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ON CERTAIN MAXIMALITY PRINCIPLES

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Abstract. We present streamlined proofs of certain maximality principles studied by Hamkins and Woodin. Moreover, we formulate an intermediate maximality principle, which is shown here to be equiconsistent with the existence of a weakly compact cardinal κ such that $V_\kappa \prec V$.
Keywords: Hamkins' maximality principle, weakly compact cardinal, consistency result.
MSC(2010): 03E35.

1. **Introduction**

In his paper "A Simple Maximality Principle" [3], Joel Hamkins introduced a maximality principle that is exactly axiom $S5$ in the sense of modal logic (see Definition 2.1). He also considered various versions of his maximality principle and proved the consistency of some of them, but the status of some others remained open. The necessary maximality principle ($\Box\mathrm{MP}(\mathbb{R})$) is a strong principle whose consistency relative to some strong hypotheses was proved by Woodin in an unpublished work; as described in [2, Theorem 4.12]. Also, in a joint work [3], Hamkins and Woodin proved that $\Box_{c.c.c.}\mathrm{MP}_{c.c.c.}(\mathbb{R})$ is equiconsistent with the existence of a weakly compact cardinal. In this paper, we present new proofs of some results of Hamkins [2] which are simpler than the original ones. Moreover we introduce $\Box_{c.c.c.}\mathrm{MP}(\mathbb{R})$ and show that it is equiconsistent with the existence of a weakly compact cardinal κ such that $V_\kappa \prec V$.

2. **Hamkins' Maximality principle**

Let $\mathbb{P}$ be a forcing notion and ϕ a sentense in the language of set theory, we write $V^{\mathbb{P}} \models \phi$ if $V[G] \models \phi$ for every $\mathbb{P}$-generic filter G over V. For a sentence ϕ in the language of set theory and a model V of ZFC, we say ϕ is *forceable* ($\Diamond\phi$) over V, if there exists a forcing notion $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models \phi$; and ϕ is *necessary* ($\Box\phi$) over V, if for all forcing notions $\mathbb{P} \in V$, $V^{\mathbb{P}} \models \phi$. When we

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restrict ourselves to the class of *c.c.c.* forcing notions, then we say ϕ is *c.c.c.*-forceable ($\Diamond_{c.c.c.}\phi$) or *c.c.c.*-necessary ($\Box_{c.c.c.}\phi$). Note that $\Diamond\phi$, $\Box\phi$, $\Diamond_{c.c.c.}\phi$ and $\Box_{c.c.c.}\phi$ are all first order expressible. Now we define some principles that are schemes in the first order language.

Definition 2.1. (The Maximality Principle) Hamkins' maximality principle ($\mathrm{MP}(X)$) asserts

$$\Diamond\Box\phi \Longrightarrow \phi,$$

whenever ϕ is a sentence in the language of set theory with parameters from X.

Definition 2.2. (The Necessary Maximality Principle) Hamkins' necessary maximality principle ($\Box\mathrm{MP}(X)$) asserts $\mathrm{MP}(X)$ is necessary, i.e., for any forcing notion $\mathbb{P}$, $V^{\mathbb{P}} \models \mathrm{MP}(X)$; in particular $V \models \mathrm{MP}(X)$.

We use the notations $\mathrm{MP}_{c.c.c.}(X)$ and $\Box_{c.c.c.}\mathrm{MP}_{c.c.c.}(X)$, when we restrict ourselves to the class of *c.c.c.*-forcing notions. We may note that Hamkins has used the notation $\Box\mathrm{MP}_{c.c.c.}(X)$ to denote our $\Box_{c.c.c.}\mathrm{MP}_{c.c.c.}(X)$.

Let $V_\delta \prec V$ be the scheme that asserts $\forall x \in V_\delta(\phi(x) \longleftrightarrow \phi(x)^{V_\delta})$, whenever $\phi(x)$ is a formula with only one free variable. The following is an immediate consequence of the Montague-Levy reflection theorem.

