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SELF-SIMILAR SOLUTIONS OF THE RIEMANN PROBLEM FOR TWO-DIMENSIONAL SYSTEMS OF CONSERVATION LAWS

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Abstract. In this paper, a new approach is applied to study the self-similar solutions of 2×2 systems of nonlinear hyperbolic conservation laws. A notion of characteristic directions is introduced and then used to construct local and smooth solutions of the associated Riemann problem.
Keywords: Self-similar solutions, Riemann problem, hyperbolic conservation laws, smooth solutions.
MSC(2010): Primary: 35L65; Secondary: 65N40, 35L55.

1. Introduction

Consider the following hyperbolic system of conservation laws

$$\partial_t U + \partial_{x_1} f_1(U) + \partial_{x_2} f_2(U) = 0, \tag{1.1}$$

where $t > 0$, $x = (x_1, x_2) \in \mathbb{R}^2$, $U \in D$ with D being an open domain of $\mathbb{R}^2$ and $f_1,\ f_2 \in \mathcal{C}^2(D, \mathbb{R}^2)$. The Riemann problem associated with the system (1.1) is a Cauchy problem where the initial data $U_0(x_1, x_2) \in D$ is discontinuous across a line Γ of class $\mathcal{C}^\infty$.

It is well known [5] that the Riemann problem in a single space variable is locally solved in terms of a succession of centered waves which consist of shock, rarefaction or contact waves. The interaction of these elementary waves is the key mechanism in determining the solutions for more general initial value problems.

In several space dimensions, a very few analytic results are available for the general Riemann problem. Majda [7] uses suitable assumptions of stability and compatibility on the initial data to prove the local existence of a solution containing a single shock surface for general systems. For 2×2 systems, Métivier [9] solves the Riemann problem when the solution has jump discontinuities

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across two shock surfaces. For quasi-linear systems in two space dimensions, Alinhac [1] obtains (with assumptions of compatibility on the initial data) the existence of a solution containing a single rarefaction wave. More recently, Pang *et al*. [10] apply a characteristic analysis method to obtain explicit solutions of a two-dimensional Riemann problem model.

It is worth noticing that many numerical techniques are available for the two-dimensional case. The works of Balsara [3], Schulz-Rinne *et al*. [11], Kurganov and Tadmor [4] and many others have provided a solid framework for further advancement.

Majda [8] observes that the characteristic surfaces delimiting a rarefaction fan are difficult to control in an approximate process. In this paper, we propose an analytical method to construct exact rarefaction waves. More precisely, we use a technique of characteristic directions to reduce the initial system (1.1) into equations with two unknowns β_1, β_2 depending on any variable $w \in \mathbb{R}^2$ and not on $U \in D$. By performing the integration for suitable β_1, β_2, we generate smooth solutions of the Riemann problem. Note that a characteristic decoupling theory is proposed in [2] in the one-dimensional case.

This paper is organized as follows. In Section 2, we introduce a coupling property and establish an existence result of real characteristic directions. In Section 3, we construct a new system of $2n$ equations ($n = 2$) with self-similar solutions. These equations may be considered as a generalization of the usual equations of rarefaction waves. Using a Schwartz (-Frobenius) integrability condition, we transform this new system into a well-posed $n \times n$ evolution system. In Section 4, we use a continuous extension technique to construct locally a foliated smooth solution of the Riemann problem.

2. Definitions and notations

Assume that the system (1.1) is strictly hyperbolic, that is for any $w \in D$ and $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2 \backslash \{0\}$, the eigenvalues of the matrix $\eta_1 df_1(w) + \eta_2 df_2(w)$ are real and simple. We shall denote by $\lambda_j(w, \eta)$ these eigenvalues, and by $r_j(w, \eta)$ the associated eigenvectors, $j = 1, 2$. Without loss of generality, we may assume that $\lambda_1(w, \eta) < \lambda_2(w, \eta)$.

