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IMPROVEMENTS OF YOUNG INEQUALITY USING THE KANTOROVICH CONSTANT

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Abstract. Some improvements of Young inequality and its reverse for positive numbers with Kantorovich constant $K(t,2)=\frac{(1+t)^2}{4t}$ are given. Using these inequalities some operator inequalities and Hilbert-Schmidt norm versions for matrices are proved. In particular, it is shown that if a,b are positive numbers and $0\leqslant\nu\leqslant1$, then for all integers $k\geqslant1$:

$$K(h^{\frac{1}{2^n}},2)^{r_n}a\sharp_\nu b\leqslant a\nabla_\nu b-\sum_{k=0}^{n-1}r_k\left((a\sharp_{\frac{m_k}{2^k}}b)^{\frac12}-(a\sharp_{\frac{m_k+1}{2^k}}b)^{\frac12}\right)^2$$
$$\leqslant K(h^{\frac{1}{2^n}},2)^{R_n}a\sharp_\nu b,$$

where $m_k=[2^k\nu]$ is the largest integer not greater than $2^k\nu$, $r_0=\min\{\nu,1-\nu\}$, $r_k=\min\{2r_{k-1},1-2r_{k-1}\}$ and $R_k=1-r_k$.
Keywords: Heinz mean, Hilbert-Schmidt norm, Kantorovich constant, Young inequality.
MSC(2010): Primary: 47A63; Secondary: 47A64, 15A42.

1. Introduction

Let a and b be positive numbers. The famous Young inequality states that

$$a^{1-\nu}b^\nu\leqslant(1-\nu)a+\nu b,$$

for every $0\leqslant\nu\leqslant1$. By defining weighted arithmetic and geometric means as

$$a\nabla b=(1-\nu)a+\nu b,\qquad a\sharp_\nu b=a^{1-\nu}b^\nu,$$

we can consider the Young inequality as weighted arithmetic-geometric means inequality. This inequality has received an increasing attention in the literature. An improvement of Young inequality, obtained by F. Kittaneh and Y. Manasrah [7], is as follows:

$$a\sharp_\nu b+r(\sqrt{a}-\sqrt{b})^2\leqslant a\nabla_\nu b,$$

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where $r = \min\{\nu, 1-\nu\}$.

The authors of [6] obtained another refinement of the Young inequality as

$$(a\sharp_\nu b)^2 + r^2(a-b)^2 \leqslant (a\nabla_\nu b)^2,$$

where $r = \min\{\nu, 1-\nu\}$.

In [12], the authors obtained another improvement of the Young inequality and its reverse as follows:

$$K(\sqrt{h}, 2)^{r'} a\sharp_\nu b \leqslant a\nabla_\nu b - r(\sqrt{a} - \sqrt{b})^2, \tag{1.1}$$

and

$$a\nabla_\nu b - R(\sqrt{a} - \sqrt{b})^2 \leqslant K(\sqrt{h}, 2)^{-r'} a\sharp_\nu b, \tag{1.2}$$

where $h = \frac{b}{a}$ and $K(t, 2) = \frac{(1+t)^2}{4t}$ is the Kantorovich constant, $r = \min\{\nu, 1-\nu\}$, $R = \max\{\nu, 1-\nu\}$ and $r' = \min\{2r, 1-2r\}$.

In addition, with the same notation as above, another type of the reverse of Young inequality, using Kantorovich constant, is presented in [10]

$$a\nabla_\nu b - r(\sqrt{a} - \sqrt{b})^2 \leqslant K(\sqrt{h}, 2)^{R'} a\sharp_\nu b, \tag{1.3}$$

where $R' = \max\{2r, 1-2r\}$.

Note that the $K(t, 2) \geq 1$ for all $t > 0$ and attains its minimum at $t = 1$. Also $K(t, 2) = K(\frac{1}{t}, 2)$.

