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Author(s):

X.F. Hu and X.B. Li

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HÖLDER CONTINUITY OF A PARAMETRIC VARIATIONAL INEQUALITY

X.F. HU AND X.B. LI*

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Abstract. In this paper, we study the Hölder continuity of solution mapping to a parametric variational inequality. At first, recalling a real-valued gap function of the problem, we discuss the Lipschitz continuity of the gap function. Then under the strong monotonicity, we establish the Hölder continuity of the single-valued solution mapping for the problem. Finally, we apply these results to a traffic network equilibrium problem.
Keywords: Hölder continuity, gap function, parametric variational inequality, traffic network equilibrium problem.
MSC(2010): Primary: 49K40; Secondary: 49K40, 90C31.

1. **Introduction**

The variational inequality of Hartman and Stampachia [14] is a well-known model in optimization theory. It has many applications in different fields (including mathematical programming and some equilibrium problems). The reader can refer to the very informative recent book by Facchinei and Pang [11] for the background information and motivations of the variational inequality. In recent thirty years, existence results and convergences for the variational inequality, stability and sensitivity for the parametric variational inequality (PVI), and some applications have been investigated extensively; see [8, 11, 12, 22–24, 29] and the references therein.

In this paper, we mainly follow with interest the Hölder continuity for PVI. There are many papers considering the stability of PVI, for example, [6, 8, 10, 13, 15, 25, 29–31]. Based on the nonexpansivity of the metric projection on closed convex sets, Dafermos [8] derived the local uniqueness, upper Lipschitz continuity and differentiability of the solution mapping of PVI. In virtue of the similar idea, Yen [29] obtained the Hölder continuity of the solution mapping for PVI. Later, Yen [30] got the local Lipschitz continuity of the solution mapping

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*Corresponding author.

of PVI on a parametric polyhedral set and then he applied the results to a traffic network equilibrium problem. Recently, Causa and Raciti [6] also made use of the metric projection to discuss the Lipschitz continuity of the solution mapping for PVI. Based on a theory of contingent derivative and by a result about differentiability property of the metric projection, Shapiro [25] discussed the continuity, locally upper Lipschitz and differentiability properties of the solution mapping for PVI where the constraint set is nonconvex. There have been many papers discussing the local uniqueness and Höder continuity of the perturbed solution sets to Ky Fan inequality KFI; see, e.g., [1, 2, 5, 17–21, 27] and references therein. Herein, having the aid of a gap function for PVI, we make the solution mapping of PVI write as a simple implicit function and then establish the Hölder continuity of the solution mapping for PVI. Finally, we apply the results to a traffic network equilibrium problem with perturbations in travel cost function and travel demand.

The rest of the paper is organized as follows. In Section 2, recalling a real-valued gap function of PVI, we discuss the Lipschitz continuity of the gap function. In Section 3, under the strong monotonicity, we establish the Hölder continuity of the single-valued solution mapping for PVI. In Section 4, we apply these results to a traffic network equilibrium problem. In Section 5, we give an open question which we consider in future.

2. The regularized gap function of PVI

In this paper, let X, P and Λ be finite dimensional, and let $\mathbb{B}$ indicate the closed unit ball of X. Given a subset $A \subset X$, we define the distance from $x \in X$ to A by $d(x, A) := \inf_{a \in A} ||x - a||$ with the convention that $d(x, \emptyset) = \infty$. Let $F : \Lambda \rightrightarrows X$ be a set-valued mapping. The effective domain and graph of F are defined by $\text{dom}F := \{\lambda \in \Lambda | F(\lambda) \neq \emptyset\}$ and $\text{gph}F := \{(\lambda, x) \in \Lambda \times X | x \in F(\lambda)\}$, respectively. F is said to be Hölder at $\bar{\lambda} \in \Lambda$ if there exist constants $\gamma > 0$, $\alpha > 0$ and $\delta > 0$ such that

$$F(\lambda) \subset F(\lambda') + \gamma ||\lambda - \lambda'||^{\alpha} \mathbb{B}, \ \forall \lambda, \lambda' \in U(\bar{\lambda}, \delta).$$

Whenever $\alpha = 1$, F is said to be Lipschitz at $\bar{\lambda} \in \Lambda$.