One of the most important features of the Riemann problem is that the solutions may be assumed to be self-similar, reducing the dimensions by one. In the current numerical studies, any collection of rays centered at some point and separated by constant states leads to a self-similar solution [3]. So we deal with self–similar smooth solutions defined by $u(\frac{x_1}{t}, \frac{x_2}{t})$ with $u \in \mathcal{C}^2(V, \mathbb{R}^2)$ and V an open domain of $\mathbb{R}^2$. We set $\sigma_j = \dfrac{x_j}{t}$, $j = 1, 2$. Then the smooth form of the system (1.1) is:

$$\big(\partial_{\sigma_1} f_1(u) - \sigma_1 \partial_{\sigma_1} u\big) + \big(\partial_{\sigma_2} f_2(u) - \sigma_2 \partial_{\sigma_2} u\big) = 0. \tag{2.1}$$

Throughout this paper, we consider the Riemann problem associated with the system (2.1) and the initial data $U_0 \in D$. We make the following coupling assumption:

For any $w \in D$, $\lambda_1(w,.)$ and $\lambda_2(w,.)$ are nonlinear functions.

Note that for any 2×2 two-dimensional system of conservation laws, the functions $\eta \mapsto \lambda_j(w,\eta)$ are both linear or both nonlinear. This is due to the formula

$$\lambda_1(w,\eta) + \lambda_2(w,\eta) = \mathrm{trace}\left(\eta_1 df_1(w) + \eta_2 df_2(w)\right).$$

Now we can define the (co)directions associated with the characteristic fields of the system (2.1).

Proposition 2.1. *Let S be the unit sphere of $\mathbb{R}^2$ and $\langle .,. \rangle$ the canonical inner product in $\mathbb{R}^2$. Then there exist two open domains $V \subset \mathbb{R}^2$ and $D' \subset D$ ($V \neq \emptyset$, $D' \neq \emptyset$) and two smooth functions $\xi^j : V \times D' \to S$ ($j = 1, 2$) such that for any $(X, w) \in V \times D'$, we have:*

1. *$\lambda_j(w, \xi^j(X,w)) = \langle \xi^j(X,w), X\rangle$ for $j = 1, 2$.*
2. *$\left\{\xi^1(X,w), \xi^2(X,w)\right\}$ is a basis of $\mathbb{R}^2$.*
3. *$\left\{r_1(w, \xi^1(X,w)), r_2(w, \xi^2(X,w))\right\}$ is a basis of $\mathbb{R}^2$.*

Proof. Let $(\alpha_0,\, \beta_0)$ be a basis of $\mathbb{R}^2$. For $w \in D$ (fixed) and $X = (\sigma_1, \sigma_2)$, the system:

$$\begin{cases} \langle \alpha_0, X\rangle &=& \lambda_1(w,\alpha_0), \\ \langle \beta_0, X\rangle &=& \lambda_2(w,\beta_0), \end{cases}$$

has a unique solution $X_0 = (\sigma_{1,0}, \sigma_{2,0})$. Set $A(w,\eta) = \ \eta_1 df_1(w) + \eta_2 df_2(w)$. Then the quadratic form $Q_{(X_0,w)} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by:

$$Q_{(X_0,w)}(\eta) = \det\bigg(A(w,\eta) - \langle \eta, X_0\rangle I_2\bigg),$$

has at least two isotropic vectors α_0 and β_0. As usual I_2 denotes the 2×2 identity matrix. Since $\lambda_j(w,.)$ are nonlinear functions, there exists $\mu = (\mu_1,\mu_2) \in S$ such that $\lambda_j(w,\mu) - \langle \mu, X_0\rangle \neq 0$, $j = 1,2$. Therefore we get $Q_{(X_0,w)}(\mu_1,\mu_2) \neq 0$, which means that $Q_{(X_0,w)} \neq 0$ and $Q_{(X_0,w)}$ is non-degenerate. Hence for (X', w') lying in a neighborhood $V \times D'$ of (X_0, w), the quadratic form $Q_{(X',w')}$ is non-degenerate and the associated characteristic fields are independent with isotropic directions proving Assertion (1).

Furthermore for any $(X', w') \in V \times D'$, the system:

$$\begin{cases} \lambda_1(w',\alpha) &=& \langle \alpha, X'\rangle, \\ \lambda_2(w',\beta) &=& \langle \beta, X'\rangle, \end{cases}$$

has a unique solution $(\alpha,\beta) \in \mathbb{R}^2\diagdown\{0\} \times \mathbb{R}^2\diagdown\{0\}$ depending continuously on (X', w'). Since the vectors α, β are linearly independent, we get the assertion (2).

