Bulletin of the Iranian Mathematical Society Vol. 30 *No. 2 (2004), pp 39-66.*

LIFTING PROBLEM IN CODIMENSION 2 AND INITIAL IDEALS†

MARGHERITA ROGGERO

Communicated by Teo Mora

ABSTRACT. Let X be a codimension 2, locally Cohen-Macaulay, integral, projective variety of degree d in $\mathbf{P}^N$. We consider the problem of finding conditions on d, N and s such that any degree s hypersurface in $\mathbf{P}^{N-1}$ containing a general hyperplane section of X lifts to a hypersurface in $\mathbf{P}^N$ containing X.

We prove general and sharp bounds on the degree of X depending on both N and s and also on the number of independent hypersurfaces of degree s containing X, especially under the additional condition that the general plane section of X does not lie on any degree $s-1$ curve.

1. Introduction

Let X be an integral, projective variety of dimension n and degree d in $\mathbf{P}^N$, defined over an algebraically closed field $\mathbf{k}$ of characteristic zero. Consider the hyperplane section $Y = X \cap K$ of X, where $K \cong \mathbf{P}^{N-1}$ is a general hyperplane in $\mathbf{P}^N$. The "lifting problem" is the problem of finding conditions on d, N, n and s such that

MSC(2000): Primary 14H50; Secondary 14M07, 14Jxx
Keywords: Lifting problem, Codimension 2 subvarieties
Received: 23 July 2004 , Revised: 27 January 2005
† Written with the support of the University Ministry funds.

any degree s hypersurface in $\mathbf{P}^{N-1}$ containing Y can be lifted to a hypersurface in $\mathbf{P}^N$ containing X.

For an integral curve C in $\mathbf{P}^3$, Laudal's "generalized trisecant lemma" states that the set of points $Z = C \cap K$ can lie on some non-liftable degree s curve in $K \cong \mathbf{P}^2$ only if $d \leq s^2+s$ (see [8]). Different proofs and improvements of this result have been afterwords obtained by several authors, using different tools of commutative algebra and algebraic geometry: we want to mention in particular the papers by Gruson-Peskine ([7]), Strano ([18]) and Green ([6]), where the main techniques were first introduced. Strano showed that the presence of a non-liftable hypersurface for an integral, locally Cohen-Macaulay subvariety X in $\mathbf{P}^N$ is a property which the general hyperplane section of X inherits; this result is the natural starting point for generalizations of Laudal's Lemma to codimension 2 subvarieties in projective spaces of higher dimension. Green combined Strano's method with the theory of generic initial ideals and sets of points in $\mathbf{P}^2$ in a uniform position.

Using these and other tools, like foci, liaisons, linear series etc., Laudal's Lemma has been generalized in (at least) two directions.

First of all, there are extensions to codimension 2 subvarieties in some projective spaces $\mathbf{P}^N$ of higher dimension. The leading idea is to bound the degree through a function $f(s, N)$ depending on a non-lifting level s and also on the dimension N of projective space (see for instance Re [13], Chiantini-Ciliberto [4], Mezzetti, Raspanti [10], [11], [12], Valenzano [20], Roggero [15], [16]); here we mention the bound $\deg(X) \leq s^2 - s + 2$ for a surface X in $\mathbf{P}^4$.

On the other hand, Tortora ([19]) found improvements of Laudal's Lemma also for curves C in $\mathbf{P}^3$ by introducing into the bounding function a new parameter, the number a of independent degree s surfaces containing C: under a few additional hypotheses he proved that the degree of C cannot exceed $\binom{s+1}{2} + \binom{s-a-1}{2} + 1$.

All these bounds are sharp, in the sense that there are subvarieties which satisfy the required conditions and whose degrees are the maximum allowed (see Example 3.8 and also [10], [19] and [20]); it is worth noting that all the examples in bibliography are given by arithmetically Buchsbaum subvarieties with very narrow deficiency modules.

Such a list of results could possibly suggest a general conjecture of the type:

if X is a codimension 2 subvariety in $\mathbf{P}^N$, s is a non-lifting level and there are at least a independent degree s hypersurfaces containing X, then:

$$\deg(X) \leq \binom{s+1}{2} + \binom{s-a-N+2}{2} + 1$$

and equality holds if X is an arithmetically Buchsbaum subvariety.

Unfortunately, the situation is far more complicated. In the present paper we only deal with the first part of the above conjecture, while border cases are considered in a following one. For what concerns this topic, we can give explicit counter examples even in the case of curves in $\mathbf{P}^3$; more precisely we show that for every whole number m and function $G(a)$ there are curves in $\mathbf{P}^3$ with non lifting level s and contained in a independent degree s surfaces, whose degree exceeds $\binom{s+1}{2} + \binom{s-a+m}{2} + G(a)$ (see Example 4.4). Nevertheless, we prove that the conjectured bound on the degree (or some bound close to it) holds under some additional condition which often concerns the number of curves containing the general plane section of the variety X. For instance, Theorem 4.8, shows that the bound holds for every $a \leq 2$ if the general plane section of X is not contained in curves of degree $s-1$, even if it does not hold in general when $a \geq 3$: Example 4.9. However, for curves C in $\mathbf{P}^3$ in Theorem 4.10 we are able to obtain general statements, proving, without any additional condition on the plane section, bounds on the degree strictly including Laudal's Lemma.

We obtain upper bounds on the degree of a codimension 2, locally Cohen-Macaulay, integral subvariety X in $\mathbf{P}^N$ depending on a non lifting level s (or a socle level s), just using a reduction to a set of points, Strano's exact sequence (2.3) and computations on both generic initial ideals and Castelnuovo functions.

In §2 we introduce notation and recall some known results that we will most often use in the following sections.

In §3 we focus on the structure of the Rao-module of a subvariety V in $\mathbf{P}^N$ of every codimension (that is on the first deficiency module of the ideal sheaf $\mathcal{I}_V$) in connection with a non lifting level or socle level s and introduce the more general notion of "generalized socle" (**Notation (5)**); we also show that some non-zero generalized socle involves the existence of hypersurfaces containing V (or its linear sections) in degrees close to s.

In §4, using generic initial ideals and results obtained in §3, we prove the main theorems, which states upper bounds on the degree of a codimension 2 subvariety X in $\mathbf{P}^N$ depending on non-lifting levels.

2. **Notation, definitions and useful results**

Unless otherwise stated:

(1) $A = \bigoplus A_i$ is the graded ring in $N+1$ variables $\mathbf{k}[x_0, \dots, x_N]$ over an algebraically closed field $\mathbf{k}$ of characteristic 0: without any further notice, elements and ideals of A are always supposed to be homogeneous. $\mathbf{P}^N$ is the projective space of dimension N over $\mathbf{k}$; we often denote by the same symbol an element $a \in A$ and the hypersurface S_a in $\mathbf{P}^N$ defined by the equation $a = 0$; $\mathcal{U}_a$ will be the open subset $\mathbf{P}^N - S_a$ and $\mathcal{U}_i = \mathcal{U}_{x_i}$.

(2) In the ring $A = \mathbf{k}[x_0, x_1, \dots, x_N]$ we consider the reverse lexicographic order on the monomials induced by $x_0 > x_1 > \dots > x_N$ and denote by $\text{gin}(G)$ and $\text{gin}(I)$ the initial term of the polynomial $G \in$A and the initial ideal of the ideal $I \subset A$ respectively; we will always assume that we have chosen general coordinates, so that $\text{gin}(I)$ is indeed the generic initial ideal of I (for generalities on generic initial ideals see for example [6]). We will sometimes perform a change of coordinates of the type: $x'_i = x_i + a_{i+1}x_{i+1} + \dots + a_N x_N$; we want to underline that it does not alter any initial term.

If $H = \mathbf{P}^r$ is the linear space in $\mathbf{P}^N$ given by $x_N = \dots = x_{r+1} = 0$ and $F \in A$, $(F)_H$ denotes the equivalence class of F in $A/I_H \cong \mathbf{k}[x_0, \dots, x_r]$. If L is the (general) line defined by

$x_2 = \cdots = x_N = 0$, and $B \subseteq A$ is a $\mathbf{k}$-vector space, we will denote by $\dim_L(B)$ the dimension of $B \cap \mathbf{k}[x_0, x_1]$. If $B = I_s$, then $\dim_L(I_s) = \dim_L((I/(x_2, \ldots, x_N))_s) = \dim(\mathrm{gin}(I) \cap k[x_0, x_1]_s)$.

(3) For any coherent sheaf $\mathcal{F}$ on $\mathbf{P}^N$ we will use the standard notation $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_{\mathbf{P}^N}(n)$, $H^i_*\mathcal{F} = \oplus_{n \in \mathbf{Z}} H^i\mathcal{F}(n)$ and (for a general hyperplane K) the standard exact sequence:

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}_K \to 0 \tag{2.1}$$

as well as its twists and cohomology exact sequences; $h^i\mathcal{F}$ is the $\mathbf{k}-$vector space dimension of $H^i\mathcal{F}$.

(4) "Subvariety" means a closed subscheme of the projective space $\mathbf{P}^N$. $I_V \subset A$ and $\mathcal{I}_V \subset \mathcal{O}_{\mathbf{P}^N}$ are the ideal and the ideal sheaf of the subvariety V, so that $I_V = \bigoplus_i H^0\mathcal{I}_V(i)$; if $\mathbf{P}^r$ is a linear subspace of $\mathbf{P}^N$ and $W = V \cap \mathbf{P}^r$, we usually denote by $\mathcal{I}_W$ the ideal sheaf of W as a subvariety of $\mathbf{P}^r$, that is $\mathcal{I}_W \subseteq \mathcal{O}_{\mathbf{P}^r}$. The ideal sheaf $\mathcal{I}_V$ of V in $\mathbf{P}^N$ is $m-$**regular** if $H^q\mathcal{I}_V(m-q) = 0$ for every $q > 0$; the ideal I_V of V is $m-$**regular** if the ideal sheaf $\mathcal{I}_V$ is $m-$regular; the **regularity** of $\mathcal{I}_V$ (or I_V) is the smallest integer ρ such that I_V is ρ-regular.