Lemma 2.3 ([2]). *If ZFC is consistent, then is $ZFC + V_\delta \prec V$.*

Definition 2.4. Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing notions. A map $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$ is a projection if

(1) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$
(2) π is order preserving, and
(3) for every $p \in \mathbb{P}$, if $q \le \pi(p)$, then there is $p' \le p$, such that $\pi(p') \le q$.

The first item of the following lemma is a special case of [1, Theorem 14.1] and the two other items easily follow from the first one.

Lemma 2.5 ([1, Theorem 14.1]). (1) *If $\mathbb{P}$ is a forcing notion of size κ and $\Vdash_{\mathbb{P}} |\kappa| = \aleph_0$, then $\mathbb{P}$ is forcing equivalent to $Coll(\omega, \kappa)$.*
(2) *If κ is an inaccessible cardinal and $\mathbb{P} \in V_\kappa$, then $V^{\mathbb{P}} \subseteq V^{Coll(\omega,<\kappa)}$.*
(3) *If $\mathbb{P}$ is a forcing notion of size $\le \kappa$, then there exists a projection from $Coll(\omega, \kappa)$ to $\mathbb{P}$.*

Lemma 2.6. *Suppose that $\mathbb{P} \in V_\delta$. If G is a $\mathbb{P}$-generic filter over V and $V_\delta \prec V$, then G is $\mathbb{P}$-generic over V_δ and $V_\delta[G] \prec V[G]$.*

Proof. Let $A \subseteq \mathbb{P}$ be a maximal antichain in V_δ, so it is a maximal antichain in V, which implies $G \cap A \neq \varnothing$, so G is $\mathbb{P}$-generic over V_δ. Suppose that $V[G] \models \exists x\phi(x)$, then there exists $p \in G$ such that $V \models p \Vdash \exists x\phi(x)$, thus by elementarity, $V_\delta \models \exists x\ p \Vdash \phi(x)$. Then one can find $x \in V_\delta[G]$ such that $V_\delta[G] \models \phi(x)$. Hence $V_\delta[G] \prec V[G]$. $\Box$

Definition 2.7. An uncountable cardinal κ is weakly compact if for every $f : [\kappa]^2 \longrightarrow 2$ there is a set H of size κ such that f is constant on $[H]^2$.

The following is a well-known characterization of weakly compactness.

Lemma 2.8 ([1])**.** *the following are equivalent for an inaccessible cardinal κ.*

(1) *κ is weakly compact.*
(2) *For every transitive set $\mathcal{M}$ of size κ with $\kappa \in \mathcal{M}$ and $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$ there are a transitive set $\mathcal{N}$ of size κ with $\mathcal{N}^{<\kappa} \subseteq \mathcal{N}$ and an elementary embedding $j : \mathcal{M} \longrightarrow \mathcal{N}$ with critical point κ.*

The following lemma is proved by Hamkins and Woodin using the method of Boolean valued models; we give a proof which avoids the use of Boolean valued models.

Lemma 2.9 ([3])**.** *Let κ be a weakly compact cardinal and let $\mathbb{P}$ be a $\kappa - c.c.$ forcing notion. Then, for any $x \in H_\kappa^{\mathbb{P}}$, there exists $\mathbb{Q} \lhd \mathbb{P}$ such that $x \in H_\kappa^{\mathbb{Q}}$ and $\mathbb{Q}$ is a forcing notion of size less than κ.*

Proof. If $|\mathbb{P}| < \kappa$, then there is nothing to prove. So, assume $|\mathbb{P}| \geq \kappa$, and choose regular $\theta > \kappa$ large enough such that $\mathbb{P} \in H_\theta$. Without loss of generality suppose that $x \subseteq \kappa$ and $|x| < \kappa$. Let τ be a $\mathbb{P}$-name for x and find an elementary substructure $\mathcal{M}$ of H_θ such that:

(1) $\kappa, \tau, \mathbb{P} \in \mathcal{M}$,
(2) $|\mathcal{M}| = \kappa$,
(3) $\mathcal{M}^{<\kappa} \subseteq \mathcal{M}$.