We denote the above solution by $(\xi^1(X', w'), \xi^2(X', w'))$ with $\xi^j = (\xi^j_1, \xi^j_2)$, $j \in \{1, 2\}$. We can assume (without loss of generality) that $(\xi_1(X', w'), \xi_2(X', w')) \in S^2$.

To prove (3), assume that for some $(X, w) \in V \times D'$ with $X = (\sigma_1, \sigma_2)$, there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $r_2(w, \xi^2(X, w)) = \lambda\, r_1(w, \xi^1(X, w))$. Then for $j \in \{1, 2\}$, we have:

$$\sum_{i=1}^{2} \xi^j_i(X, w)\Big(df^i(w) - \sigma_i I_2\Big) r_1(w, \xi^1(X, w)) = 0.$$

Using (2) above, we conclude that:

$$\big(df^i(w) - \sigma_i I_2\big) r_1(w, \xi^1(X, w)) = 0, \; i \in \{1, 2\},$$

so that, for any $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$,

$$\sum_{i=1}^{2} \eta_i \big(df^i(w) - \sigma_i I_2\big) r_1(w, \xi^1(X, w)) = 0,$$

which contradicts our coupling assumption. □

Remark 2.2. From now on, we will write (σ, τ) (respectively, $u(\sigma, \tau)$) instead of (σ_1, σ_2) (respectively, $u(\sigma_1, \sigma_2, w)$). We may assume (by reducing D) that $D' = D$ and set $R_j(\sigma, \tau, w) = r_j(w, \xi^j(\sigma, \tau, w))$ for $\xi^j = \big(\cos\theta_j(\sigma, \tau, w), \sin\theta_j(\sigma, \tau, w)\big)$ with $(\sigma, \tau, w) \in \mathbb{R}^2 \times D$ and $\theta_j : V \times D \to \mathbb{R}$ $(j \in \{1, 2\})$ a smooth function.

3. Local solution for the two-dimensional 2×2 Riemann problem

In this section, we construct (non trivial) self–similar solutions $u \in \mathcal{C}^2(V, \mathbb{R}^2)$ of the two-dimensional Riemann problem. Consider the following evolution equation for functions $\beta_1, \beta_2 : V \times D \longrightarrow \mathbb{R}$ of class $\mathcal{C}^2$:

$$\begin{aligned}
&\sum_{j=1}^{2}\{(-\sin\theta_j \partial_\sigma \beta_j + \cos\theta_j \partial_\tau \beta_j) R_j + \beta_j(\partial_\tau(\cos\theta_j R_j) - \partial_\sigma(\sin\theta_j R_j))\}(\sigma, \tau, w) \\
&+ \sum_{i \neq j} \big(\beta_i \sin(\theta_i - \theta_j)((\partial_w \beta_j) R_i) R_j\big)(\sigma, \tau, w) \\
&- \sum_{i,j} \big(\beta_i \beta_j \cos(\theta_j - \theta_i)((\partial_w \theta_j) R_i) R_j\big)(\sigma, \tau, w) \\
&+ \beta_1 \beta_2 \sin(\theta_1 - \theta_2)((\partial_w R_2) R_1 - (\partial_w R_1) R_2)(\sigma, \tau, w) = 0.
\end{aligned}$$

Observe that this equation, called “the integrability condition”, does not depend on the function u.

Lemma 3.1 (Characteristic form of System (2.1)). *Assume that the system (1.1) is strictly hyperbolic and endowed with two fields ξ^1, ξ^2 as described in Proposition 2.1. Let $u \in \mathcal{C}^2(V, \mathbb{R}^2)$ and $\beta_1, \beta_2 \in \mathcal{C}^2(V \times D, \mathbb{R})$ be functions satisfying:*

$$\partial_\sigma u(\sigma,\tau) = \sum_{j=1}^{2} (\beta_j \cos\theta_j R_j)(\sigma,\tau,u(\sigma,\tau)), \tag{3.1}$$

$$\partial_\tau u(\sigma,\tau) = \sum_{j=1}^{2} (\beta_j \sin\theta_j R_j)(\sigma,\tau,u(\sigma,\tau)). \tag{3.2}$$

Then:

1. *The function u is a solution of the Riemann problem.*
2. *The integrability condition is satisfied on the graph $T = \{(\sigma,\tau,w) \in S \times \mathbb{R}^2 : w = u(\sigma,\tau)\}$.*

Note that the equations (3.1), (3.2) generalize the one–dimensional equations of rarefaction waves [12].

