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A NECESSARY CONDITION FOR WEYL-HEISENBERG FRAMES

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Abstract. In some papers like [6] some conditions are given for a triple (g, a, b) so that it generates a Weyl-Heisenberg frame for $L^2(R)$. When G is a locally compact abelian group, in [10] it is given a necessary and sufficient condition for a function $f \in L^2(G)$ and a uniform lattice K in G such that the pair (f, K) generates a Weyl-Heisenberg frame for $L^2(G)$. But this condition depends on the Zak transform of f. Also it is stated there a *sufficient* condition for a pair $S(f, K)$ such that it generates a frame for $L^2(G)$. Here we give a necessary condition for the pair (f, K) to generate a W-H frame for $L^2(G)$ which depends only on f and K.

1. **Introduction and Preliminaries**

Frames and wavelets have been one of the most interesting fields for research in the last twenty years. A wide spectrum of researchers including pure and applied mathematicians, engineers, and physicists have been focused on them. One of the aims of these works is finding out the methods by which one can construct frames for some special Hilbert spaces. In some papers like [6] it is given conditions on a function $g \in L^2(R)$ and constants a and b such that the triple

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(g, a, b) generates some special kinds of frames for $L^2(R)$, namely Weyl-Heisenberg (abbreviated as W-H) frames. In fact these frames come from a unitary representation of a topological group on the space of all unitary operators on $L^2(R)$. G.Kutyniok in [10] has given a necessary and sufficient condition for generating such frames for $L^2(G)$ when G is an LCA group. But this condition depends on the Zak transform of some function $f \in L^2(G)$ ([10], Theorem 2.1). Also it is stated there a sufficient condition for generating W-H frames for $L^2(G)$([10],Lemma 2.2). In this paper we give a necessary condition for generating W-H frames for $L^2(G)$, independent of the Zak transform. To do so, first we give some preliminary definitions and results; then we introduce the Haar measure of a generalized W-H group. We will prove that this group is unimodular. Our main result is Theorem 2.3.2 Finally we state and prove the relation between frames for $L^2(G)$ and $L^2(\widehat{G})$.

1.1. Preliminaries

Let G be a locally compact group. A *unitary representation* of G is a continuous homomorphism π of G into the space of all unitary operators on some Hilbert space which we will show it by H_π. The space of such operators will be shown by $U(H_\pi)$. π is called *irreducible* if $\pi(x)$ has no nontrivial invariant subspace for every $x \in G$. When G is abelian, every unitary representation of G is 1-dimensional([4],Corollary 3.6). In this note we will assume that G is abelian. A *character* of G is a continuous homomorphism $\xi : G \to \mathbb{C}$ such that $\xi(x^{-1}) = \overline{\xi(x)}$. It is customer to show the action of ξ on x by $< x, \xi >$ instead of $\xi(x)$. The set of all characters of G will be shown by $\widehat{G}$. In fact $\widehat{G}$ is the set of all irreducible unitary representations of G. $\widehat{G}$ is an abelian group and moreover it is a topological group with compact convergence topology. Because of these, $\widehat{G}$ is called the *dual group* of G. In some special cases, there is an interesting relation between the topological structures of G and $\widehat{G}$:

Lemma 1.1.1. *If $G$is discrete, then $\widehat{G}$ is compact; if G is compact, then $\widehat{G}$ is discrete.*

Proof. See [4], Proposition 4.4.

Example 1.1.2. Here are three well-known groups and their dual groups.
1. $\widehat{\mathbb{Z}} \cong T$ with the pairing $< n, a >= a^n$, where T denotes the unit circle.
2. $\widehat{\mathbb{T}} \cong \mathbb{Z}$ with the pairing $< a, n >= a^n$.
3. $\widehat{\mathbb{R}} \cong \mathbb{R}$ with the pairing $< x, \xi >= e^{2\pi i x\xi}$.

