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ON RATIONAL GROUPS WITH SYLOW 2-SUBGROUPS OF NILPOTENCY CLASS AT MOST 2

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ABSTRACT. A finite group G is called rational if all its irreducible complex characters are rational valued. In this paper we discuss about rational groups with Sylow 2-subgroups of nilpotency class at most 2 by imposing the solvability and non-solvability assumption on G and also via nilpotency and non-nilpotency assumption of G'.
Keywords: Rational group, nilpotency class, Markel group.
MSC(2010): Primary: 20C15; Secondary: 20D20, 20D15.

1. Introduction and preliminaries

Let G be a finite group such that for all $\chi \in \mathrm{Irr}(G)$, and every $g \in G$, the value of $\chi(g)$ is a rational number, where $\mathrm{Irr}(G)$ is the set of all ordinary irreducible complex characters of G. Such a group G is called a rational group or a $\mathbb{Q}$-group. The structure of a finite group G, deeply depends on the structure of its Sylow subgroups and since for every non-trivial rational group G, the order of G is divisible by 2, analyzing rational groups through their Sylow 2-subgroups is important.

Let π be a set of prime numbers; by a π-group, we mean a group G, for which the set of prime divisors of the order of G is a subset of π. In 1976 Gow, [6] showed that every finite rational solvable group is a $\{2, 3, 5\}$-group. A long standing conjecture about Sylow 2-subgroups of rational groups, was that they are rational too, but Isaacs and Navarro in [11], showed that this conjecture is false. However in the same paper the authors proved that if P is a Sylow 2-subgroup of a solvable rational group, with nilpotency class 2, then P is rational. Our goal is to find some information about the structure of a rational group G, for which the Sylow 2-subgroups have nilpotency class at most 2.

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Throughout this paper we consider only finite groups and employ the following notations and terminologies. By M we mean the Markel group of order 200, i.e., the semidirect product of $\mathbb{Z}_5 \times \mathbb{Z}_5$ and the Sylow 2-subgroup of $SL(2,5)$ (which is isomorphic to Q_8, the quaternion group of order 8) via the natural action. This group is indexed as $SmallGroup(200,44)$ in the GAP-system [5]. For $\Omega = \{1,\ldots,n\}$ and the group G, the notation G^{Ω} is used for the set of all functions from Ω to G. If $K \leq \mathbb{S}_n$ (the symmetric group on Ω) then $G \wr_{\Omega} K$ is the wreath product of G and K, i.e., the set of all $(f;\sigma)$, where $f \in G^{\Omega}$ and $\sigma \in K$ and $(f;\sigma)(g;\delta) = (fg_{\sigma};\sigma\delta)$, in which for every $i \in \Omega$ we have $g_{\sigma}(i) = g(\sigma^{-1}(i))$ and $fg(i) = f(i)g(i)$. By $G \circ H$ we mean the central product of the groups G and H, and $E(2^k)$ denotes the elementary abelian 2-group of order 2^k and the notation $Es(n,p)$ is for the extra-special group of order p^{2n+1}. We denote the dihedral group of order n by D_n. By $O_p(G)$ we mean the largest normal p-subgroup of G. For a nilpotent group P, we denote the nilpotency class of P by $cl(P)$.

In the following results, we state some well-known properties of rational groups from [12], which we need for our reasoning.

Result 1.1. A Group G is rational if and only if for each $x \in G$ we have $N_G(\langle x\rangle)/C_G(\langle x\rangle) \cong Aut(\langle x\rangle)$.

Result 1.2. Let G be a rational group and $P \in Syl_2(G)$. Then the following hold

1. If G is non-trivial, then $2 \mid |G|$.
2. Every quotient of G is rational.
3. If G is abelian, then G is an elementary abelian 2-group.
4. $Z(G)$ is an elementary abelian 2-group.
5. If G is nilpotent, then G is a 2-group.
6. If P is abelian, then P is elementary abelian, G' is a 3-group and G splits over G' with P as complement.
7. $C_G(P) = Z(P)$.
8. If G is solvable, then $N_G(P) = P$.
9. G/G' is an elementary abelian 2-group. In particular every element of G having odd order lies in G'.

