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A NUMERICAL METHOD FOR DISCRETE FRACTIONAL–ORDER CHEMOSTAT MODEL DERIVED FROM NONSTANDARD NUMERICAL SCHEME

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Abstract. In this paper, the fractional–order form of three dimensional chemostat model with variable yields is introduced. The stability analysis of this fractional system is discussed in detail. In order to study the dynamic behaviours of the mentioned fractional system, the well known nonstandard (NSFD) scheme is implemented. The proposed NSFD scheme is compared with the forward Euler and fourth order Runge–Kutta methods. Numerical results show that the NSFD approach is easy and accurate when applied to fractional–order chemostat model.
Keywords: Chemostat model, fractional–order differential equation, stability, nonstandard finite difference scheme.
MSC(2010): Primary: 37M05; Secondary: 65L12, 65L20.

1. Introduction

Competition modelling is one of the important topics in the mathematical biology. The simplest form of competition, however, occurs when individuals of different species compete for the same limited source foods, in some way species inhibit each other growth. This is called exploitative competition. A simple example of this type of competition occurs in a laboratory device, called a chemostat or a continuous culture, that models competition in a very simple lake [31].

The intuitively obvious fact that many biological systems are systems with memory is now confirmed by rigorous research. Moreover, in most biological processes, in particular of the chemostat, the observed response of a cell population at a certain time instant is the combined result of various biological processes that had been initiated at different moments in the past as a response to the instantaneous environmental conditions prevailing at each particular

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time. Hence, the kinetics of the fermentation processes should be considered as depending not only on the current process state, but also on a weighted average of the states at past moments, i.e. of the culture memory [25, 26, 32]. As we know, most of the mathematical models of these biological processes are based on ordinary differential equations (ODEs) of integer–order that disregard memory and after effects. It is worth noting that describing the behaviour of these systems by fractional–order differential equations is more useful than their classical integer–order counterpart, due to good memory and hereditary properties of fractional derivatives. In other words, calculating time–fractional derivative at time t requires all previous history before time t. Also using fractional–order differential equations can help us to reduce the errors arising from the neglected parameters in modelling real life phenomena [3]. However, fractional–order differential equations are closely related to fractals which are abundant in biological systems [9]. It should be emphasized that in the literature the fractional calculus has been used as an efficient tool to simulate the true nature of so many systems in diverse and widespread fields of science and engineering. For example, fractional calculus has been successfully applied to system biology [2, 3, 9, 12, 33], physics [10, 14], chemistry and biochemistry [34], hydrology [17, 30], medicine [5, 13], and finance [6]. Hence study and use the fractional–order differential equations help us achieve a better understanding of the chemostat model behaviour in general. In the other hand, analytical solutions of these types of fractional equations cannot generally be obtained, hence good numerical schemes are playing important role in identifying the solution behaviour of such fractional equations and exploring their applications. Nevertheless, Among numericals methods, NSFD schemes can alternatively be used to obtain more qualitative results and remove numerical instabilities.

This paper is organized as follows: In next Section, we elaborate the definition and some basic properties of Günwald–Letnikov (GL) approximation as well we discuss that how NSFD scheme can be implemented for systems of ODEs. In Section 3, fractional–order form of the chemostat model is introduced and also stability theorem and fractional Routh–Hurwitz stability conditions are given for the local asymptotic stability of the fractional–order systems. Section 4 is devoted to the study of the stability analysis of the fractional–order chemostat model. In Section 5, the idea of NSFD scheme for solving the fractional–order chemostat model is presented. Finally, the theoretical results obtained in former section are compared with the other numerical methods and the simulated numerical results are given.

2. **Preliminaries**

Although the discussion of the fractional calculus is as old as integer–order calculus, the complexity and the lack of applications postponed its progress till a few decades ago. Recently, most of the dynamical systems, based on the

integer–order calculus, have been modified into the fractional–order domain due to the extra degrees of freedom and the flexibility which can be used to precisely fit the experimental data much better than the integer–order modelling.

2.1. **Grünwald–Letnikov approximation.** Derivatives of fractional–order have been introduced in several ways. In this paper we consider GL approach. The GL method for the one–dimensional fractional derivative takes the following form [28]:

$$\begin{aligned} D^{\alpha}x(t) &= f(t,x(t)), \qquad x(0)=x_0, \quad t\in[0,t_f], \\ D^{\alpha}x(t) &= \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{[\frac{t}{h}]}(-1)^j\binom{\alpha}{j}x(t-jh), \end{aligned} \tag{2.1}$$

where D^{α}, h and $[t/h]$ denote the fractional derivative, the step size and the integer part of t/h, respectively. Therefore, the Equation (2.1) is discretized as follows:

$$\sum_{j=0}^{n} c_j^{\alpha} x_{n-j} = f(t_n, x_n), \qquad n=1,2,3,...$$

where $t_n = nh$, x_{n-j} is approximation of $x(t_{n-j})$ and c_j^{α}, $j=1,2\ldots,n$ are GL coefficients that defined as:

$$c_j^{\alpha} = (1-\frac{1+\alpha}{j})c_{j-1}^{\alpha}, \qquad c_0^{\alpha}=h^{-\alpha}, \qquad j=1,2,3,...,n.$$