In this section, we consider the regularized gap function for PVI of finding $x \in K(\lambda)$ for parameters $p \in P, \lambda \in \Lambda$ such that

$$\langle f(p, x), y - x \rangle \geq 0, \ \forall y \in K(\lambda),$$

where $f : P \times X \to X$ is a vector-valued mapping, $K : \Lambda \rightrightarrows X$ is a set-valued mapping with nonempty closed convex values, and $\langle f(p, x), y - x \rangle$ denotes the value of the function $f(p, x)(\cdot)$ at $y - x$. For each $p \in P, \lambda \in \Lambda$, by $S(p, \lambda)$ we denote the solution mapping of PVI, i.e.,

$$S(p, \lambda) := \{x \in K(\lambda) : \langle f(p, x), y - x \rangle \geq 0, \ \forall y \in K(\lambda)\}.$$

Let $p \in P, \lambda \in \Lambda, x \in K(\lambda)$ and $\gamma > 0$ be arbitrarily given. The regularized gap function for PVI introduced by Fukushima [12] is defined by

$$h_\gamma(p, \lambda, x) := -\min_{y \in K(\lambda)} \{\langle f(p,x), y - x\rangle + \frac{\gamma}{2}||y - x||^2\}.$$

Let $\pi_\gamma(p, \lambda, x) := \mathrm{Proj}_{K(\lambda)}(x - \frac{f(p,x)}{\gamma})$ denote the best approximation to $x - \frac{f(p,x)}{\gamma}$ from $K(\lambda)$. Then h_γ could be written as

$$h_\gamma(p, \lambda, x) = \langle f(p,x), x - \pi_\gamma(p, \lambda, x)\rangle - \frac{\gamma}{2}||\pi_\gamma(p, \lambda, x) - x||^2.$$

Now we consider the Lipschitz property of h_γ which will be used in the next section.

Proposition 2.1. *Let $U(\bar\lambda)$ and $U(\bar p)$ be neighborhoods of $\bar\lambda \in \Lambda$ and $\bar p \in P$ respectively. Assume that*

(i) *$K(\cdot)$ is Lipschitz on $U(\bar\lambda)$ and $K(\bar\lambda)$ is bounded;*
(ii) *$f(\cdot,\cdot)$ is Lipschitz on $U(\bar p) \times K(U(\bar\lambda))$ and $f(\bar p, \cdot)$ is bounded on $K(U(\bar\lambda))$.*

Then there exist $U(\bar p, \delta_1), U(\bar\lambda, \delta_1)$ of $\bar p \in P$ and $\bar\lambda \in \Lambda$ such that h_γ is Lipschitz on $U(\bar p, \delta_1) \times U(\bar\lambda, \delta_1) \times K(U(\bar\lambda, \delta_1))$.

Proof. By the conditions, without loss of generality, we assume that there exist constants $l_1 > 0, l_2 > 0$ and $\delta_1 > 0$ such that

$$K(\lambda) \subset K(\lambda') + l_1||\lambda - \lambda'||\mathbb{B},\ \forall \lambda, \lambda' \in U(\bar\lambda, \delta_1) \tag{2.1}$$

and for any $p, p' \in U(\bar p, \delta_1), x, x' \in K(U(\bar\lambda, \delta_1))$,

$$||f(p,x) - f(p', x')|| \le l_2(||p - p'|| + ||x - x'||), \tag{2.2}$$

where $U(\bar\lambda, \delta_1) \subset U(\bar\lambda)$ and $U(\bar p, \delta_1) \subset U(\bar p)$.

We divide the proof into three steps.

(I) $K(\cdot)$ is bounded on $\lambda \in U(\bar\lambda, \delta_1)$, i.e., there exists a constant $M > 0$ such that

$$\lambda \in U(\bar\lambda), x \in K(\lambda), \|x\| \le M. \tag{2.3}$$

Indeed, it follows from the boundedness of $K(\bar\lambda)$ that there exists a constant $M' > 0$ such that

$$\forall x \in K(\bar\lambda), \|x\| \le M'. \tag{2.4}$$

From (2.1), we have

$$K(\lambda) \subset K(\bar\lambda) + l_1\delta_1\mathbb{B},\ \forall \lambda \in U(\bar\lambda, \delta_1).$$

Then for any $x \in K(\lambda)$, there exist $y \in K(\bar\lambda)$ and $b \in \mathbb{B}$ such that

$$x = y + l_1\delta_1 b,$$

which together with (2.4) yields that

$$\|x\| \leq \|y\| + l_1\delta_1\|b\| \leq M' + l_1\delta_1.$$

Then (2.3) is satisfied with $M := M' + l_1\delta_1$.