In this paper we discuss about the groups G, satisfying the condition mentioned in the title, depending on they are solvable or not, and if they are solvable their derived subgroup is nilpotent or not. Besides some properties in special cases, we will prove the following results.

Theorem 1.3. *Let G be a solvable rational group and $P \in Syl_2(G)$ with $cl(P) \leq 2$ and $K \in Syl_3(G)$. If G' is nilpotent, then G is a $\{2,3\}$-group and $G \cong K \rtimes P$ and we have the following*

(a) *If G' is abelian, then $G \cong E(3^k) \rtimes P$ for some k, and G contains a normal elementary abelian 2-subgroup H such that $G/H \cong E(3^m) \rtimes E(2^n)$, for some $m \geq 0$ and $n \geq 0$.*
(b) *G' is non-abelian if and only if K is non-abelian.*

If $5 \mid |G|$, then by [7, Theorem 1.1], G has a normal elementary abelian Sylow 5-subgroup T, and so T can be thought as a component of a semidirect product, more precisely, G splits over T. So by Result 1.2(2), the $\{2,3\}$-group G/T is rational. Therefore, in a sense, the studying solvable rational groups reduces to the studying the rational $\{2,3\}$-groups.

Theorem 1.4. *Let G be a solvable rational group and $P \in Syl_2(G)$ with $cl(P) \leq 2$. If G' is not nilpotent, then we have the following*

(a) *If G is a $\{2,5\}$-group, then $G/O_2(G) \cong \prod_{i=1}^{k} M_i$, in which, for every $i \in \{1,\dots,k\}$, M_i is a copy of Markel group M.*
(b) *If G is a $\{2,3\}$-group, $K \in Syl_3(G')$ and $H \in Syl_2(G')$ with $H \leq P$, then $K \lhd G'$ if and only if $P' = H$.*

Theorem 1.5. *Let G be a non-solvable rational group and $P \in Syl_2(G)$ with $cl(P) \leq 2$. Then every non-cyclic composition factor of G is isomorphic to A_n for $n \in \{5,6,7\}$.*

2. **Solvable rational groups**

Throughout this section we assume that G is a solvable rational group.

2.1. **Rational 2-groups.** By Result 1.2(5), every nilpotent rational group is a 2-group. In this subsection we state some properties of rational 2-groups.

Lemma 2.1. *Let G be a rational 2-group. Then the Frattini subgroup of G, coincides with its derived subgroup.*

Proof. Since G/G' is an abelian rational 2-group, it is elementary abelian by Result 1.2(3). Now by [8, Theorem 3.3.14], we have $\Phi(G) \leq G'$. But for every nilpotent group G, we know that $G' \leq \Phi(G)$, and the proof is complete. □

Lemma 2.2. *Let G be a rational 2-group. If G has nilpotency class 2, then $exp(G) = 4$ and $exp(G') = 2$.*

Proof. Since $G/\Phi(G)$ is elementary abelian and $\Phi(G) = G' \leq Z(G)$ in view of Lemma 2.1, we have $x^2 \in Z(G)$ for every $x \in G$. But by Result 1.2(4), $Z(G)$ is elementary abelian, so we have $x^4 = 1$. Now as G is non-abelian we have $exp(G) = 4$ and as $G' \leq Z(G)$ and $G' \neq 1$, we have $exp(G') = 2$. □

Lemma 2.3. *Suppose that G is a rational 2-group. Then $G' \cong \mathbb{Z}_2$ if and only if G is isomorphic to one of the following groups*

1. *$D_8 \circ D_8 \circ \cdots \circ D_8 \times E(2^k)$;*

2. $Q_8 \circ D_8 \circ \cdots \circ D_8 \times E(2^k)$.

In other words $G \cong Es(n, 2) \times E(2^k)$ *for some* n *and* k.

Proof. Suppose that $G' \cong \mathbb{Z}_2$. As $Z(G)$ is elementary abelian, we have $Z(G) = E(2^{k+1})$ for some k and so by [2, Lemma 4.2], G has one of the above mentioned forms. The converse is obvious. □

Theorem 2.4. *Suppose that* G *is a rational* 2*-group and* $cl(G) = 2$. *Then* G *is isomorphic to a subgroup of* $(\prod_{i=1}^{n} Es(n_i, 2)) \times E(2^k)$, *for some* n, n_i *and* k.