2.2. **Nonstandard finite difference schemes.** NSFD schemes were firstly proposed by Mickens [19, 24] for ODEs and successively, their use has been investigated in several fields. To describe NSFD scheme, we consider an ODE such as

$$\frac{dx}{dt} = f(t,x,\lambda), \qquad x(0)=x_0, \qquad t\in[0,t_f], \tag{2.2}$$

where λ is a parameter. Given a discretization $t_n = nh$, NSFD is constructed by following two main steps:

(i) The derivative at the left–hand side of the Equation (2.2) is replaced by a discrete form

$$\frac{dx}{dt} \approx \frac{x_{n+1}-x_n}{\phi(h,\lambda)}, \tag{2.3}$$

where x_n is an approximation of $x(t_n)$.

(ii) The nonlinear term in the Equation (2.2) is replaced by a nonlocal discrete representation $F(t,x_{n+1},x_n,\ldots,\lambda)$ depending on some of the

previous approximation. Hence the gained scheme is described as follows:

$$\frac{x_{n+1}-x_n}{\phi(h,\lambda)} = F(t, x_{n+1}, x_n, \ldots, \lambda). \tag{2.4}$$

The discrete derivative on the left–hand side of the Equation (2.4) is a generalization of the classical discrete representation for the first derivative that is obtained by using $\phi(h,\lambda) = h$.

The denominator function $\phi(h,\lambda)$, that is a function of step size, must have consistency condition

$$\phi(h,\lambda) = h + \mathrm{O}(h^2), \tag{2.5}$$

to ensure that the discrete representation of (2.3) converges to the corresponding continuous derivative as $h \to 0$. Examples of denominator functions that satisfy the condition (2.5) are h, $\sin(h)$, $1-e^{-h}$, $(1-e^{-\lambda h})/\lambda$ and so forth. The papers of Mickens [20,21,23] give a general procedure for determining $\phi(h)$ for systems of ODEs. In general, for an ODE with polynomial terms,

$$\frac{dx}{dt} = ax + (NL), \qquad \mathrm{NL} \equiv \mathrm{Nonlinearterms},$$

the NSFD discretization for the linear expression is given by Mickens [23]

$$\frac{x_{n+1}-x_n}{\phi} = ax_n + (NL)_n,$$

where the denominator function is

$$\phi(h,a) = \frac{e^{ah}-1}{a}.$$

Note that if $a = 0$ then the denominator function is just h, i.e., $\phi(h) = h$.

The first NSFD requirement is that the dependent functions should be modelled on the discrete–time computational grid. Particular examples of this include the following functions [20,22]

$$xy \approx x_{n+1}y_n, \qquad xy \approx x_n y_{n+1},$$

$$x^2 \approx 2x_{n+1}x_n - x_n^2, \qquad x^2 \approx 2x_n\frac{x_{n+1}+x_n}{2}.$$

By applying this technique and using the GL discretization method, it yields the following relation

$$x_{n+1} = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha} x_{n+1-j} + f(t_{n+1}, x_{n+1})}{c_0^{\alpha}}, \qquad n = 0, 1, 2, \ldots,$$

where $c_0^{\alpha} = \phi(h)^{-\alpha}$.

3. **Fractional–order Chemostat model**

At time t, let $s(t)$ denotes the concentration of nutrient in the vessel, $x(t)$ and $y(t)$ denote the concentration of two microorganism. The mathematical model takes the form

$$\begin{aligned} &\frac{ds}{dt} = (s_0 - s)Q - \frac{1}{\delta_1}(\frac{m_1 s}{k_1 + s} - L)x - \frac{1}{\delta_2}\frac{m_2 s}{k_2 + s}y, \\ &\frac{dx}{dt} = x(\frac{m_1 s}{k_1 + s} - L - Q), \\ &\frac{dy}{dt} = y(\frac{m_2 s}{k_2 + s} - Q), \\ &s(0) = s_0, \quad x(0) = x_0, \quad y(0) = y_0, \end{aligned} \tag{3.1}$$

where s_0 is the input concentration of nutrient, Q is the washout rate, m_i is the maximal growth rates, k_i is the Michaelis–Menton constants, δ_i is the yield coefficients and L is the intrinsic consumption rate for the first microorganism, which are all positive. This model is usually called the Monod model or the model with Michaelis Menten dynamics [4, 15, 27].

Here we investigate the system of equations (3.1) with yield coefficients $\delta_1 = A + Bs^3$, $\delta_2 = C + Ds^4$, which means that the production of the microbial biomasses is very sensitive to the concentration of the nutrient. In the system equations (3.1) we have used the growth rate functions

$$F_i(s) = \frac{m_i s}{k_i + s}, \qquad i = 1, 2,$$

which have following common features:

(i) $F_i(0) = 0$.
(ii) F_i is an increasing function of s.
(iii) F_i approaches to a constant value as s approaches to infinity.