Recently, Liao and Wu [9] obtained the following refinement of inequalities (1.1) and (1.2):

$$\begin{aligned} a\nabla_\nu b \geqslant& \nu(\sqrt{a} - \sqrt{b})^2 + r((ab)^{\frac{1}{4}} - \sqrt{a})^2 + K(h^{\frac{1}{4}}, 2)^{r_1} a\sharp_\nu b, \\ a\nabla_\nu b \leqslant& (1-\nu)(\sqrt{a} - \sqrt{b})^2 - r((ab)^{\frac{1}{4}} - \sqrt{b})^2 + K(h^{\frac{1}{4}}, 2)^{-r_1} a\sharp_\nu b, \end{aligned} \tag{1.4}$$

for $0 < \nu \leqslant \frac{1}{2}$, and

$$\begin{aligned} a\nabla_\nu b \geqslant& (1-\nu)(\sqrt{a} - \sqrt{b})^2 + r((ab)^{\frac{1}{4}} - \sqrt{b})^2 + K(h^{\frac{1}{4}}, 2)^{r_1} a\sharp_\nu b, \\ a\nabla_\nu b \leqslant& \nu(\sqrt{a} - \sqrt{b})^2 - r((ab)^{\frac{1}{4}} - \sqrt{a})^2 + K(h^{\frac{1}{4}}, 2)^{-r_1} a\sharp_\nu b, \end{aligned} \tag{1.5}$$

for $\frac{1}{2} < \nu < 1$, where $r = \min\{2(1-\nu), 1-2(1-\nu)\}$ and $r_1 = \min\{2r, 1-2r\}$.

For more related inequalities see [1, 11, 13].

Numerical inequality (1.5), leads to similar operator inequalities. See also [2]. For this purpose, let $\mathbb{B}(\mathscr{H})$ stand for the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive (denoted by $A \geqslant 0$) if A is self-adjoint with non-negative spectrum, and strictly positive (denoted by $A > 0$) if A is an invertible positive operator. The partial order $A \leqslant B$, on the class of self-adjoint operators, means that $B - A$ is a positive operator.

If $\mathscr{H}$ is a Hilbert space, of dimension n, then we identify $\mathbb{B}(\mathscr{H})$ with $\mathbb{M}_n$ consisting of all $n \times n$ complex matrices and the positive operators and strictly positive operators are the same as the positive semidefinite matrices and positive (definite) matrices, respectively.

The weighted arithmetic and geometric mean for strictly positive operators A, B, is defined by

$$A\nabla_\nu B = (1-\nu)A + \nu B, \qquad A\sharp_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}.$$

In addition, the Heinz mean of A and B is defined by

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2}.$$

See [3, 4] for more information about these means.

Using the above notations, the operator versions of Young inequality, its refinements and its reverses are proved. For instance, we have the following refinement of (1.4) and (1.5) established in [9]. The other inequalities can be stated in a similar way.

Theorem 1.1 ([9])**.** *Let $A, B \in \mathbb{B}(\mathscr{H})$ be strictly positive operators and positive real numbers m, m', M, M' satisfy either $0 < m'I \leqslant A \leqslant mI \leqslant MI \leqslant B \leqslant M'I$ or $0 < m'I \leqslant B \leqslant mI \leqslant MI \leqslant A \leqslant M'I$.*

(I) If $0 < \nu \leqslant \frac{1}{2}$, then

$$A\nabla_\nu B \geqslant 2\nu(A\nabla B - A\sharp B) + r(A\sharp B - 2A\sharp_{\frac{1}{4}} B + A) + K(h^{\frac{1}{4}}, 2)^{r_1} A\sharp_\nu B,$$

and

$$A\nabla_\nu B \leqslant 2(1-\nu)(A\nabla B - A\sharp B) - r(A\sharp B - 2A\sharp_{\frac{3}{4}} B + B) + K(h^{\frac{1}{4}}, 2)^{-r_1} A\sharp_\nu B,$$