(II) $f(\cdot,\cdot)$ is bounded on $U(\bar{p},\delta_1) \times K(U(\bar{\lambda},\delta_1))$.

By assumption (ii), we have

$$l := \sup_{x\in K(U(\bar{\lambda},\delta_1))} \|f(\bar{p},x)\| < +\infty.$$

Then from (2.2) it follows for any fixed $\epsilon > 0$, there exits a neighborhood of $\bar{p}$ denoted without loss of generality by $U(\bar{p},\delta_1)$, such that

$$\sup_{(p,x)\in U(\bar{p},\delta_1)\times K(U(\bar{\lambda},\delta_1))} \|f(p,x)\| \leq l + \epsilon < +\infty.$$

Take $l_0 := \max\{l; l+\epsilon\}$, which implies that $l_0 > 0$. Then

$$\forall p \in U(\bar{p},\delta_1), \forall x \in K(U(\bar{\lambda},\delta_1)), \|f(p,x)\| \leq l_0. \tag{2.5}$$

(III) $h_\gamma(\cdot,\cdot,\cdot)$ is Lipschitz on $U(\bar{p},\delta_1) \times U(\bar{\lambda},\delta_1) \times K(U(\bar{\lambda},\delta_1))$.

Indeed, fix $p, p' \in U(\bar{p},\delta_1), \lambda, \lambda' \in U(\bar{\lambda},\delta_1), x \in K(\lambda)$ and $x' \in K(\lambda')$. By the definition of h_γ, we have the following assertion: for any $\epsilon > 0$, there exists $y_\epsilon \in K(\lambda)$ such that

$$h_\gamma(p,\lambda,x) \leq \langle f(p,x), x - y_\epsilon\rangle - \frac{\gamma}{2}||y_\epsilon - x||^2 + \epsilon.$$

By (2.1) there exists $y' \in K(\lambda')$ such that

$$||y_\epsilon - y'|| \leq l_1||\lambda - \lambda'||. \tag{2.6}$$

Then we get

$$\begin{aligned}
&h_\gamma(p,\lambda,x) - h_\gamma(p',\lambda',x')\\
&\quad\leq \langle f(p,x), x - y_\epsilon\rangle - \frac{\gamma}{2}||y_\epsilon - x||^2 + \varepsilon - \langle f(p',x'), x' - y'\rangle + \frac{\gamma}{2}||y' - x'||^2\\
&\quad\leq \langle f(p,x) - f(p',x'), x - y_\epsilon\rangle + \langle f(p',x'), x - x' + y' - y_\epsilon\rangle\\
&\qquad + \frac{\gamma}{2}\langle y' - y_\epsilon + x - x', y' - x' + y_\epsilon - x\rangle + \epsilon,\\
&\quad\leq \|f(p,x) - f(p',x')\|(\|x\| + \|y_\epsilon\|)\\
&\qquad + \|f(p',x')\|(\|x - x'\| + \|y - y_\epsilon\|) + \frac{\gamma}{2}(\|y' - y_\epsilon\|\\
&\qquad + \|x - x'\|)(\|y'\| + \|x'\| + \|y_\epsilon\| + \|x\|) + \epsilon.
\end{aligned} \tag{2.7}$$

So, from (2.2), (2.3), (2.4), (2.5), (2.6), (2.7) and the arbitrariness of ϵ, we have

$$\begin{aligned}
&h_\gamma(p,\lambda,x) - h_\gamma(p',\lambda',x')\\
&\quad\leq 2Ml_2\|p - p'\| + (l_0 + 4M\gamma)l_1\|\lambda - \lambda'\| + (2Ml_2 + l_0 + 4M\gamma)\|x - x'\|\\
&\quad\leq l(\|p - p'\| + \|\lambda - \lambda'\| + \|x - x'\|),
\end{aligned}$$

where $l := \max\{2Ml_2, (l_0 + 4M\gamma)l_1, (2Ml_2 + l_0 + 4M\gamma)\}$. Therefore, it follows from the symmetry between (p, λ) and (p', λ') that the conclusion holds and the proof is complete. □