Let G be a topological group. By Aut(G) we mean the set of all automorphisms of G which are also homeomorphism. This is a group with the multiplication to be the composition of automorphisms. Moreover it is a topological group under the topology so called *Braconnier topology* ([8],Theorem 26.5). Let G_1 and G_2 be two locally compact groups and $\tau : G_1 \longrightarrow \text{Aut}(G_2)$ be a homomorphism such that the mapping $(x, g) \mapsto \tau_x(g)$ is a continuous homomorphism of $G_1 \times G_2$ onto G_2. Then the Cartesian product $G_1 \times G_2$ endowed with the operation

$$(x_1, g_1)(x_2, g_2) = (x_1x_2, g_1\tau_{x_1}(g_2))$$

is a group with the identity element (e_{G_1}, e_{G_2}) and the inverse: $(x, g)^{-1} = (x^{-1}, \tau_{x^{-1}}(g_2))$. In fact this is a topological group with the product topology. This group is denoted by $G_1 \times_\tau G_2$ and is called the *semidirect product* of G_1 and G_2, respectively.

1.2. Generalized Weyl-Heisenberg Group

Let $H = R \times \hat{R} \times T$. For (a_1, b_1, t_1) and (a_2, b_2, t_2) in H, the operation

$$(a_1, b_1, t_1)(a_2, b_2, t_2) = (a_1 + a_2, b_1 + b_2, t_1t_2e^{2\pi i a_1 b_2})$$

gives a group structure to H with the identity element $(0, 0, 1)$ and the inverse as $(a, b, t)^{-1} = (-a, -b, t^{-1}e^{2\pi i ab})$. On the other hand we can write $H = R \times_\tau (\hat{R} \times T)$ that, is the semidirect product of R and $\hat{R} \times T$ related to the homomorphism $\tau : R \to \text{Aut}(\hat{R} \times T)$ such that $\tau_a(b, t) = (b, te^{2\pi i ab})$. Anywhere, it is a locally compact (and nonabelian) topological group with respect to the product topology. So, by [4], Theorem 2.10, it possesses a left Haar measure.

Proposition 1.2.1. *The product measure dadbdt is the left and right Haar measure on H. In particular H is unimodular.*

Proof. See [6], Proposition 3.2.3. □

H is called *Weyl-Heisenberg* (or simply W-H) group. In [9], G. Kuytniok has given a generalized notion of the W-H group. Since we haven't seen any discussion about the Haar measure of this generalized case, for reader's convenience we include a definition and a proof to introduce such a measure.

Definition 1.2.2. Let G be a locally compact abelin group, $\widehat{G}$ its dual group, and μ and ν their Haar measure, respectively. Also let T be the unit circle. Put $H_G = G \times_\tau (\widehat{G} \times T)$, where $\times_\tau$ is the semidirect product related to the homomorphism $\tau : G \to \mathrm{Aut}(\widehat{G} \times T)$ such that $\tau_g(w, z) = (w, z.w(g))$. For (g_1, w_1, z_1) and (g_2, w_2, z_2) in H_G let

$$(g_1, w_1, z_1).(g_2, w_2, z_2) = (g_1 g_2, w_1 w_2, z_1 z_2 w_2(g_1)).$$

H_G is closed under this action. Also this action is associative. $(e, 1_{\widehat{G}}, 1)$ is the neutral element of this action, where e is the identity of G, and $(g, w, z)^{-1} = (g^{-1}, \bar{w}, z^{-1}w(g))$. This makes H_G a group. H_G is called the *Weyl-Heisenberg group associated with G.* It is a nonabelian group which is locally compact (with product topology). So, it possesses a left Haar measure (which is also a right Haar measure).