Proof. The property (1) is obvious. The property (2) is simply $\partial^2_{\sigma\tau} u(\sigma,\tau) = \partial^2_{\tau\sigma} u(\sigma,\tau)$ where $\partial_\sigma u$ and $\partial_\tau u$ are given respectively by the formulas (3.1) and (3.2). □

Now we can construct a solution $u \in \mathcal{C}^2(V, \mathbb{R}^2)$ of the Riemann problem by solving the system of equations (3.1) and (3.2). Consider the integrability condition as a system with variable $\beta = (\beta_1, \beta_2)$. From the assertion (3.1) of Proposition 2.1, this system is well-posed. It is locally hyperbolic in the direction $a\partial_\sigma + b\partial_\tau$ $\big((a,b) \in \mathbb{R}^2\big)$ [12].

Assume that $\sin\theta_1 \sin\theta_2 \neq 0$ on a neighborhood W of some point $(\sigma_0, \tau_0, w_0) \in V \times D$. Then for any smooth initial data $\beta(\sigma_0, \tau, w) = \beta_0(\tau, w)$, (τ, w) lying in a neighborhood of (τ_0, w_0), this system admits a smooth solution $\beta = (\beta_1, \beta_2)$ on W. Let us fix a solution $\beta = (\beta_1, \beta_2)$ on W and check that the system (3.1), (3.2) admits a local solution with the initial data $u(\sigma_0, \tau) = u_0(\tau)$, where τ is lying on a neighborhood of τ_0 and u_0 is a given smooth function satisfying the compatibility condition $u_0'(\tau) = \sum_{k=1}^{2} (\beta_k \sin\theta_k R_k)(\sigma_0, \tau, u_0(\tau))$. Note that such function u_0 exists by the Cauchy-Lipschitz theorem. Thus we can integrate the equation (3.1) for $(\sigma_0, \tau_0, u_0(\tau_0)) \in W$. Therefore the equation (3.2) is straightforward. In fact, set $\dfrac{d}{d\sigma} = \partial_\sigma + \partial_\sigma u.\partial_w$, $M = (\sigma, \tau, u(\sigma,\tau))$ and

$$U_1(M) \quad = \quad \sum_{j=1}^{2} (\beta_j \cos\theta_j R_j)(M)(= \partial_\sigma u(\sigma,\tau)),$$

$$\begin{aligned} U_2(M) &= \sum_{j=1}^{2}(\beta_j \sin\theta_j R_j)(M), \\ X &= \partial_\tau u - U_2. \end{aligned}$$

Let us prove that $X = 0$. The integrability condition and Equation (3.1) give:

$$\frac{dU_2}{d\sigma}(M) = \partial_\tau U_1(M) + (\partial_w U_1(M))U_2(M).$$

However we have:

$$\partial_\tau(\partial_\sigma u) = \partial_\tau U_1(M) + (\partial_w U_1(M))\partial_\tau u(\sigma,\tau).$$

Thus $\dfrac{dX}{d\sigma}(M) = \partial_w U_1(M).X(M)$. Since the compatibility condition on the initial data u_0 is $X(\sigma_0,.) = 0$, we obtain $X = 0$ for any (σ,τ) lying on a neighborhood of (σ_0,τ_0). Finally we get the following proposition.

Proposition 3.2. *Let the system (1.1) be strictly hyperbolic endowed with two fields ξ^1, ξ^2 as described in Proposition 2.1. Assume that $\sin\theta_1 \sin\theta_2 \neq 0$. Let (β_1,β_2) be a smooth solution of the integrability condition on a neighborhood W of a point (σ_0,τ_0,w_0) and $u_0 \in \mathcal{C}^2(\mathbb{R}, D)$ a function which satisfies $u_0'(\tau) = \sum_{j=1}^{2}(\beta_j \sin\theta_j R_j)(\sigma_0,\tau,u_0(\tau))$ on W. Then there exists a smooth solution of the Riemann problem with the initial data $u(\sigma_0,.) = u_0$ on a neighborhood of (σ_0,τ_0).*