Now we introduce fractional–order form of the system of equations (3.1). The new system is described by the following set of fractional differential equations of order $\alpha_1,\ \alpha_2,\ \alpha_3 > 0$, with initial population; i.e., $s(0) > 0,\ x(0) > 0,\ y(0) > 0$.

$$\begin{aligned} &D^{\alpha_1} s(t) = (s_0 - s)Q - \frac{1}{\delta_1}(\frac{m_1 s}{k_1 + s} - L)x - \frac{1}{\delta_2}\frac{m_2 s}{k_2 + s}y, \\ &D^{\alpha_2} x(t) = x(\frac{m_1 s}{k_1 + s} - L - Q), \\ &D^{\alpha_3} y(t) = y(\frac{m_2 s}{k_2 + s} - Q), \\ &s(0) = s_0, \quad x(0) = x_0, \quad y(0) = y_0, \\ &0 < \alpha_i \leq 1, \quad i = 1, 2, 3. \end{aligned} \tag{3.2}$$

Recently, many investigations were devoted to the stability analysis of fractional–order nonlinear systems [7, 8, 11, 16, 29]. In order to analyze the stability of the model, stability theorem on fractional–order systems and fractional Routh–Hurwitz stability conditions are introduced.

Theorem 3.1 ([18]). *Consider the following commensurate fractional–order system:*

$$D^{\alpha}x(t) = f(x(t)), \qquad x(0) = x_0, \qquad t \in [0, t_f], \tag{3.3}$$

where $0 < \alpha \leq 1$ *and* $x \in \mathbb{R}^n$. *Equilibrium point* E *of the system* (3.3), *calculated by solving* $f(x) = 0$, *is locally asymptotically stable if all eigenvalues of the Jacobian matrix* $J \equiv \dfrac{\partial f}{\partial x}$ *that evaluated at the equilibrium point* E, *satisfy:*

$$|\arg(\lambda)| > \alpha\frac{\pi}{2}. \tag{3.4}$$

Proposition 3.2 ([1]). *Suppose* $P(\lambda) = \lambda^2 + b\lambda + c$ *is characteristic polynomial of the Jacobian matrix* $\dfrac{\partial f}{\partial x}$, *evaluated at the equilibrium point* E. *For* $0 < \alpha \leq 1$, *the eigenvalues of Jacobian matrix* $J \equiv \dfrac{\partial f}{\partial x}$, *satisfy condition* (3.4) *in Theorem* 3.1 *if*

$$b > 0, \qquad c > 0,$$

or

$$b \leq 0, \qquad 4c > b^2, \qquad \left|\tan^{-1}(\sqrt{4c - b^2}/b)\right| > \alpha\frac{\pi}{2}.$$

4. Stability analysis of the equilibrium points

In this section we investigate the local asymptotic stability of the equilibrium points of the system of equations (3.2). The equilibrium points of this system are:

$$\begin{aligned} E_0 &= (s_0, 0, 0),\\ E_1 &= (\beta_1, (s_0 - \beta_1)(A + B\beta_1^3), 0),\\ E_2 &= (\beta_2, 0, (s_0 - \beta_2)(C + D\beta_2^4)), \end{aligned}$$

where

$$\beta_1 = \frac{k_1(L+Q)}{m_1 - (L+Q)}, \qquad \beta_2 = \frac{Qk_2}{m_2 - Q}.$$

Note that the equilibrium points E_i, $i = 0, 1, 2$, have real biological meaning if their components are non–negative. Since $s_0 > 0$, the equilibrium point E_0 exists by biological meaning. Also the equilibrium points E_1 and E_2 have real biological meaning when $0 < \beta_i < s_0$, $i = 1, 2$. Now let us verify the stability of these equilibrium points. The Jacobian matrix of the system (3.2) at the

equilibrium point $E = (s, x, y)$ is

$$J(s,x,y) = \begin{bmatrix} T(s,x,y) & -\frac{1}{A+Bs}(\frac{m_1 s}{k_1+s} - L) & -\frac{1}{C+Ds^3}\frac{m_2 s}{k_2+s} \\ \frac{m_1 k_1 x}{(k_1+s)^2} & \frac{m_1 s}{k_1+s} - L - Q & 0 \\ \frac{m_2 k_2 y}{(k_2+s)^2} & 0 & \frac{m_2 s}{k_2+s} - Q \end{bmatrix}, \tag{4.1}$$

where

$$T(s,x,y) = -Q - \frac{x}{A+Bs^3}\frac{m_1 k_1}{(k_1+s)^2} + \frac{3Bxs^2}{(A+Bs^3)^2}(\frac{m_1 s}{k_1+s} - L)$$

$$-\frac{y}{C+Ds^4}\frac{k_2 m_2}{(k_2+s)^2} + \frac{4Ds^3 y}{(C+Ds^4)^2}\frac{m_2 s}{k_2+s}.$$