Proof. We know that $Z(G)$ is elementary abelian. As $G' \leq Z(G)$, we have $G' \cong E(2^n)$ for some n. So there are subgroups $K_1, K_2, \ldots K_n$ of G' such that $|G' : K_i| = 2$, for every $i \in \{1, \ldots, n\}$, and $\bigcap_{i=1}^{n} K_i = 1$. Therefore every K_i is normal in G and the homomorphism $\varphi : G \to G/K_1 \times \cdots \times G/K_n$, for which $\varphi(g) = (gK_1, \ldots, gK_n)$, is injective. But $(G/K_i)' = G'/K_i \cong \mathbb{Z}_2$ for every $i \in \{1, \ldots, n\}$, so by Lemma 2.3, the proof is complete. □

Definition 2.5. Let H and K be subgroups of G such that K is a normal subgroup of H. We call the group $L \cong (H/K)$, a section of the group G. In other words L is a section of G if it is a homomorphic image of a subgroup of G.

Since the minimal non-abelian rational 2-groups are Q_8 and D_8, one may think that they should appear in the structure of every non-abelian rational 2-group. Our first conjecture was that every non-abelian rational 2-group has a subgroup isomorphic to quaternion or dihedral group of order 8. But we found a group of order 64 which is indexed by *SmallGroup*(64, 245) in *GAP* system [5], as the first counterexample. This group which has 60 element of order 4 and the nilpotency class 2, has no non-abelian subgroup of order 8. One may think that every non-abelian rational 2-group has a non-abelian factor group of order 8; but for the extra-special groups, *SmallGroup*(32, 49) and *SmallGroup* (32, 50), in *GAP* system, all the factor groups with order 8, are elementary abelian. However we can state the following theorem.

Theorem 2.6. *If* G *is a non-abelian rational* 2*-group, then it has a section isomorphic to* Q_8 *or* D_8.

Proof. We prove the claim using induction on the order of G. If $|G| = 8$, the claim is clearly valid. So we suppose that $|G| = 2^n$, $n > 3$, and the assertion holds for every non-abelian rational 2-group with the order 2^k, $k < n$. Now let $x \neq 1$ be an involution belonging the center of G and $H = \langle x \rangle$. Then G/H is a rational group of order 2^{n-1}. If G/H is non-abelian, then by induction hypothesis it has subgroups K/H and L/H such that $(K/H)/(L/H)$ is a non-abelian group of order 8, so the section K/L of G is isomorphic to Q_8 or D_8. Now we assume that G/H is abelian. But in this case we have $G' = H \leq Z(G)$, i.e., $G' \cong \mathbb{Z}_2$ and by Lemma 2.3, the proof is complete. □

2.2. **Groups with nilpotent derived subgroup.** In the next lemma we show that, 5 does not appear as a prime divisor of $|G|$, when G' is nilpotent.

Lemma 2.7. *Let G be a rational group for which G' is nilpotent. Then G is a $\{2,3\}$-group.*

Proof. Suppose that H, T and K are 2-, 3- and 5-Sylow subgroups of G' respectively. Then $G' = H \times T \times K$. Now T is characteristic in G', so it is normal in G and G/T is a rational $\{2,5\}$-group. Now by [7, Theorem 1.2], G/T has a quotient with a subgroup isomorphic to the Markel group M. But M' is a non-nilpotent group of order 50; This violates that G' is nilpotent and the proof is complete. □

In the sequel we discuss about the case that G' is abelian; In other words we suppose that G is metabelian.

2.2.1. *Groups with abelian derived subgroups.* At first we describe the structure of an special case of metabelian rational groups, i.e., the metacyclic $\mathbb{Q}$-groups.