Corollary 2.2. *Let $U(\bar{\lambda})$ and $U(\bar{p})$ be neighborhoods of $\bar{p} \in P$ and $\bar{\lambda} \in \Lambda$, respectively. Assume that*

(i): *$K(\cdot)$ is Lipschitz with compact values on $U(\bar{\lambda})$;*
(ii): *$f(\cdot, \cdot)$ is Lipschitz on $U(\bar{p}) \times K(U(\bar{\lambda}))$.*

Then there exist $U(\bar{p}, \delta_1)$ and $U(\bar{\lambda}, \delta_1)$ of $\bar{p} \in P$ and $\bar{\lambda} \in \Lambda$ such that h_γ is Lipschitz on $U(\bar{p}, \delta_1) \times U(\bar{\lambda}, \delta_1) \times K(U(\bar{\lambda}, \delta_1))$.

Proof. Take $U(\bar{\lambda}, \delta_1)$ and $U(\bar{p}, \delta_1)$ of (2.1) and (2.2) be compact neighborhoods. Obviously, (2.3) naturally holds as $K(\lambda)$ is compact for any $\lambda \in U(\bar{\lambda}, \delta_1)$.

Since $K(\cdot)$ is Lipschitz on $U(\bar{\lambda})$, $K(\cdot)$ is upper semicontinuous on $U(\bar{\lambda}, \delta_1) \subset U(\bar{\lambda})$. Then it follows from [3, Proposition 11] that the set $K(U(\bar{\lambda}, \delta_1))$ is compact. By (ii), $U(\bar{p}, \delta_1) \subset U(\bar{p})$ and $U(\bar{\lambda}, \delta_1) \subset U(\bar{\lambda})$, we have $f(\cdot, \cdot)$ is also Lipschitz on $U(\bar{p}, \delta_1) \times K(U(\bar{\lambda}, \delta_1))$. Therefore, we have

$$l_0 = \sup_{(p,x)\in U(\bar{p},\delta_1)\times K(U(\bar{\lambda},\delta_1))} \|f(p, x)\| < +\infty.$$

This implies that (2.5) holds. Thus, following the proof of Proposition 2.1, we can obtain the conclusion and the proof is complete. □

3. The Hölder continuity of the solution mapping for PVI

In this section, in virtue of the properties of h_γ we discuss the Hölder continuity of the solution mapping S for PVI. In the rest of paper, let $\bar{p} \in P, \bar{\lambda} \in \Lambda$ and $\bar{x} \in S(\bar{p}, \bar{\lambda})$. Since h_γ is a gap function of PVI, we have

$$S(p, \lambda) = \{x \in K(\lambda) : h_\gamma(p, \lambda, x) = 0\},$$

which illustrates the close relationships between h_γ and S.

At first, we recall the existence of the solution for PVI.

Proposition 3.1. *Suppose that f is strongly monotone with modulus $\rho > 0$ with respect to x uniformly in (p, λ) which are around $(\bar{p}, \bar{\lambda})$, namely, there exists an open neighborhood of $(\bar{p}, \bar{\lambda})$ denoted by $U(\bar{p}, \delta) \times U(\bar{\lambda}, \delta)$ for some $\delta > 0$ such that for each $p \in U(\bar{p}, \delta), \lambda \in U(\bar{\lambda}, \delta)$ and $x, x' \in K(\lambda)$ one has that*

$$\langle f(p, x) - f(p, x'), x - x'\rangle \geq \rho||x - x'||^2.$$

If $\rho > \frac{\gamma}{2}$ and f is continuous with respect to x uniformly in p which is around $\bar{p}$, then, for PVI, there exists a unique solution.