The existence of $\mathcal{M}$ is guaranteed by the inaccessibility of κ. We divide the proof into two cases:

Case 1. $|\mathbb{P}| = \kappa$: Since κ is weakly compact, there exist a transitive model $\mathcal{N}$ and an elementary embedding $j : \mathcal{M} \longrightarrow \mathcal{N}$ with $crit(j) = \kappa$. We show that

$$N \models (\exists \mathbb{Q} \lhd j(\mathbb{P}))[|\mathbb{Q}| < |j(\mathbb{P})| \wedge (\exists \sigma 1_{j(\mathbb{P})} \Vdash \sigma = j(\tau)].$$

Without loss of generality suppose that $\mathbb{P} \subseteq V_\kappa$, which implies $\mathbb{P} \subseteq j(\mathbb{P})$. Let $A \subseteq \mathbb{P}$ be a maximal antichain in $\mathcal{N}$. So $|A| < \kappa$, hence $A \in \mathcal{M}$ and

$$\mathcal{M} \models \forall p \in \mathbb{P} \exists q \in A \ q \parallel p.$$

Using j, we have

$$N \models \forall p \in j(\mathbb{P}) \exists q \in j(A) = A \ q \parallel p.$$

On the other hand, since $x \in H_\kappa^{\mathbb{P}}$, and $\mathbb{P}$ is $\kappa - c.c.$, we can assume $|\tau| < \kappa$, so $j(\tau) = \tau$. Put $\sigma = \tau$ which implies $1_{j(\mathbb{P})} \Vdash \sigma = \tau$. Thus

$$N \models (\exists \mathbb{Q} \lhd j(\mathbb{P}))[|\mathbb{Q}| < |j(\mathbb{P})| \wedge (\exists \sigma 1_{j(\mathbb{P})} \Vdash \sigma = j(\tau)].$$

Now by elementarity of j, we have in $\mathcal{M}$:

$$\exists \mathbb{Q} \lhd \mathbb{P} \ |\mathbb{Q}| < |\mathbb{P}| \wedge \exists \sigma 1_{\mathbb{P}} \Vdash \sigma = \tau.$$

Case 2. $|\mathbb{P}| > \kappa$: Consider $\mathbb{P} \cap \mathcal{M}$ and let $A \subseteq \mathbb{P} \cap \mathcal{M}$ be a maximal antichain. Then $|A| < \kappa$, hence $A \in \mathcal{M}$ and A is a maximal antichain in $\mathbb{P}$. Hence $\mathbb{P} \cap \mathcal{M} \lhd \mathbb{P}$. Then by case 1, there exist $\mathbb{Q}$ of size less than κ and a $\mathbb{Q}$-name σ such that $\mathbb{Q} \lhd \mathbb{P} \cap \mathcal{M} \lhd \mathbb{P}$ and $\vdash_{\mathbb{Q}} \tau = \sigma$. $\square$

Theorem 2.10 (Hamkins [2])**.** *The consistency of ZFC implies the consistency of ZFC* + MP*.*

Proof. Suppose that $V_\delta \prec V$. Let $\mathbb{P} = Coll(\omega, \delta)$ and let $\mathbb{P}_\lambda = Coll(\omega, \lambda)$ for $\lambda < \delta$. Now assume that G is $\mathbb{P}$-generic over V and for $\lambda < \delta$ set $G_\lambda = G \cap \mathbb{P}_\lambda$.