Theorem 2.8. *Let G be a rational group. Then G is metacyclic if and only if G is isomorphic to one of the following groups: The cyclic group $\mathbb{Z}_2$, Q_8 or D_n for $n \in \{4,6,8,12\}$.*

Proof. As G is metacyclic, there exists an element $x \in G$ such that $G/\langle x\rangle \cong \mathbb{Z}_2$. If G is abelian, then by Result 1.2(3), it is elementary abelian and so obviously G is isomorphic to $\mathbb{Z}_2$ or D_4. Now we assume that G is non-abelian. In this case since $|G| > 4$, we have $o(x) > 2$. Since by Result 1.2(4), $Z(G)$ is elementary abelian, $x \notin Z(G)$ and hence $C_G(\langle x\rangle) = \langle x\rangle$. By Result 1.1 we have

$$\mathbb{Z}_2 \cong \frac{G}{\langle x\rangle} \cong \frac{N_G(\langle x\rangle)}{C_G(\langle x\rangle)} \cong Aut(\langle x\rangle). \tag{2.1}$$

By Lemma 2.7, G is a $\{2,3\}$-group. Suppose that $o(x) = 2^\alpha 3^\beta$. If $\beta = 0$, then $\alpha \geq 2$ and we have $Aut(\langle x\rangle) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{\alpha-2}}$ and then by (2.1) we conclude that $\alpha = 2$. So $o(x) = 4$ and $|G| = 8$. But all the non-abelian rational groups of order 8 are D_8 and Q_8 which are also metacyclic. Now suppose that $\beta \neq 0$; then as $Aut(\mathbb{Z}_{3^\beta}) \cong \mathbb{Z}_{2\times 3^{\beta-1}}$, by (2.1) we conclude that $(\alpha,\beta) \in \{(0,1),(1,1)\}$, i.e., $o(x) = 3$ or $o(x) = 6$, which the first case implies that $G \cong D_6$ and the last one leads to $G \cong D_{12}$. □

By previous theorem every metacyclic rational group has a Sylow 2-subgroup of nilpotency class at most 2.

In the following we have some description for general case of metabelian rational groups.

Lemma 2.9. *Let G be a metabelian rational group. Then G is a $\{2,3\}$-group and for $K \in Syl_3(G)$, K is elementary abelian and normal in G.*

Proof. The first claim is a corollary of Lemma 2.7. Suppose that $T \in Syl_3(G)$. As G/G' is a 2-group, we have $T \leq G'$ and so T is abelian. Since Sylow subgroups of an abelian group are characteristic, T is characteristic in G' and so is normal in G. Suppose that $1 \neq x \in T$ and $o(x) = 3^\alpha$. By rationality

$$\frac{N_G(\langle x \rangle)}{C_G(\langle x \rangle)} \cong Aut(\langle x \rangle) \cong \mathbb{Z}_{2.3^{\alpha-1}}.$$

Since $G' \leq C_G(\langle x \rangle)$, the order of left hand side is not divisible by 3. So $\alpha = 1$ and we are done. □

Now we apply our condition on Sylow 2-subgroups of G in the next two results, in a special and general case respectively.

Theorem 2.10. *Let G be a rational group with a cyclic derived subgroup and a Sylow 2-subgroup P with $cl(P) \leq 2$. Then one of the following holds*

(i) $|G'| = 1$ *and G is an elementary abelian 2-group;*
(ii) $|G'| = 2$*, G is a 2-group and* $G \cong Es(2, n) \times E(2^k)$*;*
(iii) $|G'| = 3$ *and* $G \cong \mathbb{S}_3 \times E(2^k)$*;*
(iv) $|G'| = 6$ *and* $G \cong \mathbb{Z}_3 \rtimes (Es(2, n) \times E(2^k))$.

for some $n \geq 1$ and $k \geq 0$.

Proof. By the last corollary of [1, page 180], the order of G' divides 12. If $|G'| = 1$, obviously (i) is the case. Since G/G' is an elementary abelian 2-group by rationality, if $|G'| = 2$, then G is a 2-group and by Lemma 2.3, $G \cong Es(2, n) \times E(2^k)$, for some $n \geq 1$ and $k \geq 0$, and the case (ii) holds. Now if $|G'| = 4$, with a similar reason, G is a 2-group and since $cl(G) = 2$, by Lemma 2.2, we have $exp(G') = 2$; but this violates the assumption that G' is cyclic. Now let $3 \mid |G'|$ and $x \in G'$ be an element of order 3. Then as G' is cyclic, $\langle x \rangle$ is a characteristic subgroup of G' and so is normal in G and $G/\langle x \rangle$ is a rational 2-group which is isomorphic to P by Schur-Zassenhaus theorem. If $|G'| = 3$, then since G/G' is elementary abelian, the Sylow 2-subgroup P of G is abelian and hence by Result 1.2(1,6), $G \cong \mathbb{Z}_3 \rtimes E(2^{k+1})$ for some $k \geq 0$. Now $G' = \langle x \rangle$, and since $|Aut(\langle x \rangle)| = 2$, by Result 1.1 we have $(|G|/|C_G(\langle x \rangle)|) = 2$; that is, $|C_G(\langle x \rangle)| = 3 \times 2^k$. So (iii) is the case. If $|G'| = 6$, then $(G/\langle x \rangle)' \cong P' \cong \mathbb{Z}_2$ and $G \cong \mathbb{Z}_3 \rtimes P$ where $P \cong Es(2, n) \times E(2^k)$ by Lemma 2.3 and we have the case (iv). Finally if $|G'| = 12$, then $(G/\langle x \rangle)'$ is cyclic of order 4, but this is against the fact that $exp((G/\langle x \rangle)') = 2$. □