Proof. The conclusion is a parametric version of [11, Theorem 2.3.3 (b)]. We omit its proof. □

Theorem 3.2. *Let all conditions of Propositions 2.1 (or Corollary 2.2) and 3.1 hold. Then S is Hölder continuous at $(\bar{p}, \bar{\lambda})$.*

Proof. Fix $p \in U(\bar{p}, \delta), \lambda \in U(\bar{\lambda}, \delta)$ and $x \in K(\lambda)$. The following inequality holds

$$||x - S(p, \lambda)|| \leq \frac{1}{\sqrt{\rho - \frac{\gamma}{2}}} \sqrt{h_\gamma(p, \lambda, x)}, \tag{3.1}$$

since

$$\begin{aligned} h_\gamma(p, \lambda, x) &= -\min_{y \in K(\lambda)} \{\langle f(p, x), y - x\rangle + \frac{\gamma}{2}||y - x||^2\} \\ &\geq -\langle f(p, x), S(p, \lambda) - x\rangle - \frac{\gamma}{2}||S(p, \lambda) - x||^2 \\ &\geq \langle f(p, S(p, \lambda)) - f(p, x), S(p, \lambda) - x\rangle - \frac{\gamma}{2}||S(p, \lambda) - x||^2 \\ &\geq (\rho - \frac{\gamma}{2})||S(p, \lambda) - x||^2, \end{aligned}$$

where the second inequality holds thanks to $\langle f(p, S(p, \lambda)), S(p, \lambda) - x\rangle \leq 0$ and the last inequality holds due to the strong monotonicity of f.

By the conditions, it follows from Proposition 2.1 (or Corollary 2.2) that h_γ is locally Lipschitz at $(\bar{p}, \bar{x})$ with respect to (p, λ) uniformly in x, i.e., there exist constants $\delta_1 > 0$ and $l > 0$ such that for any $p, p' \in U(\bar{p}, \delta_1), \lambda, \lambda' \in U(\bar{\lambda}, \delta_1)$ and $x \in U(\bar{x}, \delta_1)$, one has

$$|h_\gamma(p', \lambda', x) - h_\gamma(p, \lambda, x)| \leq l(||p - p'|| + ||\lambda - \lambda'||). \tag{3.2}$$

Fix arbitrary $p, p' \in U(\bar{p}, \delta_2), \lambda, \lambda' \in U(\bar{\lambda}, \delta_2)$, where $\delta_2 = \min\{\delta, \delta_1\}$. By (2.1) there exists $x' \in K(\lambda')$ such that $||S(p, \lambda) - x'|| \leq l_1||\lambda - \lambda'||$. By (3.1) and (3.2), we get

$$\begin{aligned} ||S(p, \lambda) - S(p', \lambda')|| &\leq ||S(p, \lambda) - x'|| + ||x' - S(p', \lambda')|| \\ &\leq l_1||\lambda - \lambda'|| + \frac{1}{\sqrt{\rho - \frac{\gamma}{2}}} \sqrt{h_\gamma(p', \lambda', x')} \\ &= l_1||\lambda - \lambda'|| + \frac{1}{\sqrt{\rho - \frac{\gamma}{2}}} \sqrt{h_\gamma(p', \lambda', x') - h_\gamma(p, \lambda, S(p, \lambda))} \\ &\leq l_1||\lambda - \lambda'|| + \sqrt{\frac{l}{\rho - \frac{\gamma}{2}}} (||p - p'|| + ||\lambda - \lambda'|| + ||x' - S(p, \lambda)||)^{\frac{1}{2}} \\ &\leq l_1||\lambda - \lambda'|| + \sqrt{\frac{l}{\rho - \frac{\gamma}{2}}} [||p - p'|| + (l_1 + 1)||\lambda - \lambda'||]^{\frac{1}{2}} \end{aligned}$$

Thus, the conclusion holds. □

Remark 3.3. (i) Under the Hölder continuity of the best approximation mapping, Yen [29, Theorem 2.1] has obtained the Hölder continuity of the solution mapping for PVI. His methods are very different from ours, and [29, Theorem

2.1] needs not the condition: $\rho > \frac{\gamma}{2}$, while, needs additional conditions: f is locally Lipschitz at $(\bar{p}, \bar{x})$,

$$\frac{l^2}{\gamma} \leq \rho < \gamma \text{ and } \frac{l^2}{\gamma^2} - \frac{2\rho}{\gamma} + 1 > 0,$$

which are used to apply the Banach fixed point theorem, and l is the local Lipschitzian constant of f.