(II) If $\frac{1}{2} < \nu < 1$, then

$$A\nabla_\nu B \geqslant 2(1-\nu)(A\nabla B - A\sharp B) + r(A\sharp B - 2A\sharp_{\frac{3}{4}} B + B) + K(h^{\frac{1}{4}}, 2)^{r_1} A\sharp_\nu B,$$

and

$$A\nabla_\nu B \leqslant 2\nu(A\nabla B - A\sharp B) - r(A\sharp B - 2A\sharp_{\frac{1}{4}} B + A) + K(h^{\frac{1}{4}}, 2)^{-r_1} A\sharp_\nu B,$$

where $h = \frac{M}{m}, r = \min\{\nu, 1-\nu\}$ and $r_1 = \min\{2r, 1-2r\}$.

The main aim of this paper is to present a generalization of these inequalities. First, we present some generalizations of numerical inequalities. Employing them we then prove some refined operator versions of Young inequality and its reverse. Also some inequalities for Hilbert-Schmidt norm of matrices are obtained.

Throughout the paper, for $0 \leqslant \nu \leqslant 1$ and integer $k \geqslant 1$, the notation $m_k = [2^k\nu]$ stands for the largest integer not greater than $2^k\nu$, $r_0 = \min\{\nu, 1-\nu\}$, $r_k = \min\{2r_{k-1}, 1-2r_{k-1}\}$ and $R_k = 1 - r_k$.

2. **Numerical results**

Our first theorem gives a refined version of Young inequality and its reverse.

Theorem 2.1. *Let a, b be positive real numbers and $\nu \in [0,1]$. Then*

$$K(h^{\frac{1}{2^n}},2)^{r_n} a\sharp_\nu b \leqslant a\nabla_\nu b - \sum_{k=0}^{n-1} r_k \left((a\sharp_{\frac{m_k}{2^k}} b)^{\frac{1}{2}} - (a\sharp_{\frac{m_k+1}{2^k}} b)^{\frac{1}{2}}\right)^2 \leqslant K(h^{\frac{1}{2^n}},2)^{R_n} a\sharp_\nu b, \tag{2.1}$$

where $h = \frac{b}{a}$.

In addition, if $\nu = \frac{p}{2^t}$ for some $p, t \in \mathbb{N}$ with $t > 1$, then

$$K(h^{\frac{1}{2^{t-1}}},2)^{r_{t-1}} a\sharp_\nu b = a\nabla_\nu b - \sum_{k=0}^{t-2} r_k \left((a\sharp_{\frac{m_k}{2^k}} b)^{\frac{1}{2}} - (a\sharp_{\frac{m_k+1}{2^k}} b)^{\frac{1}{2}}\right)^2 = K(h^{\frac{1}{2^{t-1}}},2)^{R_{t-1}} a\sharp_\nu b.$$

Proof. First, we prove the left hand side of inequality (2.1), by using induction. For $n = 1$, we reach inequality (1.1). Let inequality (2.1) holds for n.

For $0 < \nu \leqslant \frac{1}{2}$, we have

$$\begin{aligned} a\nabla_\nu b - r_0(\sqrt{a}-\sqrt{b})^2 &= a\nabla_\nu b - \nu(\sqrt{a}-\sqrt{b})^2 \\ &= (1-\nu)a + \nu b - \nu(a - 2\sqrt{ab} + b) \\ &= 2\nu\sqrt{ab} + (1-2\nu)a \\ &= a\nabla_{2\nu}\sqrt{ab}. \end{aligned}$$