Obviously in Theorem 2.10, the direct proudcut of a group of type (ii) and a group of type (iii) is a group of type (iv); also our conjecture is that every group of type (iv) can be constructed in this way; in other words, by the assumption of previous theorem we surmise that if $|G'| = 6$, then $G \cong \mathbb{S}_3 \times Es(2, n) \times E(2^k)$.

Theorem 2.11. *Let G be a rational metabelian group and $P \in Syl_2(G)$, with $cl(P) \leq 2$. Then G contains a normal elementary abelian 2-subgroup H, such that $G/H \cong E(3^m) \rtimes E(2^n)$, for some $m \geq 0$ and $n \geq 0$.*

Proof. Since G' is an abelian $\{2,3\}$-group, $G' \cong H \times K$, where $H \in Syl_2(G')$ and $K \in Syl_3(G)$. By Lemma 2.9, $K \cong E(3^m)$ for some m and it is normal in G. So $G/K \cong P$ is a rational 2-group and $P' \cong (G/K)' = G'/K \cong H$. Now by Lemma 2.2, $exp(P') \leq 2$; so H is elementary abelian. As H is characteristic in G' it is normal in G and G/H is a rational group. But $(G/H)'$ is a 3-group isomorphic to K, and G/H splits over it and we have $G/H \cong E(3^m) \rtimes E(2^n)$. □

2.2.2. *Groups with non-abelian derived subgroups.* For the case having nilpotent non-abelian derived subgroup we can state the following lemma.

Lemma 2.12. *Let G be a rational group with a nilpotent derived subgroup, $P \in Syl_2(G)$ with $cl(P) \leq 2$ and $K \in Syl_3(G)$. Then G' is non-abelian if and only if K is non-abelian.*

Proof. By Result 1.2(9), $K \leq G'$. Suppose that G' is non-abelian. If P is abelian, then by Result 1.2(6), $G' = K$ and so K is non-abelian. Now we assume $cl(P) = 2$. Since $K \leq G'$ and G' is nilpotent, K is normal in G. But then as G is a $\{2,3\}$-group by Lemma 2.7, $G/K \cong P$ and if $H \in Syl_2(G')$, we have $G' = K \times H$ and $H \cong G'/K = (G/K)' \cong P' \neq 1$. So as $H \cong P' \leq Z(P)$, H is abelian; hence K is non-abelian. The converse is hold as $K \leq G'$. □

Here we present the proof of Theorem 1.3.

Proof. By Lemma 2.7, G is a $\{2,3\}$-group. K is contained in G' by Result 1.2(9), and it is normal in G. So G splits over K with P as a complement, i.e., $G \cong K \rtimes P$. Now if G' is abelian, then by Lemma 2.9 we have $G \cong E(3^k) \rtimes P$ for some k. By Theorem 2.11, the existence of the 2-subgroup H with mentioned property, is confirmed, and so the case (a) of theorem is proved. The case (b) is proved in Lemma 2.12. □

2.3. **Groups with non-nilpotent derived subgroup.**

Lemma 2.13. *If $G = M \wr \mathbb{Z}_2$,where M is the Markel group, and $P \in Syl_2(G)$, then $cl(P) = 4$.*

Proof. Since the Sylow 2-subgroup of M is isomorphic to Q_8, $P \cong Q_8 \wr \mathbb{Z}_2$ and the proof can be completed by a simple calculation with GAP [5]. □

Our next theorem is about the structure of a $\{2,5\}$-solvable rational group which satisfies our condition relating its Sylow 2-subgroups.