(5) $N^s_{s-i}(V, B)$ is the set of elements $\sigma \in H^1\mathcal{I}_V(s-i)$ which vanish if multiplied by each element of the vector space $B \subseteq A_i$; $n^s_{s-i}(V, B)$ is the dimension of $N^s_{s-i}(V, B)$ as a $\mathbf{k}-$vector space. If $B = A_i$, we write $\overline{N}^s_{s-i}(V)$ instead of $N^s_{s-i}(V, A_i)$; observe that $\overline{N}^s_{s-1}(V)$ is the degree $s-1$ component of the socle of $H^1_*\mathcal{I}_V$ and so we will call **generalized socle** any vector space $N^s_{s-i}(V, B)$. We write $N^s_{s-i}(V)$ instead of $N^s_{s-i}(V, B)$ when B is generated by x^i, x being a general linear form (we generally suppose $x = x_N$).

If $\dim(V) \geq 1$ and W is its general hyperplane section, then by (2.1) it follows:

$$\begin{aligned} N^s_{s-1}(V) &= \mathrm{Ker}(H^1\mathcal{I}_V(s-1) \to H^1\mathcal{I}_V(s)) \\ &= \mathrm{Coker}(H^0\mathcal{I}_V(s) \to H^0\mathcal{I}_W(s)). \end{aligned} \tag{2.2}$$

The integer s is a **non-lifting level** for V if $N^s_{s-1}(V) \neq 0$ and any non-zero element of $N^s_{s-1}(V)$ is a **non-liftable section** for V in degree s; s is a **socle level** for V if $\overline{N}^s_{s-1}(V) \neq 0$. The following result is one of the main tools in this paper. It was first proved by Strano (see [18]) and by Re (see [13] Lemma 1) and restated by Green in [6], where a particular case of the sequence (2.3) was also introduced (see [6] Proposition 4.31 and 4.37). For proofs in the most general cases, see also [19], Proposition 1.8.

Theorem 2.1. *(Strano) Let V be an equidimensional, locally Cohen-Macaulay, non-degenerate subvariety in $\mathbf{P}^N$ of dimension ≥ 1 and let W be its general hyperplane section. Then, the following sequence is exact:*

$$0 \to N^{m-i}_{m-i-1}(V) \to N^m_{m-i-1}(V) \xrightarrow{\cdot x} N^m_{m-i}(V) \xrightarrow{\pi} \overline{N}^m_{m-i}(W). \tag{2.3}$$

Moreover, if $N^m_{m-1}(V) \neq 0$, then $\overline{N}^{m'}_{m'-1}(W) \neq 0$, for some $m' \leq m$, so that a non lifting level m for V induces some non lifting level (on fact a socle level) $m' \leq m$ for W.

Proof. We only prove the latter part of the statement.

Suppose $N^m_{m-1}(V) \neq 0$ and consider the exact sequence (2.3) with $i = 1$; if the map π is injective, then also $\overline{N}^m_{m-1}(W) \neq 0$.

On the other hand, if π is not injective, then $N^m_{m-i}(V) \neq 0$, for some $i > 1$: let i_0 be the greatest i (it exists because V is equidimensional and locally Cohen-Macaulay, so that $H^1\mathcal{I}_V(t) = 0$ if $t \ll 0$).

Thus, by (2.3), $\overline{N}^m_{m-i_0}(W) \neq 0$ and then, again, $\overline{N}^{m'}_{m'-1}(W) \neq 0$ for some $m' \leq m$. In any case, thanks to the obvious inclusion $\overline{N}^m_{m-1}(W) \subset N^m_{m-1}(W)$, we can see that a non-liftable section for X induces a non-liftable section for W, in the same or in a lower degree, depending on the injectivity of π. $\square$

(6) If Z is a set of points in $\mathbf{P}^2$, the Hilbert function of Z is the integral valued function given by $h_Z(m) = h^0\mathcal{O}_{\mathbf{P}^2}(m) - h^0\mathcal{I}_Z(m)$ and the Castelnuovo function is its first difference $\Delta h_Z(m) = m + 1 - h^0\mathcal{I}_Z(m) + h^0\mathcal{I}_Z(m-1)$. Observe that we have $h^0\mathcal{I}_Z(m) - h^0\mathcal{I}_Z(m-1) = \dim_L H^0\mathcal{I}_Z(m) \leq m+1$ (see **Notation (2)**) so that $\Delta h_Z(m) = m + 1 - \dim_L H^0\mathcal{I}_Z(m) \geq 0$.

A set of points $Z \subset \mathbf{P}^2$ is in a **uniform position** (**U.P.** for short) or has the **uniform position property** (**U.P.P.** for short) if every subset of d' points $Z' \subset Z$ has the same Hilbert function: $h_{Z'}(t) = \min\{h_Z(t), d'\}$.

If I is any ideal in $\mathbf{k}[x_0, x_1, x_2]$ (for instance $I = I_Z$), $g_m(I)$ and $s_m(I)$ denote the numbers of degree m generators and first syzygies in its minimal free resolution.

Lemma 2.2. *Let Z be a set of d points in a uniform position in $\mathbf{P}^2$. If α, β and ρ are the degrees of the first and second minimal generator of I_Z and its regularity, then:*

(i) *The Castelnuovo function of Z has decreasing type, that is $\Delta h_Z(m) = m + 1$ if $0 \le m < \alpha$, $\Delta h_Z(\alpha) = \alpha + 1 - g_\alpha(I_Z)$, $\Delta h_Z(m) = \alpha$ if $\alpha \le m < \beta$, $\Delta h_Z(m + 1) \le \Delta h_Z(m) - 1$ if $\beta \le m < \rho$ (and equality holds if and only if $g_{m+1}(I_Z) = 0$), $\Delta h_Z(m) = 0$ if $m \ge \rho$.*
(ii) *$d = \sum_{m=0}^{\infty} \Delta h_Z(m) = \sum_{m=0}^{\rho-1} \Delta h_Z(m)$.*
(iii) *$\overline{n}^m_{m-1}(Z) = s_{m+2}(I_Z)$.*
(iv) *If $\overline{N}^m_{m-1}(Z) \neq 0$, then either $\Delta h_Z(m + 1) = 0$ or $\Delta h_Z(m + 1) - \Delta h_Z(m) \le -2$.*

Proof. For (i), (ii), (iii) (and also for other results used below) see for example [6] (Theorem 2.30, Proposition 4.12, 4.14 and 4.32 etc.). We only prove (iv).

Let us consider the (saturated) ideals $I = I_Z$ and $J = \mathrm{gin}(I_Z)$.

For every integer m we have $\dim I_m = \dim J_m$ so that $g_m(J) = \dim J_m - 2\dim J_{m-1} + \dim J_{m-2} = -\Delta h_Z(m) + \Delta h_Z(m-1)$; moreover $g_m(J) = s_{m+1}(J)$ and by the Cancellation Principle $g_m(I) - s_m(I) = g_m(J) - s_m(J)$. If $s_{m+2}(J) = s_{m+2}(I)$ then $g_{m+2}(J) = g_{m+2}(I)$ so that by Crystallization Principle I is $m+1-$regular and $\Delta h_Z(m + 1) = 0$.

If, in our hypothesis, we suppose I_Z not $m + 1$-regular, then $s_{m+2}(J) > s_{m+2}(I) \ge 1$ and so $\Delta h_Z(m+1) - \Delta h_Z(m) = -g_{m+1}(J) \le -2$ as requested. $\square$

(7) If $\sigma \in N^m_{m-i}(V,B), B \subseteq A_i$, then in the cohomology exact sequence of

$$0 \to \mathcal{I}_V(m-i) \to \mathcal{O}_{\mathbf{P}^N}(m-i) \to \mathcal{O}_V(m-i) \to 0 \qquad (2.4)$$

σ is the image of some $\tau \in H^0\mathcal{O}_V(m-i)$ and for every $\xi \in B$, $\xi\tau$ is cut on V by a hypersurface $F \in H^0\mathcal{O}_{\mathbf{P}^N}(m)$; so σ is a function on V given by $\frac{F}{\xi}$ on the open subset $V \cap \mathcal{U}_\xi$: by abuse of notation, we will say that $\frac{F}{\xi}$ is a **local equation** for σ on $\mathcal{U}_\xi$. We will denote any such polynomial F by $F_\xi(\sigma)$, or F_ξ for short, and any homogeneous degree $m+i$ polynomial $\xi F_\eta - \eta F_\xi$ by $F_{\xi\eta}(\sigma)$ or $F_{\xi\eta}$, for every ξ, $\eta \in B$. Note that both $F_\xi(\sigma)$ and $F_{\xi\eta}(\sigma)$ are not uniquely defined by σ, ξ and η. For instance if F_ξ is a local equation for σ on $\mathcal{U}_\xi$, also $F_\xi + G$ is so for every $G \in H^0\mathcal{I}_V(m)$; moreover "$\sigma = \sigma + G'$" for every $G' \in A_{m-i}$, so that $(F_\xi + \xi G', \mathcal{U}_\xi)$ is a local equation for σ.

If $B \subseteq A_1$ and $\xi = x_h$, $\eta = x_k$, then we will write F_h instead of F_{x_h} and F_{hk} instead of $F_{x_h x_k}$.