We show that MP holds in $V[G]$. Thus assume $V[G] \models \Diamond\Box\phi$. Then for some $p \in G$

$$p \Vdash_{\mathbb{P}} \exists \mathbb{Q}(\exists q \in \mathbb{Q}\ q \Vdash \Box\phi).$$

Let $\lambda < \delta$ be such that $p \in G_\lambda$. Now $\mathbb{P}_\lambda \in V_\delta$ and hence by Lemma 2.6, $V_\delta[G_\lambda] \prec V[G_\lambda]$. It is evident that $V[G_\lambda] \models \Diamond\Box\phi$, and hence as $p \in V_\delta$ and $V_\delta \prec V$, one can find $\dot{\mathbb{Q}}$ and $\dot{q}$ in V_δ, such that $(p, \dot{q}) \Vdash \Box\phi$, so $V^{\mathbb{P}_\lambda * \dot{\mathbb{Q}}} \models \Box\phi$. On the other hand by Lemma 2.5 and the fact that $|\mathbb{P}_\lambda * \dot{\mathbb{Q}}| < \delta$, we have $V^{\mathbb{P}_\lambda * \dot{\mathbb{Q}}} \subseteq V^{\mathbb{P}}$ which implies $V^{\mathbb{P}} \models \phi$, which completes the proof. $\square$

Theorem 2.11 (Hamkins [2])**.** *ZFC*+MP($\mathbb{R}$) *is equiconsistent with ZFC plus the existence of an inaccessible cardinal κ such that $V_\kappa \prec V$.*

Proof. (Right to Left) Assume κ is an inaccessible cardinal such that $V_\kappa \prec V$. Consider $\mathbb{P} = Coll(\omega, <\kappa)$. Let G be a $\mathbb{P}$-generic forcing over V. Suppose that $\bar{x} \in 2^\omega \cap V[G]$, so $\bar{x} \in V[G_\lambda]$, for some regular cardinal $\lambda < \kappa$, where $G_\lambda = G \cap Coll(\omega, <\lambda)$ is generic for $\mathbb{P}_\lambda = Coll(\omega, <\lambda)$. Since $V_\kappa \prec V$, by Lemma 2.6 we have $V_\kappa[G_\lambda] \prec V[G_\lambda]$. It follows that there exists $\mathbb{Q} \in V_\kappa[G_\lambda]$ such that $V[G_\lambda]^{\mathbb{Q}} \models \Box\phi(\bar{x})$, hence $V^{\mathbb{P}_\lambda * \dot{\mathbb{Q}}} \models \phi(\bar{x})$. Then $V[G] \models \phi(\bar{x})$, by Lemma 2.5(3).
The other side has appeared as [2, Lemma 3.2]. We give a proof for completness. (Left to Right) Assume MP($\mathbb{R}$). First we claim that ω_1 is inaccessible to the reals. Thus let r be a real and let $\phi(r)$ be the statement "the ω_1 of $L[r]$ is countable", that is forceably necessary. Thus it is true by $MP(\mathbb{R})$, so we have $\omega_1^{L[r]} < \omega_1$. This implies $\delta = \omega_1$ is inaccessible to the reals, and hence it is inaccessible in L. Now it is enough to show that $L_\delta \prec L$. Suppose that $L \models \exists x \phi(x, a)$, where $\phi(x, a)$ has parameters from L_δ, that are coded in a single real a. Let ψ be the following statement

"The least α such that there is an $x \in L_\alpha$ with $\phi(x, a)^{L_\alpha}$, is countable"

It is forceably necessary, thus it's already true, that means the least such α is countable, so there exists $y \in L_\delta$ with $\phi(y, a)$.

$\square$

The following theorem is proved by Hamkins and Woodin [3]:

Theorem 2.12 (Hamkins-Woodin [3]). *$ZFC + \Box_{c.c.c.}\mathrm{MP}_{c.c.c.}(\mathbb{R})$ is equiconsistent with ZFC plus the existence of a weakly compact cardinal.*

We now state and prove a generalization of the above theorem. The next result is obtained in a joint work with M. Golshani, and is presented here with his kind permission.