Theorem 2.14. *Let G be a solvable rational $\{2,5\}$-group and $P \in Syl_2(G)$. If 5 divides $|G|$ and $cl(P) = 2$, then $G/O_2(G) \cong \prod_{i=1}^{k} M_i$, in which, for each $i \in \{1,\ldots,k\}$, M_i is a copy of Markel group M.*

Proof. By [7, Theorem 1.2], $G/O_2(G) \cong M \wr K$, where $K \leq \mathbb{S}_n$ is a 2-group. Suppose that $K \neq 1$ and σ is an involution in K, say $\sigma = \prod_{l=1}^{t}(i_l, j_l)$, as a product of disjoint cycles. Let $\Omega = \{1, \ldots, n\}$ and $\Gamma = \{i_1, \ldots, i_t, j_1, \ldots, j_t\}$. We know that $M \wr_\Omega K = \{(f; \delta) | f \in M^\Omega, \delta \in K\}$. Suppose that $f, g \in M^\Omega$ such that $f(i) = g(i) = e$, for every $i \in \Omega \setminus \Gamma$, where e is the identity element of M. Then $(fg_\sigma)(i) = f(i)g_\sigma(i) = f(i)g(\sigma^{-1}(i)) = ee = e$, as $\sigma^{-1}(i) \notin \Gamma$. So if we replace every $f \in M^\Omega$, with the above property, by f_Γ (the restriction of f to Γ), we have an embedding of $M^\Gamma \rtimes \langle\sigma\rangle$ in $M^\Omega \rtimes K = M \wr K$. Now let $\tau = (i_1, j_1)$ as a cycle of length 2 and $\Delta = \{i_1, j_1\}$; then by a similar reason as above, one can see that $M \wr_\Delta \langle\tau\rangle$ is embedded in $M^\Gamma \rtimes \langle\sigma\rangle$ and so in $M \wr K$. Now by Lemma 2.13, $G/O_2(G)$ has a Sylow 2-subgroup of nilpotency class more than 2, which is a contradiction. So $K = 1$ and $G/O_2(G)$ is a direct product of M_i, $i \in \{1, \ldots, k\}$, for some $k \leq n$. □

The next corollary is a restatement of part (a) of Theorem 1.4.

Corollary 2.15. *Let G be a solvable rational $\{2, 5\}$-group, G' non-nilpotent and $P \in Syl_2(G)$ with $cl(P) \leq 2$. Then $G/O_2(G) \cong \prod_{i=1}^{k} M_i$, in which, for each $i \in \{1, \ldots, k\}$, M_i is a copy of the Markel group M.*

Proof. If $5 \nmid |G|$, then G is a 2-group, which is a contradiction. Also by Result 1.2(6), $cl(P) \neq 1$. Now as the derived subgroup of M is not nilpotent, the claim is a conclusion of Theorem 2.14. □

The group $\mathbb{S}_4$ is the only symmetric group which is solvable but its derived subgroup is not nilpotent. Using $\mathbb{S}_4$ and rational groups, satisfying our condition on Sylow 2-subgroups and having nilpotent derived subgroup, by direct product one can generate groups which are described in the title of this subsection.

The next result provides the proof of the part (b) of Theorem 1.4.

Theorem 2.16. *Let G be a solvable rational $\{2, 3\}$-group with non-nilpotent derived subgroup G'. Suppose that $P \in Syl_2(G)$, $cl(P) = 2$, $H \in Syl_2(G')$ and $K \in Syl_3(G')$. Then $K \lhd G'$ if and only if $P' \cong H$.*