Applying inequality (2.1) for positive numbers a and $\sqrt{ab}$ and $2\nu \in (0,1]$, we get

$$\begin{aligned} &a\nabla_\nu b - r_0(\sqrt{a}-\sqrt{b})^2 \\ &= a\nabla_{2\nu}\sqrt{ab} \\ &\geqslant K(\sqrt{h}^{\frac{1}{2^n}},2)^{r_{n+1}} a\sharp_{2\nu}\sqrt{ab} \\ &\quad + \sum_{k=0}^{n-1} r_{k+1}\left((a\sharp_{\frac{m_{k+1}}{2^k}}\sqrt{ab})^{\frac{1}{2}} - (a\sharp_{\frac{m_{k+1}+1}{2^k}}\sqrt{ab})^{\frac{1}{2}}\right)^2 \\ &= K(h^{\frac{1}{2^{n+1}}},2)^{r_{n+1}} a\sharp_\nu b + \sum_{k=1}^{n} r_k \left((a\sharp_{\frac{m_k}{2^k}} b)^{\frac{1}{2}} - (a\sharp_{\frac{m_k+1}{2^k}} b)^{\frac{1}{2}}\right)^2. \end{aligned}$$

For $\frac{1}{2} < \nu < 1$, we can apply the first part for $1-\nu$ and replace a and b. Note that $[2^k(1-\nu)] = 2^k - [2^k\nu] - 1$, if $2^k\nu$ is not integer. Thus, if $2^k\nu$ is not integer for any k, the inequality holds.

Now, let $\nu = \frac{p}{2^q}$ for some $q > 1$ and odd number p. Since for each $i < q$, the coefficient $r_i \leq \frac{1}{2}$ is of the form $\frac{p_i}{2^{q-i}}$, for some odd number q_i, it can be concluded that $r_{q-1} = \frac{1}{2} = R_{q-1}$. Hence the desired equality follows. A similar argument yields the second inequality. □

Changing the elements a and b in inequality (2.1), we can state the following result for Heinz mean.

Corollary 2.2. *Let a, b be positive real numbers and $\nu \in (0,1)$. Then*

$$K(h^{\frac{1}{2^n}}, 2)^{r_n} H_\nu(a,b) \leqslant a\nabla b - \sum_{k=0}^{n-1} r_k \left(H_{\frac{m_k}{2^k}}(a,b) - 2H_{\frac{2m_k+1}{2^{k+1}}}(a,b) + H_{\frac{m_k+1}{2^k}}(a,b) \right)$$
$$\leqslant K(h^{\frac{1}{2^n}}, 2)^{R_n} H_\nu(a,b),$$

where $h = \frac{b}{a}$.

In the following theorem, we state another version of the reversed Young inequality.

Theorem 2.3. *Let a, b be positive real numbers and $\nu \in (0,1)$. Then*

$$a\nabla_\nu b \leqslant K(h^{\frac{1}{2^n}}, 2)^{-r_n} a\sharp_\nu b + (\sqrt{a} - \sqrt{b})^2 - \sum_{k=0}^{n-1} r_k \left((a\sharp_{1-\frac{m_k}{2^k}} b)^{\frac{1}{2}} - (a\sharp_{1-\frac{m_k+1}{2^k}} b)^{\frac{1}{2}} \right)^2, \tag{2.2}$$

where $h = \frac{b}{a}$.

Proof. Applying the arithmetic-geometric mean inequality, we have

$$K(h^{\frac{1}{2^n}}, 2)^{-r_n} a\sharp_\nu b + K(h^{\frac{1}{2^n}}, 2)^{r_n} b\sharp_\nu a \geq 2\sqrt{ab}.$$

Using this inequality and employing inequality (2.1), we get

$$(\sqrt{a} - \sqrt{b})^2 - a\nabla_\nu b$$
$$= b\nabla_\nu a - 2\sqrt{ab} - K(h^{\frac{1}{2^n}}, 2)^{-r_n} a\sharp_\nu b + \sum_{k=0}^{n-1} r_k \left((a\sharp_{1-\frac{m_k}{2^k}} b)^{\frac{1}{2}} - (a\sharp_{1-\frac{m_k+1}{2^k}} b)^{\frac{1}{2}} \right)^2.$$

Thus we get the result. □

Corollary 2.4. *Let a, b be positive real numbers and $\nu \in (0,1)$. Then*

$$a\nabla b \leqslant K(h^{\frac{1}{2^n}}, 2)^{-r_n} H_\nu(a,b)$$
$$+ (\sqrt{a} - \sqrt{b})^2 - \sum_{k=0}^{n-1} r_k \left(H_{\frac{m_k}{2^k}}(a,b) - 2H_{\frac{2m_k+1}{2^{k+1}}}(a,b) + H_{\frac{m_k+1}{2^k}}(a,b) \right),$$

where $h = \frac{b}{a}$.