Proposition 1.2.3. *H_G is a unimodular group whose Haar measure is $d\mu d\nu dz/2\pi i$.*

Proof.

$$\int_T \int_{\widehat{G}} \int_G F((x, v, z^{'})(g, w, z)) d\mu(g) d\nu(w) dz/2\pi i$$

$$= \int_T \int_{\widehat{G}} \int_G F(xg, vw, zz^{'} w(x)) d\mu(g) d\nu(w) dz/2\pi i$$

$$= \int_T \int_{\widehat{G}} \int_G F(g, w, zz^{'} (v^{-1}w)(x)) d\mu(g) d\nu(w) dz/2\pi i$$

$$= \int_T \int_{\widehat{G}} \int_G F(g, w, zz^{'} v^{-1}(x) w(x)) d\mu(g) d\nu(w) dz/2\pi i.$$

There exist $\theta, \theta_1, \theta_2, \theta_3$ in $[0, 2\pi]$ such that $z = e^{i\theta}, z^{'} = e^{i\theta_1}$, $w(g) = e^{i\theta_2}$ and $v^{-1}(x) = e^{i\theta_3}$. So the last integral equals to

$$\int_0^{2\pi} \int_{\widehat{G}} \int_G F(g, w, e^{i(\theta+\theta_1+\theta_2+\theta_3)}) d\mu(g) d\nu(w) d\theta/2\pi$$

$$= \int_0^{2\pi} \int_{\widehat{G}} \int_G F(g, w, e^{i\theta}) d\mu(g) d\nu(w) d\theta/2\pi$$

$$= \int_T \int_{\widehat{G}} \int_G F(g, w, z) d\mu(g) d\nu(w) dz/2\pi i.$$

A similar argument shows that this is also the right Haar measure of H_G. $\square$

2. Generalized W-H frames

In this section first we introduce W-H frames. Then, according to the definition of generalized W-H group, we give the definition of the generalized W-H frame. Finally we state and prove a theorem that gives a necessary condition for existence of such frames. Most of the definitions and results in the literature are due to Daubechies, Grassman, and Meyer, Heil and Walnut, and also Kutyniok in [2], [1] ,[6] , and [10].

2.1. Frames and bases

Definition 2.1.1. A sequence$\{f_n\}$ in a separable Hilbert space H is a basis for H if for every $h \in H$ there exists a unique sequence of scalars$\{c_n\}$ such that $h = \sum c_n f_n$. $\{f_n\}$ is called an *orthonormal basis* if it is also an orthonormal set.The basis is bounded if

$$0 < \inf \parallel f_n \parallel \leq \sup \parallel f_n \parallel < \infty.$$

It is unconditional if $\sum c_n f_n$ converges unconditionally for every $h \in H$, i.e., the series converges for every rearrangement of its terms.

In Hilbert spaces all bounded unconditional bases are equivalent to an orthonormal basis. In other words, if $\{f_n\}$ is a bounded unconditional basis for H then there exists an orthonormal basis $\{e_n\}$ and a bounded invertible operator $U : H \to H$ such that $f_n = Ue_n$ for each n. The necessity of orthogonality for bases makes some difficulties and limitations for finding a suitable basis for H. Frames are a generalization of bases without these limitations. They were firstly introduced by Duffin and Scheafer in early fifties [3]. One of the advantages of frames is that they can be constructed from a single vector $f \in H$. Two of the most famous frames for $L^2(R)$ are Gabor frames (or Weyl-Heisenberg frames) and affine frames. Both of these frames come from a unitary representation of two well-known topological groups : Weyl-Heisenberg group and affine group. Here we state some definitions and results related to frames.

Definition 2.1.2. A sequence $\{f_n\}$ in a separable Hilbert space H is a frame if there exist finite constants $A, B > 0$ such that for every $h \in H$,

$$0 < A \parallel h \parallel^2 \leq \sum_n |\langle h, f_n\rangle|^2 \leq B \parallel h \parallel^2 < \infty.$$

Constants A and B are called *frame bounds*. The frame is called *tight* if A=B. It is a *Parseval frame* if A=B=1. A frame is *exact* if it ceases to be a frame whenever any single element is deleted from it. There is a close relation between exact frames and bounded unconditional bases :

Theorem 2.1.3. *A sequence $\{f_n\}$ in a Hilbert space H is an exact frame if and only if it is a bounded unconditional basis.*