The characteristic equation of the Jacobian matrix J at the equilibrium point E_0 is

$$P(\lambda) = (\lambda + Q)(\lambda^2 + b_1\lambda + c_1) = 0,$$

where

$$b_1 = L + 2Q - (\frac{m_1 s_0}{k_1+s_0} + \frac{m_2 s_0}{k_2+s_0}), \qquad c_1 = (L + Q - \frac{m_1 s_0}{k_1+s_0})(Q - \frac{m_2 s_0}{k_2+s_0}).$$

Eigenvalues of the matrix J at the equilibrium point E_0 are

$$\lambda_1 = -Q, \quad \lambda_2 = -(L + Q - \frac{m_1 s_0}{k_1+s_0}), \quad \lambda_3 = -(Q - \frac{m_2 s_0}{k_2+s_0}).$$

These eigenvalues are real, hence by Theorem 3.1 the equilibrium point E_0 is stable if $\lambda_i < 0, \quad i = 1, 2, 3$. Since $Q > 0$, the inequality $\lambda_1 < 0$ holds. Also inequalities $\lambda_2 < 0$ and $\lambda_3 < 0$ are satisfied when $\beta_i > s_0, \quad i = 1, 2$. Therefore the equilibrium point E_0 is stable if

$$\beta_i > s_0, \qquad i = 1, 2.$$

The characteristic equation of the Jacobian matrix J, evaluated at the equilibrium point E_1, is

$$P(\lambda) = (\lambda - a_2)(\lambda^2 + b_2\lambda + c_2) = 0,$$

where

$$a_2 = \frac{m_2\beta_1}{k_2+\beta_1} - Q,$$
$$b_2 = Q - (s_0 - \beta_1)(\frac{3BQ\beta_1^2}{A+B\beta_1^3} - \frac{m_1 k_1}{(k_1+\beta_1)^2}),$$
$$c_2 = (s_0 - \beta_1)\frac{m_1 k_1}{(k_1+\beta_1)^2}Q.$$

To consider the stability of the equilibrium point E_1, let

$$R_1 = \frac{3Q\beta_1^2(s_0-\beta_1)(k_1+\beta_1)^2 - Q\beta_1^3(k_1+\beta_1)^2 - m_1k_1{\beta_1}^3(s_0-\beta_1)}{Q(k_1+\beta_1)^2 + m_1k_1(s_0-\beta_1)}.$$

Note that a_2 is a real root of the characteristic equation of the Jacobian matrix J at the equilibrium point E_1, therefore conditions $a_2 < 0$, or equivalently $\beta_1 < \beta_2$ and $c_2 > 0$, are necessary for stability of the equilibrium point E_1 (Theorem 3.1 and Proposition 3.2). Since all parameters in chemostat model are positive and $s_0 > \beta_1$, the condition $c_2 > 0$ holds. We now consider the following two cases:

(i) If $b_2 > 0$ or equivalently $\dfrac{A}{B} \geq R_1$ then by Theorem 3.1, E_1 is a stable equilibrium point of the system of equations (3.2).

(ii) If $b_2 \leq 0$ or equivalently $\dfrac{A}{B} \leq R_1$ and

$$4c_2 > b_2^2, \qquad \left|\tan^{-1}\left(\frac{\sqrt{4c_2-b_2^2}}{b_2}\right)\right| > \alpha\frac{\pi}{2}, \tag{4.2}$$

then the equilibrium point E_1 is stable (Proposition 3.2).

In case (ii), the second condition in (4.2), combining with the first condition, is equivalent to

$$4\cos^2(\alpha\frac{\pi}{2})c_2 > b_2^2.$$

If we let

$$R_2 = \frac{3\beta_1^2Q(s_0-\beta_1)(k_1+\beta_1)^2}{(Q+2\cos(\alpha\frac{\pi}{2})\sqrt{c_2})(k_1+\beta_1)^2 + m_1k_1(s_0-\beta_1)} - \beta_1^3,$$

then case (ii) follows that $R_2 < \dfrac{A}{B} \leq R_1$.

The characteristic equation of the equilibrium point E_2 is

$$P(\lambda) = (\lambda - a_3)(\lambda^2 + b_3\lambda + c_3) = 0,$$

where

$$\begin{aligned} a_3 &= \frac{m_1\beta_2}{k_1+\beta_2} - L - Q, \\ b_3 &= Q - (s_0-\beta_2)\left(\frac{4QD{\beta_2}^3}{C+D\beta_2^4} - \frac{m_2k_2}{(k_2+\beta_2)^2}\right), \\ c_3 &= (s_0-\beta_2)\frac{m_2k_2}{(k_2+\beta_2)^2}Q. \end{aligned}$$

Let

$$R_3 = \frac{4Q\beta_2^3(s_0-\beta_2)(k_2+\beta_2)^2 - \beta_2^4Q(k_2+\beta_2)^2 - \beta_2^4(s_0-\beta_2)m_2k_2}{Q(k_2+\beta_2)^2 + (s_0-\beta_2)m_2k_2},$$

just as the previous case, conditions $a_3 < 0$ and $c_3 > 0$ are necessary for stability of the equilibrium point E_2. Positivity of the parameters and condition $s_0 > \beta_2$, show that $c_3 > 0$. But inequality $a_3 < 0$ holds provided that $\beta_2 < \beta_1$. Now, we have the following two cases:

(i) If $b_3 > 0$ (equivalently $\dfrac{C}{D} > R_3$) then by Theorem 3.1 the equilibrium point E_2 is stable.
(ii) If $-2\sqrt{c_3}\cos(\alpha\dfrac{\pi}{2})) < b_3 \leq 0$ then the equilibrium point E_2 is stable (Proposition 3.2).