Remark 2.5. Replacing a and b by their squares in (2.1) and (2.2), respectively, we obtain

$$(2.3)\qquad K(h^{\frac{1}{2^{n-1}}},2)^{r_n}a^2\sharp_\nu b^2 \leqslant a^2\nabla_\nu b^2-\sum_{k=0}^{n-1}r_k\left(a\sharp_{\frac{m_k}{2^k}}b-a\sharp_{\frac{m_k+1}{2^k}}b\right)^2 \leqslant K(h^{\frac{1}{2^{n-1}}},2)^{R_n}a^2\sharp_\nu b^2,$$

and

(2.4)

$$a^2\nabla_\nu b^2 \leqslant K(h^{\frac{1}{2^{n-1}}},2)^{-r_n}a^2\sharp_\nu b^2+(a-b)^2-\sum_{k=0}^{n-1}r_k\left(a\sharp_{1-\frac{m_k}{2^k}}b-a\sharp_{1-\frac{m_k+1}{2^k}}b\right)^2,$$

where $h=\frac{b}{a}$.

The following two theorems are useful to prove a version of these inequalities for the Hilbert-Schmidt norm of matrices and we apply them in Section 4.

Theorem 2.6. *Let a,b be positive real numbers and $\nu\in(0,1)$. Then*

$$K(h^{\frac{1}{2^{n-1}}},2)^{r_n}(a\sharp_\nu b)^2 \leqslant (a\nabla_\nu b)^2-r_0^2(a-b)^2-\sum_{k=1}^{n-1}r_k\left(a\sharp_{\frac{m_k}{2^k}}b-a\sharp_{\frac{m_k+1}{2^k}}b\right)^2 \leqslant K(h^{\frac{1}{2^{n-1}}},2)^{R_n}(a\sharp_\nu b)^2,$$

where $h=\frac{b}{a}$.

Proof. By a simple calculation, we have $(a\nabla_\nu b)^2-r_0^2(a-b)^2=a^2\nabla_\nu b^2-r_0(a-b)^2$. Using (2.3), we have

$$\begin{aligned}K(h^{\frac{1}{2^{n-1}}},2)^{r_n}(a\sharp_\nu b)^2 &\leqslant a^2\nabla_\nu b^2-\sum_{k=0}^{n-1}r_k\left(a\sharp_{\frac{m_k}{2^k}}b-a\sharp_{\frac{m_k+1}{2^k}}b\right)^2\\ &=(a\nabla_\nu b)^2-r_0^2(a-b)^2-\sum_{k=1}^{n-1}r_k\left(a\sharp_{\frac{m_k}{2^k}}b-a\sharp_{\frac{m_k+1}{2^k}}b\right)^2\\ &\leqslant K(h^{\frac{1}{2^{n-1}}},2)^{R_n}(a\sharp_\nu b)^2.\end{aligned}$$

□

Theorem 2.7. *Let a,b be positive real numbers and $\nu\in(0,1)$. Then*

$$(a\nabla_\nu b)^2 \leqslant K(h^{\frac{1}{2^{n-1}}},2)^{-r_n}(a\sharp_\nu b)^2+R_0^2(a-b)^2-\sum_{k=1}^{n-1}r_k\left(a\sharp_{\frac{m_k}{2^k}}b-a\sharp_{\frac{m_k+1}{2^k}}b\right)^2,$$

where $h=\frac{b}{a}$.