(8) In this paper X will always denote a codimension 2, locally Cohen-Macaulay, integral subvariety in $\mathbf{P}^N$; for every $i = 0, \dots, N-3$, X_i is the section of X with a general linear space $H_{i+2} = \mathbf{P}^{i+2}$ in $\mathbf{P}^N$ defined by $x_N = \dots = x_{i+3} = 0$ (having chosen general coordinates with respect to X). In particular:

$H_{N-1} = K$ and $X_{N-3} = Y$ denote the general hyperplane and hyperplane section of X;

$X_1 = C$ denotes the integral curve section of X with a general 3-space $H_3 = M \cong \mathbf{P}^3$;

$X_0 = Z$ denotes the set of points in U.P. section of X with the general plane $H_2 = H \cong \mathbf{P}^2$;

$L = H_1 \cong \mathbf{P}^1$ denotes a general line in $\mathbf{P}^N$.

Henceforth, we will suppose that Z is not a complete intersection; this assumption excludes only the following, few subvarieties X: complete intersections, curves of even degree on a smooth quadric surface in $\mathbf{P}^3$ and degree 4 arithmetically Buchsbaum surfaces in $\mathbf{P}^4$ with $H^1_*\mathcal{I}_X \cong \mathbf{k}$ and $H^i_*\mathcal{I}_X = 0$ for $i = 2, \dots, N-2$ that is the Veronese surface in $\mathbf{P}^4$ and some degeneration of it (see [16] Theorem 4.6).

3. Generalized socles and hypersurfaces

In this section we show some further relations between generalized socles for a subvariety V in $\mathbf{P}^N$ and hypersurfaces in $\mathbf{P}^N$ containing V or hypersurfaces in a general linear space $H \subset \mathbf{P}^N$ containing $V \cap H$.

Let σ be any non-zero element in $N^m_{m-i}(V, B)$ (see **Notation (7)**); for every $\xi, \eta \in B$, local equations $\frac{F_\xi}{\xi}$ and $\frac{F_\eta}{\eta}$ for σ must coincide on $V \cap \mathcal{U}_\xi \cap \mathcal{U}_\eta$. So $F_{\xi\eta} = \xi F_\eta - \eta F_\xi$ defines the zero function on the open subset $V \cap \mathcal{U}_\xi \cap \mathcal{U}_\eta$ of V. If we suppose that V is integral and not contained in any degree i hypersurface, $V \cap \mathcal{U}_\xi \cap \mathcal{U}_\eta$ is a dense subset of V and then $F_{\xi\eta} \in H^0\mathcal{I}_V(m+i)$.

If V is reduced, but possibly reducible, $\xi\eta F_{\xi\eta} \in H^0\mathcal{I}_V(m+3i)$; for a non reduced V, we can only say that $\xi^a\eta^b F_{\xi\eta}(\sigma) \in H^0\mathcal{I}_V(m+ i(1+a+b))$, for some positive integers a, b.

Let V be a subvariety in $\mathbf{P}^N$ of dimension ≥ 1 and let W be its general hyperplane section. By definition, a degree m non-liftable section σ for V corresponds to some hypersurface in $K = \mathbf{P}^{N-1}$ containing W, which does not come from any hypersurface containing V in $\mathbf{P}^N$: the degree m polynomial $F_N(\sigma)$ is indeed "the non-liftable section" for V. To be explicit, we can use the cohomology exact sequences of $(2.1) \otimes \mathcal{O}_{\mathbf{P}^N}(m-1)$ and $(2.1) \otimes \mathcal{O}_{\mathbf{P}^N}(m)$ and we see that $(F_N)_K =$ (the equivalence class of F_N modulo x_N) is a global section of $\mathcal{I}_W(m)$ which does not belong to the image of $H^0\mathcal{I}_V(m)$.

The following two results, proved in [16], state close relations between non-zero generalized socle of V in degree m and the number of independent hypersurfaces or minimal generators in I_V in degrees close to m.

Theorem 3.1. *Let V be an integral subvariety of $\mathbf{P}^N$ and let $B \subseteq A_1$ be a vector space of dimension $r \geq 2$, such that $N^m_{m-1}(V, B) \neq 0$.*

Then $h^0\mathcal{I}_V(m+1) \geq 2r-3$.

Lemma 3.2. *Let Z be a set of d points in $\mathbf{P}^2$ in U.P. Suppose that m is a socle level for Z and denote by ν the number of minimal generators of I_Z in degrees $\leq m+1$. Then:*

(i) *$\nu \geq 3$ or Z is a complete intersection (p,q), $p+q=m+2$. Thus, all curves in $H^0\mathcal{I}_Z(m+1)$ cannot have a common component.*

(ii) *$d \leq m^2+m+1$.*

Lemma 3.3. *Let V be an integral subvariety in $\mathbf{P}^N$ and let σ be any non zero element in $\overline{N}^s_{s-1}(V)$. Then the ideal I generated by $\{F_{xy}(\sigma)\,,\, x,y \in A_1\}$ has height at least 2.*

Proof. Let us suppose, on the contrary, that I is contained in a principal ideal $(G)=GA$ for some $G \in A_r$: then, by the way, $F_{i0}(\sigma)=x_iF_0-x_0F_i \in (G)$ for every $i=1,\dots,N$ and, in particular, $x_1F_0-x_0F_1=GM$. As $x_0,\dots,x_N$ are general coordinates, $G \notin (x_0,x_1)$ and so $M=x_1M'+x_0M''$.

Then x_0 divides $F_0-M'G$, that is $F_0-x_0P=M'G$ for a suitable $P \in A_{s-1}$. Since $(\frac{F_i-x_iP}{x_i}\,,\mathcal{U}_i)$ are also local equations for σ, we may suppose that P is 0 and G divides F_0, that is $F_0=Q_0G$. Then, for every $i=1,\dots,N$, $x_0F_i \in (G)$ so that $F_i=Q_iG$. If V is contained in the hypersurface G, then $\sigma=0$ against our hypothesis. If, on the contrary, V is not contained in G, then $x_iQ_j-x_jQ_i \in I_V$ (recall that I_V is a prime ideal) and then $\frac{Q_i}{x_i}$ are local equations for a non zero element of $\overline{N}^{s-r}_{s-r-1}(V)$ and $\sigma=G\tau=0$, against the hypothesis. □

Lemma 3.4. *Let V be an integral subvariety in $\mathbf{P}^N$ (of dimension at least 1) and let σ be any non zero element in $\overline{N}^s_{s-i}(V)$. If H is the general linear space given by $x_N=\cdots=x_{N'+1}=0$, $N' \geq \operatorname{codim} V$, and m is a degree i monomial of the type x_jm', for some $i \geq 1$ and $j > N'$, then $(F_m(\sigma))_H$ belongs to $H^0\mathcal{I}_{V\cap H}(s)$.*

Proof. Take the monomial $m'=x_{N'}^i$; as $m'F_m-mF_{m'} \in H^0\mathcal{I}_V(s+i)$, its image belongs to $H^0\mathcal{I}_{V\cap H}(s+i)$. But $(m'F_m-mF_{m'})_H=(x_{N'}^iF_m)_H$ and $I_{V\cap H}$ is saturated so that also $(F_m)_H$ belongs to

$I_{V\cap H}$. $\square$

Lemma 3.5. *Let V be an integral subvariety in $\mathbf{P}^N$ (of dimension at least 1) and σ be any non zero element of $\overline{N}^s_{s-1}(V)$, such that $\sigma_K = 0$ for the general hyperplane $K = \mathbf{P}^{N-1}$ given by $x_N = 0$. Then for a suitable choice of local equations $\frac{F_i}{x_i}$ for σ, $(F_i)_K$ belongs to $H^0\mathcal{I}_{V\cap K}(s)$ for every $i = 0, \ldots, N$.*

Proof. Thanks to Theorem 2.1, the hypothesis $\sigma_K = 0$ implies $\sigma = x_N\tau$, for some $\tau \in \overline{N}^s_{s-2}(V)$. We can chose local equations $\frac{F_i}{x_i}$ for σ so that τ is given on $\mathcal{U}_N \cap \mathcal{U}_i$ by $\frac{F_i}{x_i x_N}$; on the other hand for every monomial m of a sufficiently high degree h, τ is also defined by a local equation $\frac{G_m}{m}$ on the open subset $\mathcal{U}_m$, where $mF_i - x_i x_N G_m \in H^0\mathcal{I}_V(s+h)$.

For $m = x^h_{N-1}$ this means $x^h_{N-1}F_i - x_i x_N G \in H^0\mathcal{I}_V(s+h)$ and so $(x^h_{N-1}F_i)_K \in H^0\mathcal{I}_{V\cap K}(s+h)$; $I_{V\cap K}$ being saturated, we get $(F_i)_K \in H^0\mathcal{I}_{V\cap K}(s)$, as required. Note that the hyperplane $x_{N-1} = 0$ in $K = \mathbf{P}^{N-1}$ does not contain any component of $V \cap K$, even if $V \cap K$ is a set of points, due to the choice of general coordinates. $\square$

From now on we consider only codimension 2 subvarieties: see **Notation (8))**.

Lemma 3.6. *Let $s = s_0(X)$ the minimal non lifting level for X. If s is also the minimal non lifting level $s_0(C)$ for C, then $\dim_L H^0\mathcal{I}_Z(s) \geq \dim_L H^0\mathcal{I}_X(s) + N - 2$.*

Proof. We will just prove $\dim_L H^0\mathcal{I}_Y(s) \geq \dim_L H^0\mathcal{I}_X(s) + 1$; the complete statement easily follows by induction on N.

Let σ be a non-zero element of $N^s_{s-1}(X)$ defined on $\mathcal{U}_N$ by a local equation $\frac{F_N}{x_N}$; by construction $(F_N)_K$ belongs to $H^0\mathcal{I}_Y(s)$. We can choose F_N such that its initial term is as small as possible and, in particular, not contained in $\mathrm{gin}(H^0\mathcal{I}_X(s))$: thus to get the conclusion it is enough to prove that only x_0 and x_1 could explicitly appear in $\mathrm{gin}(F_N)$.