By analogy with the case of the equilibrium point E_1, case (ii) is equal to $R_4 < \dfrac{C}{D} \leq R_3$, where

$$R_4 = \frac{4Q\beta_2^3(s_0-\beta_2)(k_2+\beta_2)^2}{(Q+2\cos(\alpha\frac{\pi}{2})\sqrt{c_3})(k_2+\beta_2)^2 + m_2k_2(s_0-\beta_2)} - \beta_2^4.$$

Theorem 4.1. *Let $0 < \alpha \leq 1$, then for the equilibrium points E_0 , E_1 and E_2 of the system of equations (3.2), the following statements hold.*

(i) *If $\beta_i > s_0$, $i = 1, 2$, then the equilibrium point E_0 is stable.*
(ii) *If $\beta_1 < \beta_2$ and $\dfrac{A}{B} > R_1$ or $\beta_1 < \beta_2$ and $R_2 < \dfrac{A}{B} \leq R_1$, then the equilibrium point E_1 is stable.*
(iii) *If $\beta_2 < \beta_1$ and $\dfrac{C}{D} > R_3$ or $\beta_2 < \beta_1$ and $R_4 < \dfrac{C}{D} \leq R_3$, then the equilibrium point E_2 is stable.*

5. NSFD scheme for fractional–order Chemostat model

Mickens suggests that a general multistep numerical scheme that approximate the solution of the Equation (2.2) can be written in the form Equation (2.4) where the denominator function $\phi(h, \lambda)$ is of the form $h + O(h^2)$, and $F(t, x_{n+1}, x_n, \ldots, \lambda)$ is a nonlocal representation of the function $f(t_n, x(t_n), \lambda)$. The terminology of nonlocal approximation comes from the fact that the approximation of a given function $f(t, x, \lambda)$ is not only at point x_n, by $f(t_n, x_n, \lambda)$, but can eventually depends on more points of the orbits. Examples of how the rules are used to develop the discretizations presented in Section 2 (see [20, 22, 23] for more details). In this section, we present numerical simulation to illustrate the results obtained in the previous section. By using definition of

GL derivative and use NSFD scheme for the model we have:

$$\sum_{j=0}^{n+1} c_j^{\alpha_1} s_{n+1-j} = (s_0 - s_n)Q - \frac{1}{A + Bs_n^3}\left(\frac{m_1 s_{n+1}}{k_1 + s_n} - L\right)x_n - \frac{1}{C + Ds_n^4}\frac{m_2 s_{n+1} y_n}{k_2 + s_n},$$

$$\sum_{j=0}^{n+1} c_j^{\alpha_2} x_{n+1-j} = x_n\left(\frac{m_1 s_{n+1}}{k_1 + s_{n+1}}\right) - (L + Q)x_n,$$

$$\sum_{j=0}^{n+1} c_j^{\alpha_3} y_{n+1-j} = y_n \frac{m_2 s_{n+1}}{k_2 + s_{n+1}} - Qy_n,$$

where $t_n = nh$ and $c_j^{\alpha_i},\ i = 1, 2, 3$ are the GL coefficients defined as:

$$c_j^{\alpha_i} = \left(1 - \frac{1 + \alpha_i}{j}\right)c_{j-1}^{\alpha_i}, \quad c_0^{\alpha_i} = (\phi_i(h)),^{-\alpha_i} \quad j = 1, 2, \ldots, n+1, \quad i = 1, 2, 3.$$

with

$$\phi_1(h) = \frac{1 - e^{-Qh}}{Q}, \quad \phi_2(h) = \frac{1 - e^{-(Q+L)h}}{Q + L}, \quad \phi_3(h) = \frac{1 - e^{-Qh}}{Q}.$$

Comparing the previous difference equation with system of equations (3.2) we note the following:

(i) The linear term on the right–hand side of the first equation in the system of equations (3.2), $-s$ replaced by $-s_n$. Also the nonlinear terms in the numerator of the first equation, s replaced by s_{n+1}.
(ii) The linear and nonlinear terms on the right–hand side of the second and third equation in the system (3.2) replaced by s_{n+1}.
(iii) The linear and nonlinear terms x and y of the second and third equations replaced by x_n and y_n, respectively.
(iv) In order to provide an explict scheme, the nonlinear terms in the denominators of the first equation s replaced by s_n.