Proof. See [7] , [11]. □

2.2. W-H Frames

Notation. For real numbers a and b and a real function f put

$$T_a f(x) = f(x-a),$$

$$E_b f(x) = e^{2\pi ibx} f(x).$$

T_a and E_b are unitary operators on $L^2(R)$ which are called the *translation* and *modulation* operators, respectively. Also we have

$$T_a E_b f(x) = e^{2\pi ib(x-a)} f(x-a),$$

$$E_b T_a f(x) = e^{2\pi ibx} f(x-a).$$

Definition 2.2.1. Let H be the W-H group. For $(a,b,t) \in H$ and $f \in L^2(R)$ let

$$\pi(a,b,t)f(x) = t.T_a E_b f(x) = t.e^{2\pi ib(x-a)} f(x-a).$$

Then $\pi(a,b,t)$ is a unitary operator on $L^2(R)$ for every $(a,b,t) \in H$. So π is a unitary representation of H on $L^2(R)$.

If we assume that $t = 1$ and $a, b > 0$, then we can give the definition of a W-H frame.

Definition 2.2.2. Given $g \in L^2(R)$ and $a, b > 0$, we say that (g, a, b) generates a W-H frame for $L^2(R)$ if $\{E_{mb}T_{na}g\}_{m,n\in Z}$ is a frame for $L^2(R)$. The function g is called the *mother wavelet* and the numbers a, b are *frame parameters*.

Lemma 2.2.3. *$\{E_{mb}T_{na}g\}$ is a frame for $L^2(R)$ if and only if $\{T_{na}E_{mb}g\}$ is so.*

Proof. For $f \in L^2(R)$ it can be easily shown that

$$|\langle E_{mb}T_{na}g, f\rangle|^2 = |\langle T_{na}E_{mb}g, f\rangle|^2.$$

The rest of the proof is straightforward. □

Theorem 2.2.4. *Let $g \in L^2(R)$ and $a, b > 0$ be such that*
(1) *There exist constants A,B such that*

$$0 < A \leq \sum_n |g(x - na)|^2 \leq B < \infty \ a.e.,$$

(2) *g has a compact support, with* $\mathrm{supp}(g) \subset I \subset R$, *where I is some interval of length 1/b.*
Then (g,a,b) generates a W-H frame for $L^2(R)$ whit frame bounds $b^{-1}A, b^{-1}B$.

Proof. See [6], Theorem 4.1.4. □

2.3. Generalized W-H Frames

Let G be a locally compact abelian group. A *uniform lattice* in G is a discrete subgroup K of G that is also cocompact, i.e., G/K is compact. For an arbitrary subgroup K of G, the set of all ω in $\widehat{G}$ for which $\omega(k) = 1$ for every $k \in K$, is called the *annihilator* of K. We denote the annihilator of K by $\mathrm{Ann}(K)$. In [8], lemma 24.5, it is shown that $\mathrm{Ann}(K) = \widehat{G/K}$ and $\widehat{G}/\mathrm{Ann}(K) = \widehat{K}$. Because of this and according to Lemma 1.1.1, $\widehat{G/K}$ is a uniform lattice in $\widehat{G}$ when K is so in G.

Let G be as above with the Haar measure μ, $\widehat{G}$ its dual group with the Haar measure ν, K a uniform lattice in G, and $\widehat{G/K}$ its corresponding lattice in $\widehat{G}$. Also let $\Psi : H_G \rightarrow U(L^2(G))$ be the *Schrödinger representation* which is an irreducible unitary representation defined by

$$(\Psi(x, \gamma, z)f)(t) = z\gamma(t)f(xt)$$

for every $(x, \gamma, z) \in H_G$ and for every $f \in L^2(G)$. We use this representation for constructing W-H frames for $L^2(G)$. To do so we restrict ourselves to those elements of H_G that are of the form $(x, \gamma, 1)$, where $x \in K$ and $\gamma \in \widehat{G/K}$. Explicitly, for $f \in L^2(G)$ and $(k, \gamma) \in K \times \widehat{G/K}$ we define

$$\psi^f_{(k,\gamma)}(t) := (\Psi(k, \gamma, 1)f)(t) = \gamma(t)f(kt),$$

and

$$S(f,K) = \{\psi^f_{(k,\gamma)} : (k,\gamma) \in K \times \widehat{G/K}\}.$$

Now we are ready to give the definition of W-H frames in the general case.