Proof. We have

$$\begin{aligned}(a\nabla_\nu b)^2 - (1-r_0)^2(a-b)^2 &= a^2\nabla_\nu b^2 - (1-r_0)(a-b)^2\\ &\leqslant K(h^{\frac{1}{2^{n-1}}},2)^{-r_n}(a\sharp_\nu b)^2 + r_0(a-b)^2 - \sum_{k=0}^{n-1} r_k\left(a\sharp_{\frac{m_k}{2^k}}b - a\sharp_{\frac{m_k+1}{2^k}}b\right)^2\\ &\qquad \text{by inequality (2.4)}\\ &= K(h^{\frac{1}{2^{n-1}}},2)^{-r_n}(a\sharp_\nu b)^2 - \sum_{k=1}^{\infty} r_k\left(a\sharp_{\frac{m_k}{2^k}}b - a\sharp_{\frac{m_k+1}{2^k}}b\right)^2.\end{aligned}$$

□

3. Related operator inequalities

To reach the operator versions of the inequalities obtained in Section 2, we use the continuous functional calculus for self-adjoint operators.

Let X be a strictly positive operator. Then $\sigma(X)$ is a compact subset of $(0,+\infty)$. We denote by $m(X)$ and $M(X)$ the minimum and the maximum values of $\sigma(X)$.

Now, we give the first result in this section which is based on Theorem 2.1 and is a refinement of [9, Theorem 3].

Theorem 3.1. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be strictly positive operators with $M(A) \leqslant m(B)$ and $\nu \in (0,1)$. Then*

(3.1)

$$\begin{aligned}K(h^{\frac{1}{2^n}},2)^{r_n}A\sharp_\nu B &\leqslant A\nabla_\nu B - \sum_{k=0}^{n-1} r_k\left(A\sharp_{\frac{m_k}{2^k}}B - 2A\sharp_{\frac{2m_k+1}{2^{k+1}}}B + A\sharp_{\frac{m_k+1}{2^k}}B\right)\\ &\leqslant K(h^{\frac{1}{2^n}},2)^{R_n}A\sharp_\nu B,\end{aligned}$$

where $h = \frac{m(B)}{M(A)}$.

Proof. Choosing $a = 1$ in Theorem 2.1, we have

$$1-\nu+\nu b \geqslant K(b^{\frac{1}{2^n}},2)^{r_n}b^\nu + \sum_{k=0}^{n-1} r_k\left((b^{\frac{m_k}{2^k}})^{\frac{1}{2}} - (b^{\frac{m_k+1}{2^k}})^{\frac{1}{2}}\right)^2,$$

for any $b > 0$.

If $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, then $\sigma(X) \subseteq [h,+\infty)$. Due to the Kantorovich constant is increasing on $[1, +\infty)$, it follows that for all $b \geq h$,

$$1 - \nu + \nu b \geqslant K(b^{\frac{1}{2^n}}, 2)^{r_n} b^{\nu} + \sum_{k=0}^{n-1} r_k \left((b^{\frac{m_k}{2^k}})^{\frac{1}{2}} - (b^{\frac{m_k+1}{2^k}})^{\frac{1}{2}} \right)^2$$
$$\geqslant K(h^{\frac{1}{2^n}}, 2)^{r_n} b^{\nu} + \sum_{k=0}^{n-1} r_k \left((b^{\frac{m_k}{2^k}})^{\frac{1}{2}} - (b^{\frac{m_k+1}{2^k}})^{\frac{1}{2}} \right)^2.$$

By the continuous functional calculus, we get

$$(1-\nu)I + \nu X \geqslant K(h^{\frac{1}{2^n}}, 2)^{r_n} X^{\nu} + \sum_{k=0}^{n-1} r_k \left(X^{\frac{m_k}{2^k}} - 2X^{\frac{2m_k+1}{2^{k+1}}} + X^{\frac{m_k+1}{2^k}} \right).$$

Multiplying both sides by $A^{\frac{1}{2}}$, we obtain

$$A\nabla_{\nu}B \geqslant K(h^{\frac{1}{2^n}}, 2)^{r_n} A\sharp_{\nu}B + \sum_{k=0}^{n-1} r_k \left(A\sharp_{\frac{m_k}{2^k}} B - 2A\sharp_{\frac{2m_k+1}{2^{k+1}}} B + A\sharp_{\frac{m_k+1}{2^k}} B \right).$$

This completes the proof of the first inequality of (3.1). In the same way, we can prove the second one. □

The following theorem is an operator version of Theorem 2.3 and is a refinement of [9, Theorem 4].

