \quad V > 0.$$

Since (5.25) is a homogeneous Black-Scholes equation with the short rate $r + \lambda_{N-2}$, the dividend rate $\lambda_{N-2} + b$ and the volatility s_V , we use (11) of [18] to rewrite $B_{N-1}(V, T_{N-1})$ given by (5.22) as

$$B_{N-1}(V, T_{N-1}) \\ = e^{-(\lambda_{N-1} - \lambda_{N-2})\Delta T_{N-1}} [\bar{c}_N B_{K_N}^+(V, T_{N-1}; T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\ + \delta A_{K_N}^-(V, T_{N-1}; T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)] \\ + \lambda_{N-1} \int_{T_{N-1}}^{T_N} e^{-(\lambda_{N-1} - \lambda_{N-2})(\tau - T_{N-1})} [\Phi_{N-1}(\tau) B_{M_N(\tau)}^+(V, T_{N-1}; \tau; r + \lambda_{N-2}, \\ \lambda_{N-2} + b, s_V) + \delta A_{M_N(\tau)}^-(V, T_{N-1}; \tau; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)] d\tau.$$

Thus (5.26) can be written as

$$\begin{aligned}
& X(V, T_{N-1}) \\
&= e^{-(\lambda_{N-1}-\lambda_{N-2})\Delta T_{N-1}} [\bar{c}_N B_{K_N}^+(V, T_{N-1}; T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \cdot 1\{V \geq K_{N-1}\} \\
&\quad + \delta A_{K_N}^-(V, T_{N-1}; T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \cdot 1\{V \geq K_{N-1}\}] \\
&\quad + \lambda_{N-1} \int_{T_{N-1}}^{T_N} e^{-(\lambda_{N-1}-\lambda_{N-2})(\tau-T_{N-1})} [\Phi_{N-1}(\tau) B_{M_N}^+(\tau)(V, T_{N-1}; \tau; r + \lambda_{N-2}, \\
&\quad \lambda_{N-2} + b, s_V) \cdot 1\{V \geq K_{N-1}\} + \delta A_{M_N(\tau)}^-(V, T_{N-1}; \tau; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\
&\quad \cdot 1\{V \geq K_{N-1}\}] d\tau + \bar{c}_{N-1} \cdot 1\{V \geq K_{N-1}\} + \delta V \cdot 1\{V < K_{N-1}\}, \quad V > 0.
\end{aligned}$$

This is a linear combination of second or lower order binaries and therefore using the notation and second order binary option pricing formulae of [5, 15] we get the solution to problem (5.25) and (5.26) as follows:

$$\begin{aligned}
& X(V, t) \\
&= e^{-(\lambda_{N-1}-\lambda_{N-2})\Delta T_{N-1}} [\bar{c}_N B_{K_{N-1}K_N}^{++}(V, t; T_{N-1}, T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\
&\quad + \delta A_{K_{N-1}K_N}^{+-}(V, t; T_{N-1}, T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)] \\
&\quad + \lambda_{N-1} \int_{T_{N-1}}^{T_N} e^{-(\lambda_{N-1}-\lambda_{N-2})(\tau-T_{N-1})} \\
&\quad \quad [\Phi_{N-1}(\tau) B_{K_{N-1}M_N(\tau)}^{++}(V, t; T_{N-1}, \tau; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\
&\quad \quad + \delta A_{K_{N-1}M_N(\tau)}^{+-}(V, t; T_{N-1}, \tau; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)] d\tau \\
&\quad + \bar{c}_{N-1} B_{K_{N-1}}^+(V, t; T_{N-1}; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\
&\quad + \delta A_{K_{N-1}}^-(V, t; T_{N-1}; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V), \quad T_{N-2} \leq t < T_{N-1}, \quad V > 0.
\end{aligned}$$

Thus we have the representation of $B_{N-2}(V, t)$, $T_{N-2} \leq t < T_{N-1}$ as follows:

$$\begin{aligned}
& B_{N-2}(V, t) \\
&= e^{-(\lambda_{N-1}-\lambda_{N-2})\Delta T_{N-1}} [\bar{c}_N B_{K_{N-1}K_N}^{++}(V, t; T_{N-1}, T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\
&\quad + \delta A_{K_{N-1}K_N}^{+-}(V, t; T_{N-1}, T_N; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)] \\
&\quad + \bar{c}_{N-1} B_{K_{N-1}}^+(V, t; T_{N-1}; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\
&\quad + \delta A_{K_{N-1}}^-(V, t; T_{N-1}; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\
&\quad + \lambda_{N-1} \int_{T_{N-1}}^{T_N} e^{-(\lambda_{N-1}-\lambda_{N-2})(\tau-T_{N-1})} [\Phi_{N-1}(\tau) B_{K_{N-1}M_N(\tau)}^{++}(V, t; T_{N-1}, \tau; r + \lambda_{N-2}, \\
&\quad \lambda_{N-2} + b, s_V) + \delta A_{K_{N-1}M_N(\tau)}^{+-}(V, t; T_{N-1}, \tau; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)] d\tau
\end{aligned}$$

$$\begin{aligned}
 & + \lambda_{N-2} \int_t^{T_{N-1}} [\Phi_{N-2}(\tau) B_{M_{N-1}}^+(\tau)(V, t; \tau; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V) \\
 & \quad + \delta A_{M_{N-1}}^-(V, t; \tau; r + \lambda_{N-2}, \lambda_{N-2} + b, s_V)] d\tau \\
 = & e^{-\lambda_{N-2}(T_{N-1}-t)} \left\{ e^{-\lambda_{N-1}\Delta T_{N-1}} [\bar{c}_N B_{K_{N-1}K_N}^+(V, t; T_{N-1}, T_N; r, b, s_V) \right. \\
 & \quad + \delta A_{K_{N-1}K_N}^+(V, t; T_{N-1}, T_N; r, b, s_V)] \\
 & + \bar{c}_{N-1} B_{K_{N-1}}^+(V, t; T_{N-1}; r, b, s_V) + \delta A_{K_{N-1}}^-(V, t; T_{N-1}; r, b, s_V) \\
 & + \lambda_{N-1} \int_{T_{N-1}}^{T_N} e^{-\lambda_{N-1}(\tau-T_{N-1})} [\Phi_{N-1}(\tau) B_{K_{N-1}M_N}^+(\tau)(V, t; T_{N-1}, \tau; r, b, s_V) \\
 & \quad + \delta A_{K_{N-1}M_N}^+(V, t; T_{N-1}, \tau; r, b, s_V)] d\tau \left. \right\} \\
 & + \lambda_{N-2} \int_t^{T_{N-1}} e^{-\lambda_{N-2}(\tau-t)} [\Phi_{N-2}(\tau) B_{M_{N-1}}^+(\tau)(V, t; \tau; r, b, s_V) \\
 & \quad + \delta A_{M_{N-1}}^-(V, t; \tau; r, b, s_V)] d\tau.
 \end{aligned}$$

By induction we have the rest of the proof. \square

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