Definition 2.3.1. Let K be a uniform lattice in G and $g \in L^2(G)$. We say that the pair (g, K) generates a Weyl-Heisenberg frame for $L^2(G)$ if $S(g, K)$ is a frame for $L^2(G)$.

Theorem 2.3.2. *Let G be a locally compact abelian group, μ its Haar measure, K a uniform lattice in G and $g \in L^2(G)$ be compactly supported with $\mu(\text{supp}(g)) = b$. If there exist constants A and B such that*

$$0 < A \leq \sum_{k\in K} \mid g(xk) \mid^2 \leq B < \infty \text{ a.e (*)},$$

Then (g, K)generates a W-H frame for $L^2(G)$ with frame bounds bA and bB.

Proof. Let the support of g be the set $I \subset G$. With our assumption $\mu(I) = b < \infty$. The operator $T_x : G \to G$ defined by $T_x(g) = gx$ is the translation operator. It is clear that for every $f \in L^2(G)$ the function $f.T_x\bar{g}$ is supported in $I_k = \{xk^{-1} : x \in I\}$ for every $k \in K$ and $\mu(I_k) = \mu(I) = b$. Also, if we put $\gamma'_k = \gamma\mid_{I_k}$, then the collection $\{b^{-1/2}\gamma'_k\}_{\gamma\in\widehat{G/K}}$ will be an orthonormal basis for $L^2(I_k)$ for each fixed $k \in K$. We explain it here.

First we will show that $\{b^{-1/2}\gamma'_k\}$ is an orthonormal set. If $\gamma'_{k_1} \neq \gamma'_{k_2}$ on I_k, there exists $x_0 \in I_k$ such that $\gamma'_{k_1}(x_0) \neq \gamma'_{k_2}(x_0)$, i.e., $\gamma'_{k_1}(x_0)\overline{\gamma'_{k_2}(x_0)} \neq 1$, where 1 denotes the identity element of $\mathbb{C}$. So,

$$\langle\gamma'_{k_1}, \gamma'_{k_2}\rangle = \int_{I_k} \gamma'_{k_1}(x)\overline{\gamma'_{k_2}(x)}d\mu(x) = \int_{I_k} \gamma'_{k_1}(x_0x)\overline{\gamma'_{k_2}(x_0x)}d\mu(x_0x)$$

$$= \gamma'_{k_1}(x_0)\overline{\gamma'_{k_2}(x_0)} \int_{I_k} \gamma'_{k_1}(x)\overline{\gamma'_{k_2}(x)}d\mu(x) = \gamma'_{k_1}(x_0)\overline{\gamma'_{k_2}(x_0)}\langle\gamma'_{k_1}, \gamma'_{k_2}\rangle,$$

which implies that $\langle\gamma'_{k_1}, \gamma'_{k_2}\rangle = 0$. Normality follows simply (similar to the Prop. 4.3, [1]). Next we show the completeness of

$\{b^{-1/2}\gamma_k^{'}\}_{k\in K}$. Since $(\gamma_k^{'}\omega_k^{'})(x) = \gamma_k^{'}(x)\omega_k^{'}(x)$, the finite linear combinations of these restricted characters is an algebra . By Gelfand-Raikov theorem and also Pontriagin duality theorem (see [4]) the set of all characters separate points on G. Now, if there exist x_1k and x_2k in I_k such that $x_1k \neq x_2k$, since k is fixed, it will follow that $x_1 \neq x_2$ and with the above discussion there exists a γ such that

$$\gamma(x_1k) = \gamma(x_1)\gamma(k) = \gamma(x_1) \neq \gamma(x_2) = \gamma(x_2)\gamma(k) = \gamma(x_2k).$$