On the contrary, let i be the greatest index such that x_{i+2} appears; the image of F_N in $H^0\mathcal{I}_{X_i}(s)$ contains x_{i+2} as a factor that is $(F_N)_H = x_{i+2}(G)_{H_{i+2}}$ for some $G \in A$; I_{X_i} being saturated, G must belong to $H^0\mathcal{I}_{X_i}(s-1)$.

As $s = s_0(X) = s_0(C)$, every restriction map $H^0\mathcal{I}_{X_j}(s-1) \to H^0\mathcal{I}_{X_{j-1}}(s-1)$ is surjective and then we can choose $G \in H^0\mathcal{I}_X(s-1)$ so that $F_N - x_N G \in (x_{i+1}, \dots, x_N)$; but $\frac{F_N - x_N G}{x_N}$ also defines σ on $\mathcal{U}_N$ and $\mathrm{gin}(F_N - x_N G)$ is lower than $\mathrm{gin}(F_N)$, against its minimality. $\square$

The following result will be a key point in §4. Note that it is sharp for what concerns both $a = h^0\mathcal{I}_X(s)$ and the bound on $h^0\mathcal{I}_Z(s+1)$. If the general plane section Z of X is not contained in curves of degree lower than s (s being a socle level), then $h^0\mathcal{I}_Z(s+1) \geq 2h^0\mathcal{I}_Z(s) + 2 \geq 2(h^0\mathcal{I}_X(s) + N - 2) + 2 = 2(a + N - 2) + 2$; Proposition 3.7 shows that this is in fact a strict inequality provided $a = h^0\mathcal{I}_X(s) \leq 2$. On the contrary, for every $a \geq 3$, and $s > a$ there are curves in $\mathbf{P}^3$ with socle level s and $a = h^0\mathcal{I}_C(s)$ such that equality $h^0\mathcal{I}_Z(s+1) = 2h^0\mathcal{I}_Z(s) + 2 = 2(a+1) + 2$ holds : see Example 4.9. Moreover, there are codimension 2 subvarieties X in $\mathbf{P}^N$ for every possible N, a and s as in the hypotheses of Proposition 3.7 for which the minimum allowed $2(a+N-2)+3$ for $h^0\mathcal{I}_Z(s+1)$ holds: see Example 3.8.

Proposition 3.7. *Let σ be any non zero element in $\overline{N}^s_{s-1}(X)$ with local equations $\frac{F_i}{x_i}$. Suppose that $s > a + N - 2$, $h^0\mathcal{I}_X(s) = a \leq 2$, $h^0\mathcal{I}_Z(s) = a + N - 2$ and $h^0\mathcal{I}_Z(s-1) = 0$. Then the subspace of $H^0\mathcal{I}_Z(s+1)$ generated by*

$$\{x_i H^0\mathcal{I}_X(s)_H,\ i = 0,1,2\} \cup \{F_{ij}(\sigma)_H, i = 0,1,2, j = 0,\dots,N\}$$

has dimension $\geq 2(a+N-2)+3$.

Proof. We may suppose $a = 2$ (if $a < 2$ we consider the general hyperplane section of X: see Lemma 3.4). So, let us fix generators F_{N+2}, F_{N+1} for $H^0\mathcal{I}_X(s)$.

First of all, observe that in our hypotheses, $(F_{N+2})_H, (F_{N+1})_H, (F_N)_H, \dots, (F_3)_H$ are free generators for $H^0\mathcal{I}_Z(s)$; thus we can choose the F_i's such that $\mathrm{gin}(F_{N+2}) > \mathrm{gin}(F_{N+1}) > \mathrm{gin}(F_N) > \cdots > \mathrm{gin}(F_3)$ and, more precisely:

$$x_0^s = \mathrm{gin}(F_{N+2})\,,\ x_0^{s-1}x_1 = \mathrm{gin}(F_{N+1})\,,$$

$$x_0^{s-2}x_1^2 = \mathrm{gin}(F_N)\dots, x_0^{s-N+1}x_1^{N-1} = \mathrm{gin}(F_3).$$

Let us denote by F_{ij} the degree $s+1$ polynomials $F_{ij}(\sigma)$ for $0 \le i < j \le N$ (see **Notation (7)**); for sake of simplicity, we also use F_{ij} meaning x_iF_j if $0 \le i \le N$ and $N+1 \le j \le N+2$.

Let I'_X be the ideal in $k[X_0, \dots, X_N]$ generated by $\{F_{N+2}, F_{N+1}\} \cup \{F_{ij}, 1 \le i.j \le N\}$ and let J'_X be its generic initial ideal. Moreover let I'_Z and J'_Z be respectively the ideal in $k[X_0, X_1, X_2]$ generated by $\{(F_{N+2})_H, \dots (F_3)_H, (F_{01})_H, (F_{02})_H, (F_{12})_H\}$ and its generic initial ideal.

If the dimension of the vector space $(I'_Z)_s$ is $2N+1$, then J'_Z would not have any new generator in degree $s+1$ and I would be s-regular (see [6] Proposition 2.28) against the hypothesis $N < s$.

Moreover if $\sigma_H \neq 0$, then the dimension is at least $2N+3$.

So, **suppose that the dimension is** $2N+2$ **and** $\sigma_H = 0$.

As $\sigma_H = 0$, we can assume that F_0, F_1, F_2 belong to the ideal generated by $x_3, \dots, x_N$.

In degree $s+1$, J'_Z is generated by the $2N+2$ monomials:

$$x_0^{s+1}, x_1(x_0^s\,,\ \dots\ ,\ x_0^{s-N+1}x_1^{N-1})\,,\ x_2(x_0^s\,,\ \dots, x_0^{s-N+1}x_1^{N-1})\,,\ x_0^{s-N}x_1^{N+1}.$$

Then, in degree $s+1$, I'_Z is generated by the $2N+2$ polynomials:

$$(F_{0N+2})_H,\ \ (F_{1N+2})_H, (F_{1N+1})_H \dots, (F_{13})_H,\ \ (F_{2N+2})_H,$$

$$(F_{2N+1})_H \dots, (F_{23})_H,\ \ (F_{0t})_H,$$

where t is the smallest integer k, $3 \le k \le N+1$ such that $x_0^{s-N}x_1^{N+1} = \mathrm{gin}(F_{0k} - \sum_{j=3}^{k+1} b_{kj}F_{1j})$, b_{kj} being suitable constant coefficients.

Such an integer t does in fact exist: first of all, for every k we can find coefficients b_{kj} such that $\mathrm{gin}(F_{0k} - \sum_{j=3}^{k+1} b_{kj}F_{1j}) \leq x_0^{s-N}x_1^{N+1}$; if strict inequality holds, then $(F_{0k} - \sum_{j=3}^{k+1} b_{kj}F_{1j})_H = x_2 G$ for some $G \in H^0\mathcal{I}_Z(s)$ and so $(F_{0k} - \sum_{j=3}^{k+1} b_{kj}F_{1j} - \sum \beta_{kj}F_{2j})_H = 0$. If this would happen for every $k = 3, \dots N+1$, then there would be $N-1$ independent syzygies in degree $s+1$ both for I'_Z and for J'_Z, then I'_Z would be s-regular (see [1] Theorem 2.4).

Easy calculations show that the dimension of I'_X in degree $s+1$ is $\frac{(N+2)(N+1)}{2}$. Thus the following monomials are a bases for $(J'_X)_{s+1}$:

(1) $x_0^s x_i = \mathrm{gin}(F_{iN+2}) \quad , \quad 0 \leq i \leq N$
(2) $x_0^{s-1}x_1x_i = \mathrm{gin}(F_{iN+1}) \quad , \quad 1 \leq i \leq N$
(3) $x_0^{s-N+j-2}x_1^{N-j+2}x_i = \mathrm{gin}(F_{ij}) \; , \quad 1 \leq i < j \leq N \; , \quad j \geq 3$
(4) $x_0^{s-N}x_1^{N+1} = \mathrm{gin}(F_{0t} - \sum_{j=3}^{t+1} b_{tj}F_{1j})$,

because they are $\frac{(N+2)(N+1)}{2}$ linearly independent monomials in $(J'_X)_{s+1}$.

The corresponding polynomials F_{ij} form a basis for $(I'_X)_{s+1}$ as a $\mathbf{k}$-vector space and

$$\mathcal{B} = \{F_{N+2}, F_{N+1}\} \cup \{F_{ij}, \;\; 1 \leq i < j \leq N, \;\; j \geq 3\} \cup \{F_{0t}\}$$

is a set of minimal generators for I'_X in degrees $\leq s+1$.

Let S_{ijk} be the degree $s+2$-relation between elements of $\mathcal{B}$ corresponding to the identity:

$$x_i F_{jk} - x_j F_{ik} + x_k F_{ij} = 0.$$

Claim: the $\frac{N^3-N}{6}$ relations S_{ijk} are linearly independent.