Manipulating previous discretization we get the following equations:

$$
\begin{aligned}
s_{n+1} &= \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_1} s_{n+1-j} + (s_0 - s_n)Q + \dfrac{Lx_n}{A + Bs_n{}^3}}{c_0^{\alpha_1} + \dfrac{m_1 x_n}{(A + Bs_n{}^3)(k_1 + s_n)} + \dfrac{m_2 y_n}{(C + Ds_n^4)(k_2 + s_n)}}, \\
x_{n+1} &= \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_2} x_{n+1-j} + x_n(\dfrac{m_1 s_{n+1}}{k_1 + s_{n+1}} - L - Q)}{c_0^{\alpha_2}}, \\
y_{n+1} &= \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_3} y_{n+1-j} + y_n(\dfrac{m_2 s_{n+1}}{k_2 + s_{n+1}} - Q)}{c_0^{\alpha_3}}.
\end{aligned} \tag{5.1}
$$

Remark 5.1. Since $c_0^{\alpha_i} > 0$, and using the subsequent recursive formula

$$c_j^{\alpha_i} = (1 - \frac{1 + \alpha_i}{j}) c_{j-1}^{\alpha_i},\ j = 1, 2, \ldots, n+1,$$

it is understood that $c_j^{\alpha_i} < 0$ for $j = 1, 2, \ldots, n+1$. It is not difficult to see that if the inequalities

$$
\begin{aligned}
(-c_1^{\alpha_1} - Q) &= (\alpha_1 c_0^{\alpha_1} - Q) > 0, \\
(-c_1^{\alpha_2} - L - Q) &= (\alpha_2 c_0^{\alpha_2} - L - Q) > 0, \\
(-c_1^{\alpha_3} - Q) &= (\alpha_3 c_0^{\alpha_3} - Q) > 0,
\end{aligned} \tag{5.2}
$$

together with $s_n > 0$, $x_n > 0$ *and* $y_n > 0$ are satisfied, then $s_{n+1} > 0$, $x_{n+1} > 0$ *and* $y_{n+1} > 0$. On the other hand, if $h^\alpha < \dfrac{\alpha}{L+Q}$, where $\alpha = \min\limits_{1 \leqslant i \leqslant 3} \alpha_i$ and $0 < h \leqslant 1$, then $h^{\alpha_2} < \dfrac{\alpha_2}{L+Q}$ and $h^{\alpha_i} < \dfrac{\alpha_i}{Q}$, $i = 1, 3$. Moreover, since $\dfrac{1 - e^{-ah}}{a} \leqslant h$ for $a > 0$ and $0 < h \leqslant 1$, it follows that inequalities (5.2) hold. As a matter of fact, according to the above discussion as well as exploiting the principal of induction a sufficient condition to ensure positivity of the NSFD scheme is $h^\alpha \leqslant \min\{1, \dfrac{\alpha}{L+Q}\}$.

6. **Numerical results**

Analytical studies always remain incomplete without numerical verification of the results. In this section we present numerical simulation to illustrate the result obtained in previous sections. Now we consider the fractional–order chemostat model in several cases. For the parameter values $m_1 = 1.5$, $m_2 = 1.25$, $k_1 = 1.6$, $k_2 = 1.75$, $Q = 0.5$ and initial conditions $s_0 = 0.75$, $x_0 = 0.5$, $y_0 = 0.4$, numerical solutions of the system of equations (3.2) converge to the equilibrium point E_0 (Figure 1).

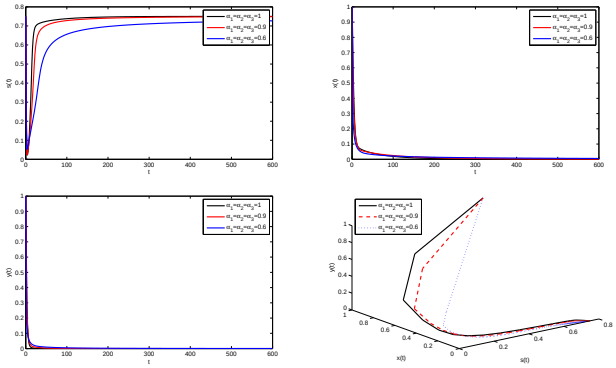

FIGURE 1. Numerical solutions of the system of equations (3.2) converge to the equilibrium point E_0 for different α_i with step size $h = 1.5$.

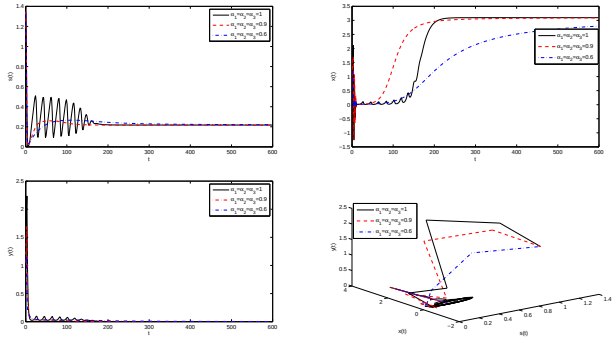

FIGURE 2. Numerical solutions of the system of equations (3.2) converge to the equilibrium point E_1 for different α_i with step size $h = 1.5$.