The second and the third equalities hold because $\gamma \in \widehat{G/K}$, the annihilator of K. Hence the algebra mentioned above separate points on I_k. Also this algebra is conjugate closed and vanishes at no point of I_k. So by Stone-Weierstrass theorem this algebra is dense in $C(I_k) = C_c(I_k)$ in the uniform norm and hence in the L^2- norm. $C(I_k)$ itself is dense in $L^2(I_k)$ ([5], Theorem 7.9). So $\{b^{-1/2}\gamma_k^{'}\}$ is complete and hence an orthonormal basis for $L^2(I_k)$. So

$$\sum_{(k,\gamma)} |\langle f, \psi^g_{(k,\gamma)}\rangle|^2 = \sum_{(k,\gamma)} |\langle f, \gamma(.)T_k g\rangle|^2 =$$

$$\sum_{(k,\gamma)} |\langle f.T_k\bar{g}, \gamma\rangle|^2 = \sum_{(k,\gamma)} |\int_G f(x)\bar{g}(xk)\bar{\gamma}(x)d\mu(x)|^2$$

$$= \sum_{(k,\gamma)} |\int_G f(x)\bar{g}(xk)\gamma(x)d\mu(x)|^2$$

$$= \sum_{(k,\gamma)} |\int_{I_k} f(x)\bar{g}(xk)\gamma(x)d\mu(x)|^2$$

$$= \sum_{(k,\gamma_k^{'})} |\int_{I_k} f(x)\bar{g}(xk)\gamma_k^{'}(x)d\mu(x)|^2$$

$$= \sum_{(k,\gamma_k^{'})} |\langle f.T_k\bar{g}, \gamma_k^{'}\rangle|^2 = \sum_k \sum_{\gamma_k^{'}} |\langle f.T_k\bar{g}, \gamma_k^{'}\rangle|^2$$

$$= b\sum_k \sum_{\gamma_k^{'}} |\langle f.T_k\bar{g}, b^{-1/2}\gamma_k^{'}\rangle|^2 = b\sum_k \| f.T_k\bar{g} \|_2^2$$

$$= b\sum_k \int_{I_k} |(f.T_k\bar{g})(x)|^2 d\mu(x) = b\sum_k \int_{I_k} |f(x)\bar{g}(xk)|^2 d\mu(x)$$

$$= b\sum_k \int_{I_k} |f(x)|^2 |g(xk)|^2 d\mu(x) = b\int_G (|f(x)|^2 \sum_k |g(xk)|^2) d\mu(x).$$

Now from (*) the result follows. $\square$

Example 2.3.3. If we put $G = \mathbb{R}$, then Theorem 2.2.4 will be a special case of this theorem with $ab = 1$.

Example 2.3.4. If we put $G = \mathbb{R}^d$ and we consider it with its usual Haar measure, the Lebesgue measure dx, and $K = \mathbb{Z}^d$, then $G/K = \prod_{i=1}^d [0,1)$which is isomorphic to T^d, the d-times product of T. Everything is similar to the above example with the exception of modulation operator which is: $E_y(x) = e^{2\pi i y x^t}$ where x^t denotes the transpose of x. Also a and b are as above and if $n = (n_1, ..., n_d)$ is in $\mathbb{Z}^d$, then $na = (n_1 a, ..., n_d a)$.