We can write the F_{ij} which don't belong to $\mathcal{B}$ as linear combinations of elements in $\mathcal{B}$ as follows:

$$\left\{\begin{array}{lcl} F_{0k} & = & (\sum_{j=3}^{k+1} b_{kj}F_{1j}) + d_k(F_{0t} - \sum_{j=3}^{t+1} b_{tj}F_{1j}) + (.) \\ & & \text{for} \quad t-1 \leq k \leq N+1, \\ \\ F_{0k} & = & (\sum_{j=3}^{k+1} b_{kj}F_{1j}) + (.) \quad \text{for} \quad 3 \leq k \leq t-1, \\ \\ F_{01} & = & \sum_{jk} p_{2jk}F_{jk} \quad \text{for} \quad 3 \leq j \leq N, \quad 3 \leq k \leq N+2, \\ \\ F_{02} & = & \sum_{jk} p_{1jk}F_{jk} \quad \text{for} \quad 3 \leq j \leq N \quad 3 \leq k \leq N+2, \\ \\ F_{12} & = & \sum_{jk} p_{0jk}F_{jk} \quad \text{for} \quad 3 \leq j \leq N, \quad 3 \leq k \leq N+2, \end{array}\right. \tag{3.1}$$

where the (.) contain combinations of F_{ij} with $i \geq 2$.

Suppose that $\sum_{ijk} \alpha_{ijk}S_{ijk} = 0$, $\alpha_{ijk} \in \mathbf{k}$, is a relation among the S_{ijk}. Then, $0 = \psi(\sum_{ijk} \alpha_{ijk}S_{ijk}) = \sum_{ijk} \alpha_{ijk}\psi(S_{ijk}) = \sum_{ij} H_{ij}e_{ij} + Qe_{N+2} + Q'e_{N+1}$ and all the H_{ij}'s, which are linear forms $H_{ij} = \sum_{k=0}^{N} h_{ijk}x_k$, and also Q and Q', which are degree 2 forms, must be zero.

We will compute a few coefficients h_{ijk} in terms of the α_{ijk} and of the constant coefficients that appear in (3.1). To the aim of uniforming our notation, we extend the definition of α_{ijk} to every set of three integers $i, j, k \in \{0, 1, \ldots, N+2\}$ by: $\alpha_{ijk} = \alpha_{jik} = \cdots = \alpha_{kji}$ and also $\alpha_{ijk} = 0$ if at least two of the indices are equal or one of them is greater than N.

If j is either 1 or 2, and $k \geq 3$, then $h_{jk0} = \alpha_{0jk}$: thus $\alpha_{01k} = \alpha_{02k} = 0$ for every $k \geq 3$.

Moreover $h_{2k1} = \alpha_{12k}$ and so $\alpha_{12k} = 0$ for every $k \geq 3$.

Let us now consider indices i, j, k such that $F_{ij} \in \mathcal{B}$ and $k \geq 3$ (we do not suppose $j < k$); by what just proved, h_{ijk} is sum of α_{ijk} and a linear combination of some α_{0lk}'s; if we prove that $\alpha_{0lk} = 0$ for every l, we also obtain that $\alpha_{ijk} = 0$. Let us use descent on k.

If $k = N+2$, then $\alpha_{ijk} = 0$ by definition. Now, let k be any integer $3 \leq k \leq N+1$; we suppose that $\alpha_{ijl} = 0$ for every i, j, l so that $F_{ij} \in \mathcal{B}$ and $l \geq k+1$ and we show that also $\alpha_{ijk} = 0$.

If $k = t$, for every l we have $0 = h_{0tl} = \alpha_{0tl} + \sum_{m \geq t+1} d_m\alpha_{0ml} = \alpha_{0tl}$ and from this $\alpha_{ijt} = 0$ for every i, j such that $F_{ij} \in \mathcal{B}$.

If $k \neq t$, for every l we have $0 = h_{1k+1l} = \alpha_{1lk+1} + \alpha_{0lk} = \alpha_{0lk}$; then $\alpha_{0lk} = 0$ for every l and so $\alpha_{ijk} = 0$ for every ij such that $F_{ij} \in \mathcal{B}$.

We have just proved that all the α_{ijk} are zero, unless possibly α_{012}.

Let us suppose $\alpha_{012} \neq 0$. For every i, j such that $F_{ij} \in \mathcal{B}$, we have $0 = h_{ij0} = \alpha_{012} p_{0ij}$; so p_{0ij} must be zero for every i, j such that $F_{ij} \in \mathcal{B}$.

$$\left\{\begin{array}{lcl} F_{01} & = & (\sum_{j=3}^N x_j p_{2jN+1})F_{N+1} + (\sum_{j=3}^N x_j p_{2jN+2})F_{N+2} \\ & = & y_2 F_{N+1} + z_2 F_{N+2} \\ \\ F_{02} & = & (\sum_{j=3}^N x_j p_{1jN+1})F_{N+1} + (\sum_{j=3}^N x_j p_{1jN+2})F_{N+2} \\ & = & y_1 F_{N+1} + z_1 F_{N+2} \\ \\ F_{12} & = & (\sum_{j=3}^N x_j p_{0jN+1})F_{N+1} + (\sum_{j=3}^N x_j p_{0jN+2})F_{N+2} \\ & = & y_0 F_{N+1} + z_0 F_{N+2} \end{array}\right. \tag{3.2}$$

Thus S_{012} leads to the identity: $(y_2x_2 - y_1x_1 + y_0x_0)F_{N+1} + (z_2x_2 - z_1x_1 + z_0x_0)F_{N+2} = 0$. But F_{N+1} and F_{N+2} are irreducible polynomials of degree $s > 2$ and then also $p_{2jN+2} = p_{1jN+2} = p_{0jN+2} = p_{2jN+1} = p_{1jN+1} = p_{0jN+1} = 0$ and so $F_{01} = F_{02} = F_{12} = 0$, which is not allowed (on the contrary σ should vanish).

Then I'_X has at least $(N+1)N(N-1)/6$ syzygies in degree $s+2$. On the other hand, by direct computations, we can see that J'_X has exactly this number of independent syzygies in degree $s+2$: then by Crystallization Principle (see [6] Proposition 2.28) I'_X is $s+1$-regular and $s = N$, against the assumption. $\square$

Example 3.8. (see also [2] and [3]) For every three integers N, a, s such that $N \geq 3$, $0 \leq a \leq 2$ and $s > a + N - 2$, let ϕ be a general map :

$$\phi\colon\ \mathcal{O}_{\mathbf{P}^N}(a + N - 2 - s) \oplus \mathcal{O}_{\mathbf{P}^N}^{a+N-2} \longrightarrow \Omega_{\mathbf{P}^N}(2) \oplus \mathcal{O}_{\mathbf{P}^N}(1)^a.$$

As $(\Omega_{\mathbf{P}^N}(2) \oplus \mathcal{O}_{\mathbf{P}^N}(1)^a) \otimes (\mathcal{O}_{\mathbf{P}^N}(a + N - 2 - s) \oplus \mathcal{O}_{\mathbf{P}^N}^{a+N-2})^\vee$ is generated by global sections, ϕ is injective and degenerates on an integral, codimension 2 subvariety X (see [5]):

$$0\to\mathcal{O}_{\mathbf{P}^N}(a+N-2-s)\oplus\mathcal{O}_{\mathbf{P}^N}^{a+N-2}\xrightarrow{\phi}\Omega_{\mathbf{P}^N}(2)\oplus\mathcal{O}_{\mathbf{P}^N}(1)^a$$
$$\longrightarrow\mathcal{I}_X(s+1)\to 0.$$

Thus the ideal sheaf of the general plane section Z of X has resolution:

$$0\longrightarrow\mathcal{O}_{\mathbf{P}^2}(a+N-2-s)\oplus\mathcal{O}_{\mathbf{P}^2}^{a+N-2}\longrightarrow\Omega_{\mathbf{P}^2}(2)\oplus\mathcal{O}_{\mathbf{P}^2}(1)^{a+N-2}$$
$$\longrightarrow\mathcal{I}_Z(s+1)\longrightarrow 0$$

so that $h^0\mathcal{I}_Z(s)=a+N-2$ and $h^0\mathcal{I}_Z(s+1)=(a+N-2)(h^0\mathcal{O}_{\mathbf{P}^2}(1)-h^0\mathcal{O}_{\mathbf{P}^2})+h^0\Omega_{\mathbf{P}^2}(2)=2(a+N-2)+3$.

4. **Bounds on the degree**

In this section, X, Y, C, Z K, M, H and L will be as in **Notation (8)**.

The main goal of the paper is to give upper bounds on the degree of an integral subvariety X depending on a pair of suitable integers (s,k) (s being either a non-lifting level or a socle level for X and k a non negative integer connected to the number of independent, degree s hypersurfaces containing X and/or its sections). According to this aim, we introduce the function $D(s,k)$, defined on every pair of positive integers (s,k), $k\le s+1$ by:

$$D(s,k)=\binom{s+1}{2}+\binom{s-k+1}{2}+1,$$

that is:

$$D(s,k)=s^2-s(k-1)+\binom{k}{2}+1.$$

Lemma 4.1. *Let Z a set of d points in U.P. in $\mathbf{P}^2$.*

(i) *If $h^0\mathcal{I}_Z(s)-h^0\mathcal{I}_Z(s-1)\ge k+1$, then $d\le D(s,k)-1$.*

(ii) *If $h^0\mathcal{I}_Z(s)-h^0\mathcal{I}_Z(s-1)=k$ and $h^0\mathcal{I}_Z(s+1)-h^0\mathcal{I}_Z(s)\ge k+3$, then $d\le D(s,k)$.*

Proof. To prove (i) we use Lemma 2.2 and observe that in any case the Castelnuovo function $\Delta h_Z(m)$ is at most $m+1$; moreover in our hypothesis, $\Delta h_Z(s) \leq s-k$ and, having decreasing type, $\Delta h_Z(s+t) \leq \max\{s-k-t\,,\,0\}$ for every $t \geq 1$. An easy computation gives the desired inequality. The proof of (ii) is quite similar to the previous one. □

Lemma 4.2. *Let Z be a set of d points in $\mathbf{P}^2$ in a U.P.*

(i) *If* $N^m_{m-1}(Z,B) \cap \overline{N}^{m+t+1}_{m-1}(Z) \neq 0$ *with* $\dim(B) = 2$ *and* $t \geq 0$ *then either Z lies on a degree $m+1$ curve or* $d \leq \frac{(m+t+3)(m+t+2)}{2}$.