Theorem 2.13. *The following conditions are equiconsistent:*

1. *ZFC plus the existence of a weakly compact cardinal κ such that $V_\kappa \prec V$.*
2. *$ZFC + \Box_{c.c.c.}\mathrm{MP}(\mathbb{R})$.*

Remark 2.14. Clearly, $\Box_{c.c.c.}\mathrm{MP}(\mathbb{R})$ is stronger than $\Box_{c.c.c.}\mathrm{MP}_{c.c.c.}(\mathbb{R})$, but weaker than $\Box\mathrm{MP}(\mathbb{R})$. It is easily seen that the condition (1) of Theorem 2.10 is strictly stronger than the existence of a weakly compact cardinal, but it is consistent to have this condition, if, for example, one assumes the existence of a measurable cardinal.

We need the following theorem of Leo Harrington and Saharon Shelah.

Theorem 2.15 ([4]). *Assume* MA *holds. Then either there is a real a such that $\aleph_1 = \aleph_1^{L[a]}$ or $\aleph_1$ is weakly compact in L.*

Proof of Theorem 2.13. ($1 \Longrightarrow 2$). Let κ be a weakly compact cardinal such that $V_\kappa \prec V$. Let $\mathbb{P} = Coll(\omega, <\kappa)$ and let G be a $\mathbb{P}$-generic filter over V. By Theorem 2.11, $V[G] \models \mathrm{MP}(\mathbb{R})$. Now let $\mathbb{Q}$ be an arbitrary *c.c.c.* forcing in $V[G]$ and let H be a $\mathbb{Q}$-generic filter over $V[G]$. Our aim is to show that $V[G][H] \models \mathrm{MP}(\mathbb{R})$. Thus assume $r \in \mathbb{R} \cap V[G][H]$ and $V[G][H] \models \Diamond\Box\phi(r)$.

Clearly $\mathbb{P} * \dot{\mathbb{Q}}$ is $\kappa - c.c.$, so Lemma 2.9 guarantees the existence of a $\kappa - c.c.$ forcing notion $\mathbb{S} \lhd \mathbb{P} * \dot{\mathbb{Q}}$ such that $|\mathbb{S}| < \kappa$ and $\mathbb{S}$ capture r. Assume, without loss of generality, that $\mathbb{S} \in V_\kappa$.

Let K be an $\mathbb{S}$-generic filter over V such that $r \in V[\mathbb{S}]$ and $V[K] \subseteq V[G*H]$. Since $V[G*H] \models \Diamond\Box\phi(r)$, we have $V[K] \models \Diamond\Box\phi(r)$. On the other hand we have $V_\kappa[K] \prec V[K]$ by Lemma 2.6, so $V_\kappa[K] \models \Diamond\Box\phi(r)$. Thus there exist $\mathbb{T} \in V_\kappa[K]$ and a $\mathbb{T}$-generic filter L such that $V_\kappa[K][L] \models \Box\phi(r)$. But we have a canonical projection $\pi : Coll(\omega, <\kappa) \longrightarrow \mathbb{S} * \dot{\mathbb{T}}$, which implies $V[G*H] \models \phi(r)$.

($2 \Longrightarrow 1$). Assume $V \models \Box_{c.c.c.}\mathrm{MP}(\mathbb{R})$, and let $\mathbb{P}$ be some *c.c.c.* forcing notion which forces $MA + \neg CH$. Then $V^{\mathbb{P}} \models \mathrm{MP}(\mathbb{R}) + \Box_{c.c.c.}\mathrm{MP}_{c.c.c.}(\mathbb{R})$. It follows from the proof of Theorem 2.11 that $\delta = \omega_1^{V^{\mathbb{P}}}$ is inaccessible in L and $L_\delta \prec L$. Since MA holds, by Harrington-Shelah's theorem we have ω_1 is weakly compact in L. $\Box$

As a corollary, we obtain a simpler proof for the consistency of $\Box_{c.c.c.}\mathrm{MP}_{c.c.c.}(\mathbb{R})$.

Corollary 2.16. *If "$ZFC+$ there is a weakly compact cardinal κ such that $V_\kappa \prec V$" is consistent, then is so "$ZFC + \Box_{c.c.c.}\mathrm{MP}_{c.c.c.}(\mathbb{R})$".*

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