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FURTHER INEQUALITIES FOR OPERATOR SPACE NUMERICAL RADIUS ON 2×2 OPERATOR MATRICES

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Abstract. We present some inequalities for operator space numerical radius of 2×2 block matrices on the matrix space $\mathcal{M}_n(X)$, when X is a numerical radius operator space. These inequalities contain some upper and lower bounds for operator space numerical radius.
Keywords: Operator space, numerical radius operator space, operator space numerical radius, block matrix.
MSC(2010): Primary: 47A12; Secondary: 46L15, 47A30, 47A63, 47L25.

1. Introduction

An operator space is a complex linear space X together with a sequence of norms $\mathcal{O}_n(\cdot)$ $(n = 1, 2, \ldots)$ on the $n \times n$ matrix space $\mathcal{M}_n(X)$ which satisfies the Ruan's axioms so that, $\mathcal{O}_{m+n}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \max\left\{\mathcal{O}_m(x), \mathcal{O}_n(y)\right\}$ and $\mathcal{O}_n(\alpha x \beta) \leq \|\alpha\| \mathcal{O}_m(x) \|\beta\|$, for all $x \in \mathcal{M}_m(X)$, $y \in \mathcal{M}_n(X)$, $\alpha \in \mathcal{M}_{n,m}(\mathbb{C})$ and $\beta \in \mathcal{M}_{m,n}(\mathbb{C})$ (cf. [7]). Before going to a numerical radius operator space (NROS), we will explain some notions.
Let H^n be the n-direct sum of H and $\mathcal{B}(H^n)$ the bounded operators on H^n which is identified with the $n \times n$ matrix space $\mathcal{M}_n(\mathcal{B}(H))$. Recall that for $a \in \mathcal{M}_n(\mathcal{B}(H))$ we denote $w_n(a)$ (resp. $\|a\|_n$) the numerical radius norm (resp. the operator norm) which is defined as

$$w_n(a) = \sup\{|\langle ax, x\rangle| : x \in H^n, \|x\| = 1\}.$$

In 2006 the definition of a numerical radius operator space introduced by Itoh and Nagisa [4] (see, e.g., [5]). Recall that X is a numerical radius operator space if a complex linear space X admits a sequence of norms $\mathcal{W}_n(\cdot)$ on $\mathcal{M}_n(X)$ for each $n \in \mathbb{N}$, which satisfies a couple of conditions (1.1), (1.2) where (1.1) is

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the same as the first Ruan's axiom however (1.2) is a slightly weaker condition than the second Ruan ones as follows:

$$\mathcal{W}_{m+n}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \max\{\mathcal{W}_m(x), \mathcal{W}_n(y)\}, \tag{1.1}$$

$$\mathcal{W}_n(\alpha x \alpha^*) \leq \|\alpha\|^2 \mathcal{W}_m(x) \tag{1.2}$$

for all $x \in \mathcal{M}_m(X)$, $y \in \mathcal{M}_n(X)$ and $\alpha \in \mathcal{M}_{n,m}(\mathbb{C})$, where α^* is the conjugate transpose of α.

Note that if $(X, (\mathcal{W}_n))$ is a numerical radius operator space, then $\mathcal{W}_n$ induces an operator space norm $\mathcal{O}_n$ on X by setting

$$\mathcal{O}_n(x) := 2\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \quad \text{for } x \in \mathcal{M}_n(X), \tag{1.3}$$

in the sense that X becomes an operator space. Recent some inequalities for operator space numerical radius on the 2×2 block matrices have been proved in [6]. Furthermore, some inequalities for the off-diagonal part $\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}$ of the 2×2 bolck matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ defined on $\mathcal{M}_2(\mathcal{M}_n(X))$ have been obtained.

Several inequalities for the block matrices of the form $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and also its off-diagonal part have established in [2,3,6]. For more information about the numerical radius inequalities, the reader may consult [1,8,9].

The main purpose of this paper is to give some inequalities for operator space numerical radius on the 2×2 block matrices. These inequalities include upper and lower bounds for $W_n(\cdot)$ with entries in an appropriate matrix space as well.