(ii) *If* $\overline{n}^{m+1}_m(Z) = 1$ *and* $\overline{n}^{m+1}_{m-1}(Z) \geq 2$, *then either Z lies on two degree $m+1$ curves or* $d \leq \frac{(m+3)(m+2)}{2}$.

Proof. (i) Let us choose a general basis x, y for the vector space B, so that $Z \cap \{x=0\} = Z \cap \{y=0\}$ contains at most the point $P := (x=y=0)$. Let us take any non-zero element $\sigma \in N^m_{m-1}(Z,B)$: it has local equations $\frac{F_x}{x}$ and $\frac{F_y}{y}$ on the open sets $\mathcal{U}_x$ and $\mathcal{U}_y$ respectively and $\frac{F_z}{z^{2+t}}$ on the open set $\mathcal{U}_z$ for a general linear form z (see **Notation (7)**). Then, $yF_x - xF_y \in H^0\mathcal{I}_{Z'}(m+1)$, where $Z' = Z - \{P\}$ contains at least $d-1$ points of Z.

If x divides F_x, let $F_x = xF$, then σ is (the isomorphic image of) the class modulo $H^0\mathcal{O}_{\mathbf{P}^2}(m)$ of some global section $\tau \in H^0\mathcal{O}_Z(m)$ which is defined on Z' by F (however, $\tau \neq F$ on Z, since $\sigma \neq 0$). But in this case we can also consider σ as the class of $\tau - F$, which is the 0-function on Z'. Thus, $F_z - z^{t+2}F \in H^0\mathcal{I}_{Z'}(m+t+1)$, while $F_z - z^{t+2}F \notin H^0\mathcal{I}_Z(m+t+1)$ and, by the U.P.P., we get the desired bound $d-1 = h_{Z'}(m+t+1) < h_Z(m+t+1) \leq \frac{(m+t+3)(m+t+2)}{2}$.

If on the contrary x does not divide F_x, then either $yF_x - xF_y$ defines a curve containing Z or $H^0\mathcal{I}_Z(m+1) \neq H^0\mathcal{I}_{Z'}(m+1)$; again by the U.P.P., we get the bound $d-1 \leq d' = h_{Z'}(m+1) < h_Z(m+1) = \frac{(m+3)(m+2)}{2}$.

(ii) If $\overline{N}^m_{m-1}(Z) \neq 0$, then $h^0\mathcal{I}_Z(m+1) \geq 3$.

If $\overline{N}^m_{m-1}(Z) = 0$, every σ in $\overline{N}^{m+1}_{m-1}(Z)$ also belongs to $N^m_{m-1}(Z, B_\sigma)$ for some 2-dimensional vector space B_σ, where $B_\sigma \neq B_{\sigma'}$ when σ and σ' are linearly independent. As shown in the proof of the previous point, either the bound on the degree holds or every $\sigma \in \overline{N}^{m+1}_{m-1}(Z)$ corresponds to some non-zero $F_\sigma \in H^0\mathcal{I}_Z(m+1) \cap B_\sigma A$. If $h^0\mathcal{I}_X(m+1) = 1$ then every F_σ belongs to $\cap_\sigma B_\sigma A$ which is at most a principal ideal generated by a linear form x so that $H^0\mathcal{I}_Z(m+1) = \mathbf{k}xG$: this is not allowed by the U.P.P. $\square$

In [19] (Theorem 0.1) Tortora proved the following result:

Theorem 4.3. *Let d and s be the degree and a non lifting level for $X \subset \mathbf{P}^N$ and let $a = h^0\mathcal{I}_X(s)$. If the following two conditions hold:*
(1) $H^0\mathcal{I}_Z(s-1) = 0$
(2) $N^s_{s-1}(X) \to \overline{N}^s_{s-1}(Z)$ *is not the zero map*
then $d \leq D(s, N+a-2)$.

The two conditions (1) and (2) are crucial points in Tortora's proof: (2) implies $\overline{N}^s_{s-1}(Z) \neq 0$ which means that there is at least a double down step of the Castelnuovo function $\Delta h_Z(s+1)$ (see Lemma 2.2 (iv)) beyond (at least) $a+N-3$ down steps in degree s which are consequence of (1). (To be precise, in [19] Condition (2) is written in the slightly different manner: "$N^s_{s-1}C) \to \overline{N}^s_{s-1}(Z)$ not zero", but, in that contest, the two conditions are equivalent.)

Though in the proof of Theorem 4.3 conditions (1) and (2) look both necessary and their outcomes completely independent, they are strictly related so that assuming both turns out to be in some sense redundant.

Namely condition (2) says something stronger than $\overline{N}^s_{s-1}(Z) \neq 0$ allowing a more general statement, not requiring (1) (see Theorem 4.5) and in the last part of the paper we obtain bounds on the degree assuming that only (1) holds (see Corollary 4.6 and Proposition 4.7).

However, we cannot completely avoid both conditions (1) and (2), as the following example shows.

Example 4.4. Curves with non lifting-level s and degree $d > D(s, a-m)$ for every m and $a = h^0\mathcal{I}_C(s) \geq 3$.

Let Y be the union of a pair of skew lines in $\mathbf{P}^3$ and consider, for every positive integer l, the reflexive sheaf $\mathcal{F}$ defined as an extension by:

$$0 \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{F}(-l) \to \mathcal{I}_Y(-2l) \to 0.$$

Easy computations on Chern classes give $c_1(\mathcal{F}) = 0$ and $c_2(\mathcal{F}) = 2 - l^2$. If t is any whole number ($t \geq l+2$), then a general section of $\mathcal{F}(t)$ degenerates on an integral curve C of degree $2 - l^2 + t^2$ and for which $s = t + l + 1$ is a socle level. Moreover $a = h^0\mathcal{I}_C(s) = h^0\mathcal{F}(l+1) = h^0\mathcal{O}_{\mathbf{P}^3}(2l+1)$ does not depend on t and exceeds l^3, so that for $t \gg l \gg 0$ the degree $d = s^2 - 2s(l+1) + 2l + 3$ is greater than $s^2 - s(a-m) + G(a)$ for every fixed m and function $G(a)$.

Theorem 4.5. *Let X be a codimension 2, locally Cohen-Macaulay, integral subvariety in $\mathbf{P}^N$, $N \geq 3$, and let d be the degree of X, s a non-lifting level for X, $a = \dim_L H^0\mathcal{I}_X(s)$ and $b = \dim_L H^0\mathcal{I}_Y(s)$.*

(i) *If s is also the minimal non-lifting level $= s_0(C)$ for C, then*

$$d \leq D(s, a+N-3) - 1 = s^2 - s(a+N-4) + \binom{a+N-3}{2}$$

and equality holds if and only if $h^0\mathcal{I}_Z(s-1) = 0$, $h^0\mathcal{I}_Z(s) = a+N-2$ and moreover, for every $t \geq s$, $\Delta h_Z(t+1)$ is either $\Delta h_Z(t) - 1$ or 0.

(ii) *If both $s = s_0(C)$ and $\overline{N}^s_{s-1}(Z) \neq 0$ hold, then*

$$d \leq D(s, a+N-2) = s^2 - s(a+N-3) + \binom{a+N-2}{2} + 1$$

and equality holds if and only if $h^0\mathcal{I}_Z(s-1) = 0$, $h^0\mathcal{I}_Z(s) = a+N-2$, $h^0\mathcal{I}_Z(s+1) = 2(a+N-2)+3$ and moreover, for every $t \geq s+1$, $\Delta h_Z(t+1)$ is either $\Delta h_Z(t) - 1$ or 0.

(iii) *If $N \geq 4$ and $\overline{N}^s_{s-1}(Y) \to \overline{N}^s_{s-1}(Z)$ is not the zero map, then*

$$d \leq D(s, b+N-3) = s^2 - s(b+N-4) + \binom{a+N-3}{2} + 1.$$

Proof. A proof of (i) immediately follows from Lemma 3.6 and Lemma 4.1 (i), while for (ii) we need Lemma 3.6, Lemma 2.2 (iv) and Lemma 4.1 (ii).

So we just have to prove (iii).

Let us take an element $\sigma \in \overline{N}^s_{s-1}(Y)$ such that $\sigma_H \neq 0$. If $\frac{F_i}{x_i}$ are local equations for σ on $K \cap \mathcal{U}_i$, then $(F_i)_H \in H^0\mathcal{I}_Z(s)$ for every $i = N-1, \dots, 3$ (Lemma 3.4). We can choose the F_i's such that their initial terms are as small as possible, and in particular not contained in $\text{gin}(H^0\mathcal{I}_Y(s))$. Furthermore, under a suitable change of coordinates (which does not alter K, H and L and initial terms: see **Notation (2))**), we may also suppose $\text{gin}(F_{N-1}) > \dots > \text{gin}(F_2)$.

Claim: the initial terms of $F_{N-1}, \dots, F_3$ are of the type $x_0^\bullet x_1^\bullet$.

If not, some x_i, $i \geq 2$, would certainly appear in the initial term of F_3 and then consequently in the initial term of F_2 which is smaller: so, in $\mathbf{k}[x_0, x_1, x_2]$, $(F_2)_H = x_2 G$. But $\frac{(F_2)_H - x_2 G}{x_2}$ is also a local equation for σ_H on $\mathcal{U}_2 \cap H$. Moreover, by the generality of coordinates, no points of Z lies on the line $L = \{x_2 = 0\}$ and so σ_H would be zero, against the hypothesis.