These values satisfy conditions $\beta_1 > s_0$ and $\beta_2 > s_0$ (for these set of data $\beta_1 = 0.8490$ and $\beta_2 = 1.1667$). For parameter values $A = 2$, $B = 2$, $k_1 = 0.4$, $k_2 = 1$, $m_1 = 2.4$, $L = 0.2$, $Q = 0.64$ and initial conditions $s_0 = 1.75$, $x_0 = 1$, $y_0 = 1$, numerical solutions of the system of equations (3.2) converge to the equilibrium point E_1 (Figure 2). For these values we have $\dfrac{A}{B} > R_1$ and $\beta_1 < \beta_2$ that ensure solutions converge to the equilibrium point E_1 (for this set of data $\beta_1 = 0.2554$, $\beta_2 = 0.2712$, $\dfrac{A}{B} = 1, R_1 = 0.0202$). Finally for parameter values $C = 0.25$, $D = 3.7$, $k_2 = 3.25$, $m_2 = 3$, $L = 0.02$, $Q = 0.625$ and initial

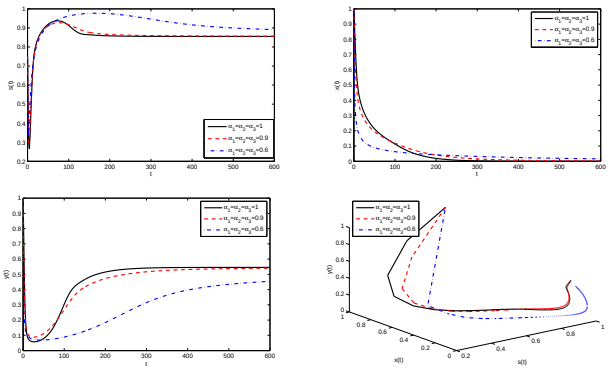

FIGURE 3. Numerical solutions of the system of equations (3.2) converge to the equilibrium point E_2 for different α_i with step size $h = 1.5$.

conditions $s_0 = 1.1$, $x_0 = 1$, $y_0 = 1$, we have $\dfrac{C}{D} > R_2$ and $\beta_1 > \beta_2$ (for this set of data, $\dfrac{C}{D} = 0.0676$, $R_2 = -0.0357$, $\beta_1 = 1.1026$ and $\beta_2 = 0.8553$). With these conditions numerical solutions converge to the equilibrium point E_2 . All numerical solutions illustrated in Figures 1$\sim$ 3, gained by step size $h = 1.5$, this step size shows high accuracy of NSFD scheme.

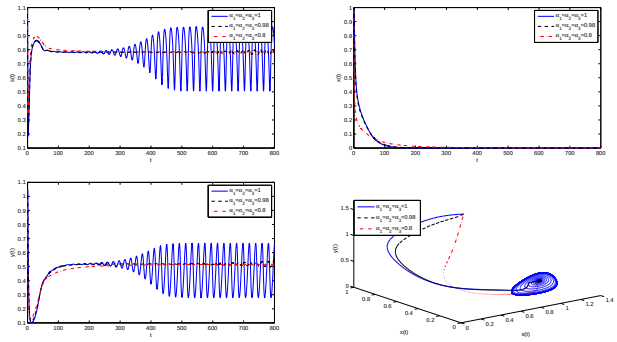

FIGURE 4. Numerical solutions of the system of equations (3.2) converge to the equilibrium point E_1 for some α_i with step size $h = 0.1$.

Remark 6.1. Note that the conditions b_2, $b_3 > 0$ are necessary for stability of the equilibrium points E_1 and E_2 when $\alpha_i = 1$, but in obtained fractional–order model b_1 *and* b_2 might be less or equal to zero and the equilibrium

points E_1 and E_2 are stable yet. We observe that for parameter values $A = 2$, $B = 4$, $C = 0.015$, $D = 0.98$, $k_1 = 1$, $k_2 = 3.5$, $m_1 = 1$, $m_2 = 1$, $L = 0.02$, $s_0 = 2.35$ and $Q = 0.6$, the sing of b_2 is negative ($b_2 = -0.0070$), but for some $0 < \alpha_i < 1$ the numerical solutions of the system of equations (3.2) converge to the equilibrium point E_1. Also by using the values, $A = 1$ $B = 33.02$, $C = 2$, $D = 3.7$, $k_1 = 3$, $k_2 = 3$, $m_1 = 2.4$, $m_2 = 3$, $L = 0.02$, $s_0 = 1.1$ and $Q = 0.62$, the value of b_3 is less than zero ($b_3 = -0.274$), but the equilibrium point E_2 is stable yet (Proposition 3.2 and Figures 4 and 5).

In Tables 1–3 the NSFD scheme for the system of equations (3.2) with forward Euler and fourth order Runge–Kutta methods are compared when $\alpha_i = 1$, $i = 1, 2, 3$. This comparison is done for different time step size h. As we observe, the numerical solutions obtained from the NSFD scheme convergent for high step size h. Therefore the numerical solutions of NSFD scheme have better manner than Euler and Runge–Kutta methods. Also in Figures 6 and 7, forward Euler and fourth order Runge–Kutta methods are compared with NSFD scheme graphically.