Before going further, note that the compatibility condition is not restrictive. In fact, by the assumption $\sin\theta_1\sin\theta_2 \neq 0$ on a neighborhood of $(\sigma_0,\tau_0,u_0(\sigma_0,\tau_0))$, we get for any $u_0 \in \mathcal{C}^2(\Lambda, D)$ with $\Lambda \subset \mathbb{R}$, a smooth function $\beta_0 = (\beta_{1,0},\beta_{2,0})$ which depends on (τ,w) and satisfies the compatibility condition:

$$u_0'(\tau) = \sum_{j=1}^{2} \beta_{j,0}(\tau,u_0(\tau))(\sin\theta_j R_j)(\sigma_0,\tau,u_0(\tau))).$$

Remark 3.3. The problem is even more complicated for a space dimension $d \geq 3$. It requires writing of $nd(d-1)/2 > n$ integrability conditions (on n variables $\beta_1,\ldots,\beta_n$).

Consider for example the case $d = 3$ and $n \geq 2$. Set

$$C(X,w) = \{\xi \in \mathbb{R}^d : \det\left(\sum_{j=1}^{d}\big(\xi_j(df^j(w) - \sigma_j I_n)\big)\right) = 0\},$$

for $(X,w)=(\sigma_1,\dots,\sigma_d,w)\in\mathbb{R}^d\times\mathbb{R}^n$. Assume that $C(X,w)\neq\emptyset$ and write $C(X,w)=\bigcup_{j=1}^{n}C_j(X,w)$ with

$$C_j(X,w)=\{\xi\in C(X,w):\lambda_j(w,\xi)=\langle\xi,X\rangle\}.$$

For any $(X,w)\in\mathbb{R}^d\times\mathbb{R}^n$, we denote by $M_j(X,w)$ ($j\in\{1,\dots,n\}$) a curve included in $C_j(X,w)\cap S$ (S denotes the unit sphere) and depending on the parameter $s\in[0,1]$. The generic point of $M_j(X,w)$ is denoted by $\xi^j(X,w,s)$. As stated before, we set $R_j(X,w,s)=r_j(X,w,\xi^j(X,w,s))$. Then the system (3.1), (3.2) can be generalized as follows ($i\in\{1,\dots,d\}$):

$$\partial_{\sigma_i}u(X)=\sum_{j=1}^{n}\langle\mu_j(X,u(X)),\xi_i^j(X,u(X),.)R_j(X,u(X),.)\rangle.$$

Here $(X,w)\mapsto\mu_j(X,w)$ is a measure–valued smooth function. Note that if $\mu_j(X,w)$ is absolutely continuous for the Lebesgue measure on $[0,1]$, it is possible to write $nd(d-1)/2$ integrability conditions on the differentials of density functions $\rho_j(X,w,.)=\dfrac{d\mu_j(X,w)}{ds}$. Nevertheless we cannot say whether or not this new system is well-posed.

4. Extension of smooth solutions

4.1. Foliated solutions.

First we have:

Proposition 4.1. *Let $u=(u_1,u_2)\in\mathcal{C}^2(V,\mathbb{R}^2)$ be a solution of system (2.1). Assume that*

1. $\nabla u_1\neq 0$;
2. *The function u is constant along the leaves of a smooth foliation F with single codimension.*

Then these leaves are included in straight lines.

Proof. Let $f^1=(f_1,f_2)$ and $f^2=(g_1,g_2)$. By hypothesis, there exists a function $\Phi:V\to\mathbb{R}$ such that $\nabla u_2=\Phi\nabla u_1$. Moreover we have:

$$\partial_\sigma\Phi\partial_\tau u_1-\partial_\tau\Phi\partial_\sigma u_1=\partial^2_{\sigma\tau}u_2-\partial^2_{\sigma\tau}u_2=0.$$

Then Φ is constant along the level lines of u_1. In particular Φ is constant along the leaves of F.