(ii) If we set $f(x, y, p) = \langle f(x, p), y - x\rangle$, then PVI becomes a case of PKFI in [1, 2, 5, 17–21, 27]. In these papers, the strongly monotonicity (or pseudomonotonicity) and the Hölder related assumptions of Ky Fan function and the properties of the solution have been employed directly to obtain the sufficient conditions for Hölder continuity of the solutions of PKFI. However, in this paper, we use a regularized gap function of PVI to write the solution mapping of PVI as a simple implicit function and then we establish the Hölder continuity of the solution mapping for PVI. Therefore, our method is very different from corresponding ones of [1, 2, 5, 17–21, 27].

4. **Application to a traffic network equilibrium problem**

Consider a traffic network equilibrium $G = (N, A)$, where N and A denote the set of nodes and directed arcs, respectively. Let W denote the set of origin-destination (OD) pairs and let $d = (d_w)_{w\in W}$ denote the demand vector, where d_w denotes the demand of traffic flow on OD pair w. Let $a \in A$ denote an arc of the network connecting a pair of nodes and p denote a path, assumed to be acyclic, consisting of a sequence of arcs connecting an OD pair. For each $w \in W$, let P_w denote the set of available paths joining the OD pair w. Let $\alpha = |A|$, $\omega = |W|$, $\pi = \sum_{w\in W} |P_w|$ and $P = \cup_{w\in W} P_w$. For a given path $p \in P_w$, let q_p denote the traffic flow on this path and $q = (q_p)_{p\in P} \in R^\pi$ denote the path flow. The path flow vector q induces an arc flow x_a on each arc $a \in A$ given by

$$x_a = \sum_{p\in P} \delta_{ap} q_p, \tag{4.1}$$

where $\delta_{ap} = 1$ if arc a is contained in path p and 0, otherwise. Let $x = (x_a)_{a\in A} \in R^\alpha$ denote the arc flow. Suppose that the demand of network flow is denoted by d_w for each OD pair w, and $d = (d_w)_{w\in W}$ denotes the travel demand. We say that a path flow q satisfies demands if

$$\sum_{p\in P_w} q_p = d_w,\ \forall w \in W. \tag{4.2}$$

Then the path flow q is called a feasible path flow. If there exists a feasible path flow $q = (q_p)$, then the arc flow $x = (x_a)$ with $x_a = \sum_{p\in P} \delta_{ap} q_p$ for each arc a is called a feasible arc flow. Let $f : R^\alpha \to R^\alpha$ be the travel cost function and

$f_a(x)$ denote the cost on the arc a. Then the travel disutility on path $p \in P$ is denoted by

$$F_p = \sum_{a \in A} f_a \delta_{ap}. \tag{4.3}$$

The traffic network equilibrium conditions, following the standard theory (see [4,9,22] and the references therein), take on the following form: a feasible path flow q is called an equilibrium path flow if, for each OD pair w and each path $p \in P_w$, the following Wardrop conditions (see [28]) hold:

$$F_p \begin{cases} = \lambda_w & \text{if } x_p > 0 \\ \geq \lambda_w & \text{if } x_p = 0, \end{cases} \tag{4.4}$$

where λ_w is an indicator, whose value is not known a priori. A feasible arc flow x is an equilibrium arc flow if there exists an equilibrium path flow q satisfying (4.1). It is easy to verify (see also [7,23,26]) that x is an equilibrium arc flow if and only if it solves the following variational inequality: find $x \in K$ such that

$$\langle f(x), y - x \rangle \geq 0, \ \forall y \in K,$$

where K is the set of feasible arc flows, i.e.,

$$K = \{x \in R^\alpha | \text{for some } q \in R_+^\pi \text{ satisfying (4.1) and (4.2)}\}.$$

The arguments in Smith [26] also show that if x is an equilibrium arc flow, then each $q \in R_+^\pi$ satisfying (4.1) and (4.2) is an equilibrium path flow.