Thus, $\dim_L(H^0\mathcal{I}_Z(s)) = h^0\mathcal{I}_Z(s) - h^0\mathcal{I}_Z(s-1) \geq \dim_L(H^0\mathcal{I}_Y(s)) + N - 3 = b + N - 3$; moreover $\Delta h_Z(s+1) \leq -2$ as $\overline{N}^s_{s-1}(Z) \neq 0$ (see Lemma 2.2 (iv)) and we can conclude thanks to Lemma 4.1 (ii). □

Now we assume that the general plane section Z of the variety X is not contained in curves of degree lower than s (s a non-lifting level).

First of all observe that in this assumption, s is also the minimal non-lifting level for C and $a = h^0\mathcal{I}_X(s) = \dim_L H^0\mathcal{I}_X(s) \leq h^0\mathcal{I}_Y(s) - 1 = b - 1$; thus the bounds on the degree that we have obtained in Theorem 4.5 (i) and (ii) (for (ii) see also [19]) can be reformulated in the following way:

Corollary 4.6. *Let X be a codimension 2, integral subvariety of degree d in $\mathbf{P}^N$, $N \geq 3$. Suppose that s is a non-lifting level for X*

and that no degree $s-1$ plane curve contains Z. Put $a := h^0\mathcal{I}_X(s)$ and $r := a+N-2$.

(i) *If either $N \geq 4$ or $N=3$ and $\overline{N}^s_{s-1}(C) \neq 0$, then*

$$d \leq D(s, r-1) - 1 = s^2 - s(r-2) + \binom{r-1}{2}.$$

(ii) *If $\overline{N}^s_{s-1}(Z) \neq 0$, then*

$$d \leq D(s,r) = s^2 - s(r-1) + \binom{r}{2} + 1.$$

But the assumption $h^0\mathcal{I}_Z(s-1) = 0$ allows further and stronger bounds on the degree. Let us first consider two special cases: as in our hypothesis $h^0\mathcal{I}_Z(s) = \Delta h_Z(s)$ and $\Delta h_Z(s) \leq s+1$, then $h^0\mathcal{I}_Z(s)$ is at most $s+1$. Thus, if s is a non-lifting level for $X \subset \mathbf{P}^N$, the two higher (and special) values for $h^0\mathcal{I}_X(s)$ are $s-N+3$ and $s-N+2$.

Proposition 4.7. *Let X be a codimension 2, locally Cohen-Macaulay, integral subvariety in $\mathbf{P}^N$ of degree d. Suppose that s is a non-lifting level for X and $h^0\mathcal{I}_Z(s-1) = 0$.*

(i) *If $h^0\mathcal{I}_X(s) = s-N+3$ then $s \geq 3$, $d = \frac{s^2+s}{2} = D(s,s)-1$ and $\overline{N}^s_{s-1}(Z) = 0$.*
(ii) *If $h^0\mathcal{I}_X(s) = s-N+2$, then either one of the following two cases happens:*
   - *$d = \frac{s^2+s}{2}$ (when $s \geq 3$ and either $h^0\mathcal{I}_Z(s) = s+1$ or $h^0\mathcal{I}_Z(s) = s$ and $h^0\mathcal{I}_Z(s+1) = 2s+3$);*
   - *$d = \frac{s^2+s}{2} + 1 = D(s,s)$ (when $h^0\mathcal{I}_Z(s) = s$, $h^0\mathcal{I}_Z(s+1) = 2s+2$).*

Proof. (i) As $h^0\mathcal{I}_Z(s-1) = 0$, then s is also a non-lifting level for X_i, $i = N-2, \dots, 1$ and $h^0\mathcal{I}_Z(s) \geq (s-N+3)+(N-2) = s+1$; on the other hand $h^0\mathcal{I}_Z(s) = \Delta h_Z(s) \leq s+1$: then $h^0\mathcal{I}_Z(s) = s+1$. Computations on Castelnuovo function give $d = \frac{s^2+s}{2} = D(s,s) - 1$; moreover $\mathcal{I}_Z(s)$ is globally generated and then $H^1\mathcal{I}_Z(s-1) = \overline{N}^s_{s-1}(Z) = 0$.

Finally, note that the socle-level s cannot be 2 because a curve of degree 3 in $\mathbf{P}^3$ is contained in at least 3 independent quadrics.

(ii) As above, s is also a non-lifting level for every X_i so that $a \le h^0\mathcal{I}_Z(s) \le s+1$; computations on Castelnuovo function give $\frac{s^2+s}{2}+1 \ge d \ge \frac{s^2+s}{2}$.

The higher value $d=\frac{s^2+s}{2}+1$ happens if and only if $h^0\mathcal{I}_Z(s)=s$ and $h^0\mathcal{I}_Z(s+1)=2s+2$. $\square$

The following theorem, which is the main result of this paper, shows that, for every N, if $a=h^0\mathcal{I}_X(s)\le 2$, then the bound $d\le D(s,N+a-2)$ holds also without assuming condition (2) of Theorem 4.3. Conversely, if $a\ge 3$, then in general the bound does not hold either for curves in $\mathbf{P}^3$, not even under the stronger hypothesis that s is a socle level: see Example 4.9.

Theorem 4.8. *Let X be a codimension 2, integral subvariety of degree d in $\mathbf{P}^N$ with a socle level s. Suppose that no plane curve of degree $s-1$ contains the general plane section Z of X. Then:*

(i)
$$d \le s^2 - s(N-3) + \binom{N-2}{2} + 1.$$

(ii) *If X is contained in some hypersurface of degree s then:*

$$d \le s^2 - s(N-2) + \binom{N-1}{2} + 1.$$

(iii) *If moreover s is a socle level for X and X is contained in a complete intersection (s,s), then:*

$$d \le s^2 - s(N-1) + \binom{N}{2} + 1.$$

Proof. Put $a=h^0\mathcal{I}_X(s)$ and $r=\min\{N,a+N-2\}$. Computations on the Castelnuovo function $\Delta h_Z(t)$ easily leads to stronger upper bounds when either $H^0\mathcal{I}_Z(s)\ge r+1$ or both $H^0\mathcal{I}_Z(s)=r$ and $H^0\mathcal{I}_Z(s+1)\ge 2r+4$.

If the restriction map $\overline{N}^s_{s-1}(C) \to \overline{N}^s_{s-1}(Z)$ is not zero, the statement has been already proved in Theorem 4.6.

So suppose $H^0\mathcal{I}_Z(s) = r$, $H^0\mathcal{I}_Z(s+1) \leq 2r+3$ and $\overline{N}^s_{s-1}(C)_H = 0$.

In these hypotheses we have $h^0\mathcal{I}_X(s) \leq 2$, $h^0\mathcal{I}_{X_{i-1}}(s) = h^0\mathcal{I}_{X_i}(s)+1$ and $N^s_{s-1}(X_i) = \overline{N}^s_{s-1}(X_i) \cong \mathbf{k}$ for every $i = N-2, \dots, 1$.

This can not happen when $N = 3$; in fact as $\overline{N}^s_{s-2}(C) \neq 0$, then $N^s_{s-2}(C, B) \neq 0$, for a 3-dimensional $\mathbf{k}$-vector space $B \subset A_1$, and $h^0\mathcal{I}_C(s) \geq 3$ (see Theorem 2.1 and Theorem 3.1).

If $N \geq 4$ we can suppose that s is a socle level for X and $h^0\mathcal{I}_X(s) = 2$, so that $r = N$: if $h^0\mathcal{I}_X(s) \leq 1$, we consider instead of X its general hyperplane section Y of X and prove the bounds on the degree $\deg(Y) = \deg(X)$ depending on $a(Y)$ and s, because $a(Y) = a(X) + 1$ and s is a socle level for Y.

In this situation, we can apply Lemma 3.7 and get the opposite inequality $H^0\mathcal{I}_Z(s+1) \geq 2r+3$. Again, computations on Castelnuovo function show that the wanted upper bound on the degree holds; moreover the degree reaches the higher value if the ideal sheaf of X is globally generated in degree $s+1$ by the global sections $F_{ij}(\sigma)$, σ being a fixed non zero element in $\overline{N}^s_{s-1}(X)$. □

Example 4.9. Curves C with $h^0\mathcal{I}_Z(s-1) = 0$, socle level s, $a = h^0\mathcal{I}_C(s)$ and $d > D(s, a+1)$ for every pair of integers $s > a \geq 3$. We use induction on a.

Case $a = 3$. Let Y be the disjoint union of a line and a conic in $\mathbf{P}^3$ and let $\mathcal{F}$ be a rank 2 normalized reflexive sheaf on $\mathbf{P}^3$ such that Y is the zero scheme of a global section in $H^0\mathcal{F}(1)$: $\mathcal{I}_Y$ and $\mathcal{F}$ are connected by the following standard exact sequence:

$$0 \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{F}(1) \to \mathcal{I}_Y(2) \to 0.$$

Through this exact sequence we can compute Chern polynomials $c_t(\mathcal{F}) = c_t(\mathcal{I}_Y(2))$ and Rao modules $H^1_*\mathcal{F} = H_*\mathcal{I}_Y(2)$ so that especially $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = 2$ and $H^1_*\mathcal{F} = H^1\mathcal{F}(-1) \oplus H^1\mathcal{F} \cong \mathbf{k} \oplus \mathbf{k}$. Let us consider an integral curve C zero scheme of a general section of $\mathcal{F}(t)$, for some $t \gg 0$ (more precisely for $t \geq 3$, F being 2-regular: see [14] Teorema 3.9).