Now we fix a leaf $\mathcal{F}$ and denote by s the oriented arc length on $\mathcal{F}$. Let $X(s)$ be the point of $\mathcal{F}$ with arc length s. We write Frenet formulas on $\mathcal{F}$:

$$T(s)=X'(s)\text{ and }T'(s)=c(s)N(s).$$

Since the normal vector $N(s)$ is proportional with $\nabla u_1(s)$, we get from the system (2.1):

$$\langle A(s)-X(s),N(s)\rangle=0,$$

where $A(s)$ denotes the vector field $\Big(\partial_{w_1} f_1(u(X(s)) + \Phi(X(s))\partial_{w_2} f_1(u(X(s)),$ $\partial_{w_1} g_1(u(X(s)) + \Phi(X(s))\partial_{w_2} g_1(u(X(s))\Big)$. The function A being constant, we obtain from the previous equality the existence of a function $\alpha : J \subset \mathbb{R}^2 \to \mathbb{R}$ such that $(X - \alpha T) = 0$. Hence we have:

$$(1 - \alpha'(s))T(s) - (\alpha c)(s)N(s) = 0.$$

Thus $\alpha c = 0$, $\alpha'(s) = 1$ and so $c = 0$. □

4.2. **Construction of smooth solutions.** Our goal here is to extend (as far as possible) the previous smooth local solution u of System (2.1). We introduce a continuous extension of u, foliated (when it does not coincide with u) in the sense of Proposition 3.2. More precisely, let $X_0 \in \mathbb{R}^2$ and $\mathcal{U}$ be a sphere of center X_0 and radius $R > 0$ small enough. We denote by $\mathcal{E}$ a smooth curve, defined by $\varphi : I \to \mathbb{R}^2$, containing X_0 and dividing $\mathcal{U}$ into two subsets $\mathcal{U}_-$ and $\mathcal{U}_+$. Here $I = (a, b)$ is a segment of $\mathbb{R}$.

For $u \in C^1(\mathcal{U}_-, \mathbb{R}^2) \cap C^0(\overline{\mathcal{U}_-}, \mathbb{R}^2)$ ($\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$ in $\mathbb{R}^2$) solution of the system (2.1), we will define a function $\rho \in C^0(\mathcal{U}, \mathbb{R}^2)$ which coincides with u on $\mathcal{U}_-$. Assume that $\mathcal{U}_+$ is foliated into lines $(H_s)_{s\in I}$ of $\mathbb{R}^2$ and $\mathcal{E} \cap H_s = \{\varphi(s)\}$ for any $s \in I$. Then the function $(\sigma_1, \sigma_2) \mapsto s$ is constant along the leaves H_s. We set $v = u \circ \varphi$ with $\varphi = (\varphi_1, \varphi_2)$. To determine the function ρ on $\mathcal{U}_+$, it is sufficient to determine the function φ and the family $(H_s)_{s\in I}$. Let us first write the system (2.1) on $\mathcal{U}_+$ in the following way:

$$\sum_{i=1}^{2} \big((f^i \circ v)' - \varphi_i v'\big)\frac{\partial s}{\partial \sigma_i} = 0. \tag{4.1}$$

Consider then the system:

$$\nabla s(\varphi(s)) \wedge \xi^j(\varphi(s), v(s)) = 0, \tag{4.2}$$

$$v'(s) \wedge r_j(v(s), \xi^j(\varphi(s), v(s))) = 0, \tag{4.3}$$

for $j \in \{1, 2\}$ and ξ^1, ξ^2 co-directions as described in Proposition 2.1. It is easy to check that a solution of this system is also a solution of (4.1).
Let us solve the equations (4.2), (4.3). Consider the system

$$\varphi'(s) = \xi^{q\perp}(\varphi(s), u \circ \varphi(s)), \tag{4.4}$$

$$\varphi(s_0) = X_0, \tag{4.5}$$

on variable $\varphi : I \to \mathbb{R}^2$ with $s_0 \in I$, $q \in \{1, 2\} \setminus \{k\}$ and $\xi^{q\perp} = (-\sin\theta_q, \cos\theta_q)$. The curves H_s are defined such that for any $s \in I$, H_s is contained in a straight line orthogonal to $\xi^k(\varphi(s), v(s))$. So we write H_s^k instead of H_s.