2. Main results

To perform our goals of presenting some inequalities for $\mathcal{W}_n(\cdot)$ we need the following lemma. This basic lemma has also plentyful applications in inequalities of the usual numerical radius. Therefore, this lemma can be proved easily by inequality (1.2) and we omit its proof.

Lemma 2.1. *If $(X, (\mathcal{W}_n))$ is an NROS and $u \in \mathcal{M}_n$ is a unitary, then for any $x \in \mathcal{M}_n(X)$*

$$\mathcal{W}_n(u^* x u) = \mathcal{W}_n(x). \tag{2.1}$$

In fact, $\mathcal{W}_n(\cdot)$ is weakly unitarily invariant. Note that $\mathcal{O}_n(\cdot)$ is unitarily invariant, namely $\mathcal{O}_n(uxv) = \mathcal{O}_n(x)$ for all unitary $u, v \in \mathcal{M}_n$ and $x \in \mathcal{M}_n(X)$. The first result reads as follows.

Proposition 2.2. *Let $(X, (\mathcal{W}_n))$ be an NROS. Then for each $x, y \in \mathcal{M}_n(X)$*

(a) $\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}\right) = \mathcal{W}_{2n}\left(\begin{bmatrix} 0 & xu^* \\ uy & 0 \end{bmatrix}\right)$ *for any unitary* $u \in \mathcal{M}_n(\mathbb{C})$;

(b) $\mathcal{W}_{2n}\left(\begin{bmatrix} y & xu^* \\ ux & uyu^* \end{bmatrix}\right) = \max\left\{\mathcal{W}_n(x+y), \mathcal{W}_n(x-y)\right\}$ *for any unitary* $u \in \mathcal{M}_n(\mathbb{C})$;

In particular, for any complex number γ *with* $|\gamma| = 1$, *we have*

$$\begin{aligned} \mathcal{W}_{2n}\left(\begin{bmatrix} y & x \\ \gamma^2 x & y \end{bmatrix}\right) &= \mathcal{W}_{2n}\left(\begin{bmatrix} \overline{\gamma} y & x\overline{\gamma} \\ \gamma x & \overline{\gamma} y \end{bmatrix}\right) \\ &= \max\left\{\mathcal{W}_n(x+\overline{\gamma} y), \mathcal{W}_n(x-\overline{\gamma} y)\right\}. \end{aligned} \tag{2.2}$$

Proof. Part (a) is concluded by using (1.2), Lemma (2.1) and the relation

$$\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} 0 & xu^* \\ uy & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & u \end{bmatrix}.$$

Part (b) follows from applying (1.1), (1.2), Lemma (2.1) and the relation

$$\begin{bmatrix} x+y & 0 \\ 0 & u(y-x)u^* \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & u^* \\ -u & I \end{bmatrix} \begin{bmatrix} y & xu^* \\ ux & uyu^* \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -u^* \\ u & I \end{bmatrix}.$$

By a similar fashion as the above identity, it can be easily concluded the relations (2.2). □

The next consequence is a general inequality for $\mathcal{W}_n(\cdot)$, which contains some inequalities as special cases.

Theorem 2.3. *Let* $(X, (\mathcal{W}_n))$ *be an NROS. Then for each* $x, y, z, w \in \mathcal{M}_n(X)$ *and* $\alpha, \beta \in \mathcal{M}_n(\mathbb{C})$

$$\mathcal{W}_n(\alpha x\alpha^* + \alpha y\beta^* + \beta z\alpha^* + \beta w\beta^*) \le \|\alpha\alpha^* + \beta\beta^*\| \mathcal{W}_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right).$$

When $x = w = 0$, *we have*

$$\mathcal{W}_n(\alpha y\beta^* + \beta z\alpha^*) \le 2\|\alpha\|\|\beta\| \mathcal{W}_{2n}\left(\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}\right).$$