Proof. By Result 1.2(9), $K \in Syl_3(G)$. If $K \lhd G'$, then $K \lhd G$ and $P \cong G/K$ is a rational 2-group and we have $P' \cong G'/K \cong H$. So $P' \cong H$. Conversely, suppose that $P' \cong H$, we show that G' has a normal 2-complement, which is equivalent to $K \lhd G$. If it is not the case, then by [10, Lemma 5.25], there are elements $x, y \in H$ that are conjugate in G' and they are not conjugate in H. Now we can choose P such that $P' = H$ and then we have $x, y \in P'$ so that x, y are conjugate in G and they are not conjugate in P'. But, as $cl(P) = 2$, we have $P' \subseteq Z(P) = C_G(P)$ which the last equality deduced from Result 1.2(7). Now $x, y \in C_G(P)$ and since $C_G(P)$ is elementary abelian, x and y are not conjugate in $C_G(P)$, and by [10, Lemma 5.12], there exists $t \in N_G(P)$ such that $x^t = y$. Also by Result 1.2(8), we have $N_G(P) = P$, i.e., $t \in P$. But this violates the fact that $x \in Z(P)$. This contradiction completes the proof. □

Here we mention that the result of previous theorem does not hold if we drop the rationality assumption of G. A counterexample in this case is $G = SmallGroup(216, 153)$ for which, by the above notation $H \ncong P'$ but $K \lhd G'$.

2.4. **Examples.** In this section we show that an interesting class of rational groups, i.e., the class of all the rational Frobenius groups satisfies in the condition of the title of this paper. For the definition of a Frobenius group one can see [9, Def. 7.1]. Here we state the main theorem of [3], which classifies the rational Frobenius groups.

Theorem 2.17. *If G is a Frobenius $\mathbb{Q}$-group, then exactly one of the following occurs:*

(i) *We have $G \cong E(3^n) \rtimes \mathbb{Z}_2$, where $n \geq 1$ and $\mathbb{Z}_2$ acts on $E(3^n)$ by inverting every nonidentity element.*
(ii) *We have $G \cong E(3^{2m}) \rtimes \mathbb{Q}_8$, where $m \geq 1$ and $E(3^{2m})$ is a direct sum of m copies of the 2-dimensional irreducible representation of $\mathbb{Q}_8$ over the field with 3 elements.*
(iii) *We have $G \cong E(5^2) \rtimes \mathbb{Q}_8$, where $E(5^2)$ is the 2-dimensional irreducible representation of $\mathbb{Q}_8$ over the field with 5 elements.*

Corollary 2.18. *Let G be a rational Frobenius group. Then G is solvable and for the Sylow 2-Subgroup P of G, we have $cl(P) \leq 2$.*

In the classification theorem of Frobenius rational groups, Theorem 2.17, the class (i) consists of groups with nilpotent derived subgroups and both the second and third classes contain the groups with non-nilpotent derived subgroups.

3. Non-solvable rational groups

The next theorem, concerning the non-abelian simple composition factors of a rational group, is achieved by Feit and Seitz.

Theorem 3.1 (See [4])**.** *Let G be a non-cyclic finite simple group. Then G is a composition factor of a rational group if and only if G is isomorphic to an alternating group or one of the following groups:*

(i) $PSp_4(3)$, $Sp_6(2)$, $O_8^+(2)$.
(ii) $PSL_3(4)$, $PSU_4(3)$.

Here we restate and prove Theorem 1.5.

Theorem 3.2. *Let G be a non-solvable rational group, $P \in Syl_2(G)$ and $cl(P) \leq 2$. Then a non-cyclic composition factor of G is isomorphic to an alternating group A_n, where $n \in \{5, 6, 7\}$.*

Proof. Suppose that H/K is a non-abelian composition factor of G. Obviously the nilpotency class of a Sylow 2-subgroup of H/K is not greater than that of a Sylow 2-subgroup of G. Now H/K is isomorphic to one of the groups mentioned in Theorem 3.1. But all such groups contain a Sylow 2-subgroup of nilpotency class greater than 2, except A_5, A_6 and A_7 and the proof is complete. □

It is worthwhile to mention here that $\mathbb{S}_5$, $\mathbb{S}_6$ and $\mathbb{S}_7$ are non-solvable rational groups having class 2 Sylow 2-subgroups.

Remark 3.3. There is a result analogous to Theorem 3.1, due to Thompson, which is about the cyclic composition factors of a finite rational group. (See [13].) If p is a prime such that some rational finite group has a composition factor of order p, then $p \leq 11$. In the cited article, Thompson has mentioned that, he has not found any rational groups with a composition factor of order p if $p \in \{7, 11\}$, but he has not been able to eliminate this possibility either.

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