Then $H^1_*\mathcal{I}_C = H^1\mathcal{I}_C(t-1) \oplus H^1\mathcal{I}_C(t) \cong \mathbf{k} \oplus \mathbf{k}$, so that $s = t+1$ is a socle level for C; moreover $a = h^0\mathcal{I}_C(s) = h^0\mathcal{F}(1) = h^0\mathcal{I}_Y(2)+1 = 3$ and $h^0\mathcal{I}_C(s-1) = h^0\mathcal{F} = 0$ (actually: $h^0\mathcal{I}_Z(s-1) = h^0\mathcal{F}_H = 0$, because a general plane section of a stable reflexive sheaf, which is not a null-correlation bundle, is still stable). The degree of C is $d = c_2(\mathcal{F}(t)) = t^2 + c_1 t + c_2 = s^2 - 2s + 3 > D(s, a+1)$.

Suppose that there are an integral curve C_0 in $\mathbf{P}^3$ and whole numbers $a_0 = a(C_0)$ and $s_0 = s(C_0)$ which satisfy the following conditions:

(1) $h^0\mathcal{I}_{C_0}(s_0 - 1) = 0$, $h^0\mathcal{I}_{C_0}(s_0) = a_0$;
(2) C_0 is s_0-regular;
(3) $h^0\omega_{C_0}(4 - s_0) \neq 0$.

Then there is also an integral curve C which satisfies (1), (2) and (3) for $a = a(C) = a_0 + 1$, $s = s(C) = s_0 + 1$. To see this it is sufficient to consider the exact sequence defined by a non-zero section of $\omega_{C_0}(4 - s_0)$:

$$0 \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{F}(k_0) \to \mathcal{I}_{C_0}(s_0) \to 0,$$

where $\mathcal{F}$ is a reflexive sheaf, $c_1 = c_1(\mathcal{F})$ is either 0 or -1 and $2k_0 + c_1 = s$. As $\mathcal{F}$ is k_0-regular (thanks to (2)), we obtain an integral curve C satisfying the required conditions through a general global section of $\mathcal{F}(k_0 + 1)$.

If, moreover, C_0 has degree $D(s_0, a_0) - 1$, then $\deg(C) = D(s_0 + 1, a_0 + 1) - 1 = D(s, a) - 1$.

Finally, when X is a curve C, then we can extend the previous results to the most general case, without any assumption on $h^0\mathcal{I}_Z(s-1)$. The first item of Theorem 4.10 is the well known Laudal's Lemma.

Theorem 4.10. *Let C be a degree d integral curve in $\mathbf{P}^3$. Suppose that s is a non-lifting level for C and put $a := h^0\mathcal{I}_C(s)$. Then:*

(i) $d \leq d_0 := s^2 + 1$.
(ii) $d \leq d_1 := s^2 - s + 2$ *when* $a \geq 1$.
(iii) $d \leq s^2 - 2s + 5$ *when* $a \geq 2$.

(iv) $d \leq d_2 := s^2 - 2s + 4$ *when* $a \geq 2$ *and* s *is a socle level for* C.

Proof. First of all, let us consider a curve C such that Z is a complete intersection (p, q): then $p = 2$, $q = s$ (see either [17] or [13] or [16]) and $d = 2s \leq d_2 \leq d_1 < d_0$ (more precisely $d = 2s < d_2$ except for $s = 2$).

So, suppose that Z is not a complete intersection.

Proof of (i) and (ii). Since s is a non-lifting level for C (we may suppose it is the lowest one), then some $s' \leq s$ is a socle level for Z (see Theorem 2.1).

If $\overline{N}^s_{s-1}(Z) = 0$ (so that $s' < s$), we find $d \leq s^2 - s + 1$ (see Lemma 3.2 (ii)), which is a strictly lower bound than both $d \leq d_0$ and $d \leq d_1$.

On the other hand, if s itself is a socle level for Z, all the bounds follow from Theorem 4.5 (ii): note that we have $\dim_L H^0\mathcal{I}_C(s) = h^0\mathcal{I}_C(s)$ if $h^0\mathcal{I}_C(s) \leq 2$, $\dim_L H^0\mathcal{I}_C(s) \geq 2$ otherwise.

Proof of items (iii) and (iv): let $h^0\mathcal{I}_C(s) \geq 2$.

If $s = 2$, then either C is a plane curve (not possible because plane curves have no non-lifting levels) or it is contained in a complete intersection $(2, 2)$ so that $d \leq 4 \leq d_2$. So, assume $s \geq 3$.

If s is a socle level for Z, then $d \leq d_2$ (see Theorem 4.5 (ii)); if $h^0\mathcal{I}_Z(s-1) \neq 0$ then $d \leq s^2 - 2s + 2$. Thus, suppose $\overline{N}^s_{s-1}(Z) = 0$ and $h^0\mathcal{I}_Z(s-1) = 0$. If s is a socle level for C, then $h^0\mathcal{I}_Z(s) \geq 4$ and $d < d_2$; we have in fact either $n^s_{s-1}(C) \geq 2$ (so that $h^0\mathcal{I}_Z(s) \geq h^0\mathcal{I}_C(s) + 2$) or $\overline{n}^s_{s-1}(C) = \overline{n}^s_{s-2}(C) = 1$ so that $\overline{N}^s_{s-2}(C, B) \neq 0$, where $B \subset A_1$ is a 3-dimensional $\mathbf{k}$-vector space (see Theorem 2.1) which implies $h^0\mathcal{I}_C(s) \geq 3$ (see Theorem 3.1).

Computations on Castelnuovo functions and what just proved, show that $d \leq s^2 - 2s + 5$ except when $s \geq 4$ (for $s \leq 3$, see Lemma 4.7), s is not a socle level neither for C nor for Z and I_Z has 3 minimal generators in degree s and none in degrees $s+1$, $s+2$ and $< s$, so that $\overline{N}^{s-1}_{s-2}(Z) \cong \mathbf{k}$ and $\overline{N}^s_{s-1}(Z) = \overline{N}^{s+1}_s(Z) = 0$.

We claim that the above situation cannot happen. In fact, as $\overline{N}^s_{s-1}(C) \neq N^s_{s-1}(C)$, then $N^{s+1}_s(C) \neq 0$, so that, by the exact sequence (2.3), we find $N^{s+1}_{s-2}(C) \geq 2$ and from this, using again (2.3), we get $\overline{N}^{s-1}_{s-3}(Z) \neq 0$. But any non-zero element of $\overline{N}^{s-1}_{s-3}(Z) \neq 0$ also belongs to $N^{s-2}_{s-3}(Z, B)$, for some 2-dimensional vector space $B \subset A_1$, and this is not allowed by Lemma 4.2. $\square$

References

[1] D. Bayer, M. Stillman, A criterion for detecting m-regularity, *Invent. Math.* **87** (1987), 1–11.

[2] M. Chang, Characterization of arithmetically Buchsbaum subschemes of codimension 2 in $\mathbf{P}^n$, *J. Differential Geom.* **31** (1990), 323–341.

[3] M. Chang, On the hyperplane sections of certain codimension 2 subvarieties in $\mathbf{P}^n$, *Arch. Math.*, **58** (1992), 547–550.

[4] L. Chiantini, C. Ciliberto, A few remarks on the lifting problem, *Asterisque* **218** (1993), 95–109.

[5] W. Fulton, *Intersection Theory*, Springer-Verlag, New York, 1984.

[6] M. Green, Generic initial ideals , *Summer School on Commutative Algebra*, 1996, 11–85.

[7] L. Gruson, C. Peskine, Section plane d'une courbe gauche: postulation, In: *Enumerative Geometry and Classical Algebraic Geometry*, Progress in Mathematics, 1982, 33–35.

[8] O. Laudal, A generalized trisecant lemma, In: *Algebraic Geometry Proceedings*, LNM 687 Springer, 1978, 112–149.

[9] R. Maggioni, A. Ragusa, The Hilbert function of generic plane sections of curves of $\mathbf{P}^3$, *Invent. Math.* (1988) 253–258.

[10] E. Mezzetti, The border cases for lifting problems in $\mathbf{P}^4$, *J. Reine Angew. Math.* **433** (1992), 101–111.

[11] E. Mezzetti, Differential-geometric methods for lifting problem and linear systems on plane curves, *J. Algebraic Geom.* **3** (1994), 375–398.

[12] E. Mezzetti, I. Raspanti, A Laudal type theorem for surfaces in $\mathbf{P}^4$, *Rend. Sem. Mat. Univ. Politec. Torino* **48** (1990), 529–537.

[13] R. Re, Sulle sezioni iperpiane di una varietà proiettiva, *Le Matematiche*, **42** (1987), 211–218.

[14] M. Roggero, Teoremi di Bertini. fasci riflessivi e curve sottocanoniche, *Seminario Matematico di Bologna-Seminari di Geometria*, 1987-88, 115-125.

[15] M. Roggero, Lifting problem for codimension 2 subvarieties in $\mathbf{P}^{n+2}$: border cases, Geometric and combinatorial aspects of commutative algebra, *Lecture Notes in Pure and Appl. Math.* **207** (2001), 309–326.

[16] M. Roggero, Laudal-type theorem in $\mathbf{P}^N$, *Indag. Math.* (*N.S.*) **14** (2) (2003), 249-262.

[17] R. Strano, A characterization of complete intersection curves in $\mathbf{P}^3$, *Proc. Amer. Math. Soc.* **104** (1988), 711–715.

[18] R. Strano, On generalized Laudal's lemma, In: Complex Projective Geometry(Trieste, 1989) *London Math. Soc. Lecture Note Ser.* **179** (1992), 284-293.

[19] A. Tortora, On the lifting problem in codimension 2, *Le Matematiche Vol LII, Fasc. I* (1997), 41–51.

[20] M. Valenzano, Bounds on the degree of two-codimensional integral varieties in projective space, *J. Pure Appl. Algebra* **158** (2001), 111–122.

M. Roggero
Dipartimento di Matematica dell' Università
Via Carlo Alberto 10
10123 Torino, Italy
e-mail:margherita.roggero@unito.it