Proof. We use (1.2) and the following facts:

$$\begin{aligned} \mathcal{W}_n(\alpha x\alpha^* + \alpha y\beta^* + \beta z\alpha^* + \beta w\beta^*) &= \mathcal{W}_n\left(\begin{bmatrix} \alpha & \beta \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} \alpha^* \\ \beta^* \end{bmatrix}\right) \\ &\le \|\begin{bmatrix} \alpha & \beta \end{bmatrix}\|^2 \mathcal{W}_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \\ &= \|\alpha\alpha^* + \beta\beta^*\| \mathcal{W}_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right). \end{aligned} \tag{2.3}$$

When $x = w = 0$, we replace α and β by $\sqrt{\|\beta\|/\|\alpha\|}\alpha$ and $\sqrt{\|\alpha\|/\|\beta\|}\beta$, respectively, in inequality (2.3). Then the following inequality

$$\|\begin{bmatrix} \sqrt{\|\beta\|/\|\alpha\|}\alpha & \sqrt{\|\alpha\|/\|\beta\|}\beta \end{bmatrix}\|^2 \le 2\|\alpha\|\|\beta\|$$

yields that

$$\mathcal{W}_n(\alpha y\beta^* + \beta z\alpha^*) \le 2\|\alpha\|\|\beta\|\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix}\right).$$

□

According to the previous theorem, we find lower bounds for $\mathcal{W}_{2n}(\cdot)$ on operator matrix of the form $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

Corollary 2.4. *If $(X, (\mathcal{W}_n))$ is an NROS and $x, y, z, w \in \mathcal{M}_n(X)$, then*

$$\begin{aligned} &\mathcal{W}_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \\ &\ge \max\left\{\mathcal{W}_n(\alpha x\alpha^* + \alpha y\beta^* + \beta z\alpha^* + \beta w\beta^*) : \alpha, \beta \in \mathcal{M}_n(\mathbb{C}), \|\alpha\alpha^* + \beta\beta^*\| = 1\right\} \\ &\ge \frac{1}{2}\max\left\{\mathcal{W}_n(x + yu^* + uz + uwu^*) : u \in \mathcal{M}_n(\mathbb{C}), uu^* = 1\right\}. \end{aligned}$$

In particular,

$$\begin{aligned} \mathcal{W}_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \ge \frac{1}{2}\max\Big\{&\mathcal{W}_n(x + y + z + w), \mathcal{W}_n(x - y - z + w), \\ &\mathcal{W}_n\big(x + w + i(y - z)\big), \mathcal{W}_n\big(x + w + i(z - y)\big)\Big\}. \end{aligned}$$

Proof. These inequalities follow immediately from Theorem (2.3). Note that if we choose u as $I, -I, -iI$ or iI, then $\mathcal{W}_n(x + yu^* + uz + uwu^*)$ becomes

$\mathcal{W}_n(x+y+z+w), \mathcal{W}_n(x-y-z+w), \mathcal{W}_n\big(x+w+i(y-z)\big), \mathcal{W}_n\big(x+w+i(z-y)\big)$ as required. □

The next result is an another estimation for $\mathcal{W}_{2n}(\cdot)$ on a certain 2×2 block matrix. It should be mentioned here that the following result has been proved in [6, Remark 3.6], and we want to provide a simple proof.

Theorem 2.5. *Let $(X, (\mathcal{W}_n))$ be an NROS. Then*

$$\mathcal{W}_{2n}\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) \le \max\{\mathcal{W}_n(x), \mathcal{W}_n(w)\} + \mathcal{W}_n\left(\frac{y+z}{2}\right) + \mathcal{W}_n\left(\frac{y-z}{2}\right)$$

for all $x, y, z, w \in \mathcal{M}_n(X)$.

Proof. Based on part (b) of Proposition (2.2), one can conclude that $\mathcal{W}_{2n}\left(\begin{bmatrix} y & x \\ x & y \end{bmatrix}\right) = \max\left\{\mathcal{W}_n(x + y), \mathcal{W}_n(x - y)\right\}$, for any x and $y \in \mathcal{M}_n(X)$. In particular,

$$\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}\right) = \mathcal{W}_n(x). \tag{2.4}$$

So, the desired inequality follows from (1.1), the identity (2.4) and the following facts:

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix} + \begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & y+z \\ y+z & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & y-z \\ z-y & 0 \end{bmatrix}.$$

□

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