Remark 6.2. In the Figures 6 and 7 numerical results of the proposed NSFD scheme for $\alpha_1 = \alpha_2 = \alpha_3 = 1$, which corresponds to the integer–order chemostat model, illustrated. From the numerical results in these figures and Tables 1–3 it is concluded that the approximation solutions obtained by the NSFD scheme is good agreement with the asymptotically stability of the equilibrium points. Also these results show robustness of the NSFD scheme in the high step size.

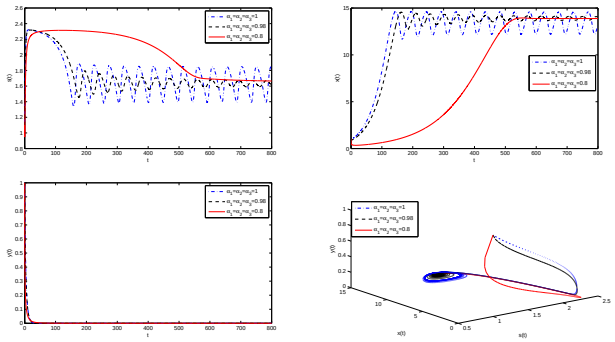

FIGURE 5. Numerical solutions of the system of equations (3.2) converge to the equilibrium point E_2 for some α_i with step size $h = 0.1$.

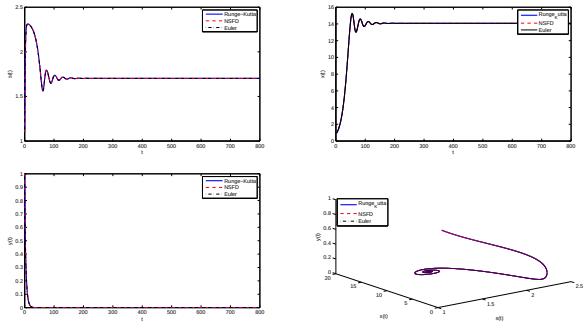

FIGURE 6. Numerical solutions of forward Euler and fourth order Runge–Kutta and NSFD methods converge to the equilibrium point E_1 with step sizes $h = 0.01$ for NSFD scheme and $h = 0.001$ for Euler and fourth–order Runge–Kutta .

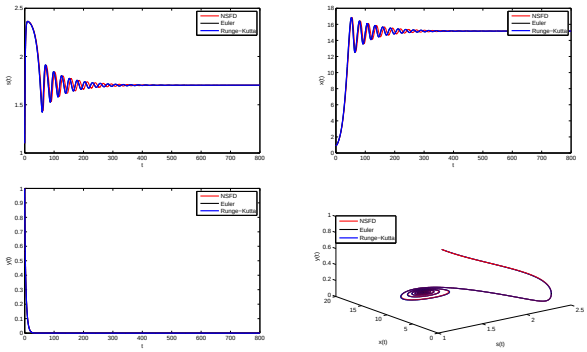

FIGURE 7. Numerical solutions of forward Euler and fourth order Runge–Kutta and NSFD methods converge to the equilibrium point E_1 with step sizes $h = 0.1$ for NSFD scheme and $h = 0.001$ for Euler and fourth–order Runge–Kutta .

7. **Conclusion**

In this paper the fractional form of the chemostat model with variable yields is introduced. The local stability analysis as well as the dynamic behaviour of mentioned system are studied. Moreover, a NSFD scheme has been investigated for the numerical solution of the model. Positivity of the presented NSFD scheme discussed. The obtained numerical results of NSFD scheme are compared with the forward Euler and fourth order Runge–Kutta methods in integer–order case. Numerical results show that NSFD schemes are useful tool for detecting the local stability of equilibrium points.

h	Euler	Runge–Kutta	NSFD
0.1	Convergence	Convergence	Convergence
0.2	Convergence	Convergence	Convergence
0.5	Divergence	Convergence	Convergence
2.5	Divergence	Convergence	Convergence
5	Divergence	Divergence	Convergence
10	Divergence	Divergence	Convergence

TABLE 1. Qualitative result of the equilibrium point E_0 for different step sizes h.

h	Euler	Runge–Kutta	NSFD
0.1	Convergence	Convergence	Convergence
0.2	Convergence	Convergence	Convergence
0.3	Convergence	Convergence	Convergence
0.5	Divergence	convergence	Convergence
5	Divergence	Divergence	Convergence
10	Divergence	Divergence	Convergence

TABLE 2. Qualitative result of the equilibrium point E_1 for different step sizes h.

h	Euler	Runge–Kutta	NSFD
0.1	Convergence	Convergence	Convergence
0.2	convergence	convergence	Convergence
0.5	Divergence	convergence	Convergence
3.5	Divergence	Divergence	Convergence
5	Divergence	Divergence	Convergence
8	Divergence	Divergence	Convergence

TABLE 3. Qualitative result of the equilibrium point E_2 for different step sizes h.

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