Since $\nabla s(\varphi(s))$ is proportional to $\xi^k(\varphi(s), v(s))$, we use the system (3.1), (3.2) to write:

$$\begin{aligned} v'(s) &= \frac{d}{ds}(\varphi(s))\varphi'(s) \\ &= \sum_{j=1}^{2} \big((-\cos\theta_j \sin\theta_q + \sin\theta_j \cos\theta_q)\, \beta_j r_j\big)(\varphi(s), v(s)) \\ &= \big(\sin(\theta_k - \theta_q)\beta_k r_k\big)(\varphi(s), v(s)). \end{aligned}$$

Thus φ is a solution of the system (4.2), (4.3) denoted by φ_k in what follows.

Now let us construct the extended solution $\rho \in \mathcal{C}^0(\mathcal{W}, \mathbb{R}^2)$ of System (2.1), where $\mathcal{W}$ is a subset of $\mathcal{U}$, to be determined below. We draw the curve families $L_k(X_0)$ defined by $s \mapsto \varphi_k(s, X_0)$ where $X_0 \in \mathcal{U}$ and $k \in \{1, 2\}$. Note that if $\varphi_1(s_1, X_1) = \varphi_2(s_2, X_2)$ for $(s_1, s_2) \in \mathbb{R}^2$ and $(X_1, X_2) \in \mathbb{R}^2$, then $(\partial_{s_1}\varphi_1(s_1, X_1), \partial_{s_2}\varphi_2(s_2, X_2))$ is a basis of $\mathbb{R}^2$ (Proposition 2.1).

Let $\mathcal{P}$ be a parallelogram in $\mathcal{U}$ with sides contained in $L_1(X_1)$, $L_2(X_2)$, $L_1(X_3)$, $L_2(X_1)$, for some $(X_1, X_2, X_3) \in V^3$ with $X_2 \in L_1(X_1)$ and $X_3 \in L_2(X_2)$. We set $\rho = u$ inside $\mathcal{P}$. Then we draw (as far as possible) the affine foliation $(H_s^k)_s$ described above, from each side of $\mathcal{P}$ and extend the function ρ continuously such that it is constant along the leaves H_s^k. Note that each leaf H_s^k is tangent to $L_q(H_s^k \cap L_k(X_i))$ for $q \neq k$ and $i \in \{1, 2, 3\}$. The set $\mathcal{W}$ is chosen in $\mathcal{U}$ such that it contains $\mathcal{P}$.

Finally let us check that ρ is a solution of the system (2.1) on the support of the leaves H_s^k for $k \in 1, 2$ and $i \in \{1, 2, 3\}$. We set $v_k = \rho \circ \varphi_k$ and we use equation (3.2) to write:

$$\begin{aligned} v_k'(s) &= \rho'(\varphi_k(s))\varphi_k'(s) \\ &= \sum_{j=1}^{2} \big[-(\cos\theta_j \sin\theta_q)(\varphi_k(s), v_k(s))(\beta_j r_j)(v_k(s), \xi^j(\varphi_k(s), v_k(s))) \\ &\quad +(\sin\theta_j \cos\theta_q)(\varphi_k(s), v_k(s))(\beta_j r_j)(v_k(s), \xi^j(\varphi_k(s), v_k(s)))\big] \\ &= \sin(\theta_k - \theta_q)(\varphi_k(s), v_k(s))(\beta_k r_k)(v_k(s), \xi^k(\varphi_k(s), v_k(s))). \end{aligned}$$

Hence we get

$$v_k'(s) \wedge r_k(v_k(s), \xi^k(\varphi_k(s), v_k(s))) = 0.$$

Otherwise, the function $(\sigma, \tau) \mapsto s$ is defined by "(σ, τ) and $\varphi_k(s)$ belong to the same leaf", so that:

$$\nabla s(\varphi_k(s)) \wedge \xi^k(\varphi_k(s), v_k(s)) = 0.$$

Consequently

$$\sum_{i=1}^{2} \big[\partial_{\sigma_i} f^i(\rho) - \sigma_i \partial_{\sigma_i}\rho\big] = \sum_{i=1}^{2} \big[(f^i \circ v_k)' - \varphi_{k\,i} v_k'(s)\big] \frac{\partial s}{\partial \sigma_i} = 0,$$

and the solution $\rho \in \mathcal{C}^0(\mathcal{W}, D)$ is constructed.

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