Now we introduce the parameter to be present in the sensitivity analysis: they are denoted by $\mu \in R^c$, where the dimensions $c \in R_+ \backslash \{0\}$. The parameter μ may conclude many factors, for example, the weather, the tastes of the travelers, traffic regulations, etc. These factors all could influence the travel cost. Consequently, in this paper, we assume that the travel cost function makes perturbations with the parameter μ which takes values in a subset U of R^c. Moreover, we treat the travel demand d as a parameter of this problem and let it take values in a subset V of R^ω.

For the parameters μ and d, we shall consider the sensitivity of the equilibrium arc flow which is the solution set of the parametric variational inequality: find $x \in K(d)$ such that

$$\langle f(\mu, x), y - x \rangle \geq 0, \ \forall y \in K(d), \tag{4.5}$$

where $K(d) = \{x \in R^\alpha | \text{for some } q \in R_+^\pi \text{ satisfying (4.1) and (4.2)}\}$. Obviously, $K(d)$ is a compact convex set for each d. For $d \in R^\omega$ and $\mu \in R^c$, let $S(\mu, d)$ denote the solution mapping of (4.5).

Theorem 4.1. *Let $\bar{d} \in R^\omega, \bar{\mu} \in R^c$ and $\bar{x} \in S(\bar{\mu}, \bar{d})$. Assume that f is locally Lipschitz in μ at $(\bar{\mu}, \bar{x})$ uniformly in x and that f is locally strongly monotone with modulus $\rho > 0$ in x at $(\bar{\mu}, \bar{x})$ uniformly in μ. If $\rho > \frac{\gamma}{2}$, then there exists a neighborhood $U(\bar{x})$ of $\bar{x}$ such that $S(\cdot, \cdot) \cap U(\bar{x})$ is a single-valued mapping and is Hölder continuous at $(\bar{\mu}, \bar{d})$.*

Proof. To prove the conclusion, it follows from Theorem 3.2 that we only need prove that K is locally Lipschitz at $\bar{d}$ which implies the Lipschitz-likeness of K at $(\bar{d}, \bar{x})$. For an arbitrary neighborhood $U(\bar{d})$ of $\bar{d}$, let $d', d \in U(\bar{d})$ and $x' \in K(d')$. Then there exists $q' \in R^{\pi}_{+}$ such that $x'_a = \sum_{p \in P} \delta_{ap} q'_p$ and $\sum_{p \in P_w} q'_p = d'_w$, $\forall w \in W$. Let $\Delta = [\delta_{ap}] \in R^{\alpha \times \pi}$ and $\Lambda = [\Lambda_{wp}] \in R^{\omega \times \pi}$ denote the arc-road incidence matrix and the OD-road incidence matrix, respectively, where $\Lambda_{wp} = 1$ if the road $p \in P_w$ and $\Lambda_{wp} = 0$ otherwise. Then $x' = \Delta q'$ and $d' = \Lambda q'$. Then there exists a matrix $G \in R^{\pi \times \omega}$ such that $q' = Gd'$. Set $q := Gd, x := \Delta q$. Then $x \in K(d)$. Thus,

$$||x' - x|| = ||\Delta G(d' - d)|| \leq ||\Delta G|| \cdot ||d' - d||.$$

This completes the proof. □

5. Conclusions

In this paper, we study Hölder continuity of the single-valued solution mapping to PVI by using a gap function. While, generally speaking, the solution mapping for PVI is set-valued. Now a very interesting and valuable topic arises in a natural way: How to establish the Hölder continuity (Lipschitz property) of the set-valued solution mapping of PVI? In fact, by using a different method from here, they have discussed Lipschitz property of the set-valued solution mapping for PVI [16]. However, the first condition of [16, Theorem 5.1] which is added in the function $\langle f(p, \cdot), \cdot \rangle : x \mapsto \langle f(p, x), x \rangle$ is hard to be checked. And [16, Theorem 5.1] also need that K is compact valued at the reference point. Therefore, under suitable conditions and avoiding the compactness of the constraint set, it is worth investigating the Lipschitz property of the set-valued solution mapping for PVI in our future work.

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(Xian-Fu Hu) Department of Electronic Engineering, Chongqing City Management College, Chongqing, 401331, China.
E-mail address: xianhufu@126.com

(Xiao-Bing Li) College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, China.
E-mail address: cqjtulixb@yeah.net