Example 2.3.5. Let $G = T$, the 1-dimentional torus. Then, its dual group is $\mathbb{Z}$ and $K = \{e^{\frac{in\pi}{2}} : n \in \mathbb{Z}\}$ is a discrete subgroup of T. Also $T/K = \{e^{i\theta} : 0 \leq \theta < \frac{\pi}{2}\}$ which is compact with respect to the quotient topology. For $n \in \mathbb{Z} = \widehat{T}$, let $n(t) = t^n$, where $t \in T$. Then $n(t_1 t_2) = (t_1 t_2)^n = t_1^n t_2^n = n(t_1)n(t_2)$ for every t_1, t_2 in T. So this action is well-defined for characters of T. Accordingly, $\psi^f_{(k,\gamma)}(t) = t^\gamma f(kt)$ and hence $\mathrm{Ann}(K) = \{m \in \mathbb{Z} : m = 4l, l \in \mathbb{Z}\} = 4\mathbb{Z}$. Now if $g \in L^2(T)$ satisfies the conditions of theorem 2.3, then (g, K) generates a W-H frame for $L^2(T)$.

Example 2.3.6. Let $G = T^n$, the n-dimensional torus. Then $\widehat{G} = \mathbb{Z}^n$ and $K = \{(e^{\frac{im_1\pi}{2}}, e^{\frac{im_2\pi}{2}}, ..., e^{\frac{im_n\pi}{2}}) : m_j \in \mathbb{Z}, j = 1, ..., n\}$ is a uniform lattice in $\widehat{G}$. Also if $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{Z}^n$ and $t = (t_1, ..., t_n) \in T^n$, then $\gamma(t) = \prod_{i=1}^n t_i^{\gamma_i}$. As in previous example for the case n=1, the annihilator of K is the set $4\mathbb{Z}^n = \{4m : m \in \mathbb{Z}^n\}$. The rest is as above.

2.4. Frames for $L^2(\widehat{G})$

In the previous section we showed the construction of W-H frames of $L^2(G)$. Now it is the time to discus about W-H frames for $L^2(\widehat{G})$. In fact it is easy to construct a W-H frame for $L^2(\widehat{G})$ when we have such a frame for $L^2(G)$. Here we compute the Fourier transform of $\psi^f_{(k,\gamma)}$ for some fixed $f \in L^2(G)$ and $(k,\gamma) \in K \times \widehat{(G/K)}$. By this computing we will prove the following theorem.

Theorem 2.4.1 $\{\psi^f_{(k,\gamma)}\}$ *is a frame for* $L^2(G)$ *if and only if* $\{\psi^{\hat{f}}_{(\bar{\gamma},k)}\}$*is a frame for* $L^2(\widehat{G})$.

Proof.

$$\widehat{(\psi^f_{(k,\gamma)})}(\xi) = \int_G \psi^f_{(k,\gamma)}(x)\bar{\xi}(x)d\mu(x) = \int_G \gamma(x)f(xk)\bar{\xi}(x)d\mu(x)$$

$$= \int_G \gamma(x)\gamma(k)f(xk)\overline{\xi}(x)d\mu(x) = \int_G \gamma(xk)f(xk)\bar{\xi}(x)d\mu(x)$$

$$= \int_G \gamma(x)f(x)\bar{\xi}(xk^{-1})d\mu(x) = \int_G \gamma(x)f(x)\bar{\xi}(x)\bar{\xi}(k^{-1})d\mu(x)$$

$$= \int_G \xi(k)f(x)\bar{\xi}\gamma(x)d\mu(x) = \langle k,\xi\rangle \int_G f(x)\overline{\xi\bar{\gamma}(x)}d\mu(x)$$

$$= \langle k,\xi\rangle \hat{f}(\xi\bar{\gamma}) \ = \psi^{\hat{f}}_{(\bar{\gamma},k)}(\xi).$$

The third equality holds because $\gamma \in \widehat{(G/K)}$ and so $\gamma(k) = 1$ for every $k \in K$. So we showed that $\widehat{(\psi^f_{(k,\gamma)})} = \psi^{\hat{f}}_{(\bar{\gamma},k)}$. Hence by the unitary nature of the Fourier transform,$\{\psi^f_{(k,\gamma)}\}$ is a frame for $L^2(G)$ if and only if $\{\psi^{\hat{f}}_{(\hat{\gamma},k)}\}$ is a frame for $L^2(\widehat{G})$. $\square$

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