Theorem 3.2. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be strictly positive operators with $M(A) \leqslant m(B)$ and $\nu \in (0,1)$. Then*

$$A\nabla_{\nu}B \leqslant K(h^{\frac{1}{2^n}}, 2)^{-r_n} A\sharp_{\nu}B + (A - 2A\sharp B + B)$$
$$- \sum_{k=0}^{n-1} r_k \left(A\sharp_{\frac{m_k}{2^k}} B - 2A\sharp_{\frac{2m_k+1}{2^{k+1}}} B + A\sharp_{\frac{m_k+1}{2^k}} B \right),$$

where $h = \frac{m(B)}{M(A)}$.

Proof. Utilizing Lemma 2.3, and employing the same ideas as used in the proof of Theorem 3.1, we can get the desired double inequality. □

Corollary 3.3. *Let $A, B \in \mathbb{B}(\mathscr{H})$ be strictly positive operators with $M(A) \leqslant m(B)$ and $\nu \in (0,1)$. Then*

$$K(h^{\frac{1}{2^n}}, 2)^{r_n} H_{\nu}(A,B) \leqslant A\nabla B$$
$$- \sum_{k=0}^{n-1} r_k \left(H_{\frac{m_k}{2^k}}(A,B) - 2H_{\frac{2m_k+1}{2^{k+1}}}(A,B) + H_{\frac{m_k+1}{2^k}}(A,B) \right)$$
$$\leqslant K(h^{\frac{1}{2^n}}, 2)^{R_n} H_{\nu}(A,B),$$

and

$$A\nabla B \leqslant K(h^{\frac{1}{2^n}},2)^{-r_n} H_\nu(A,B) + (A - 2A\sharp B + B) - \sum_{k=0}^{n-1} r_k \left(H_{\frac{m_k}{2^k}}(A,B) - 2H_{\frac{2m_k+1}{2^{k+1}}}(A,B) + H_{\frac{m_k+1}{2^k}}(A,B) \right),$$

where $h = \frac{m(B)}{M(A)}$.

4. Matrix Young and reverse inequalities for the Hilbert-Schmidt norm

In this section, we present some inequalities for the Hilbert-Schmidt norm. It is known that every positive semidefinite matrix is unitarily diagonalizable. Thus for positive semidefinite $n \times n$ matrices A and B, there exist unitary matrices U and V such that $A = U\text{diag}(\lambda_1, \ldots, \lambda_n)U^*$ and $B = V\text{diag}(\mu_1, \ldots, \mu_n)V^*$.

In what follows, we use the Hadamard product of two matrices, which is defined as the entrywise product $A \circ B = [a_{ij}b_{ij}]$.

Applying Theorem 2.6, we get the following theorem that is a generalization of the inequalities in [9, Theorem 5].

Theorem 4.1. *Suppose $A, B, X \in \mathbb{M}_n$ such that A and B are positive matrices and $\nu \in (0,1)$. Let*

$$\underline{K_t} = \min\left\{ K\left((\frac{\mu_j}{\lambda_i})^{\frac{1}{2^{t-1}}}, 2 \right)^{r_t} : \ i,j = 1,2,\ldots,n \right\},$$

and

$$\overline{K_t} = \max\left\{ K\left((\frac{\mu_j}{\lambda_i})^{\frac{1}{2^{t-1}}}, 2 \right)^{R_t} : \ i,j = 1,2,\ldots,n \right\},$$

for all $t \in \mathbb{N}$. Then

$$\begin{aligned} \underline{K_t}\|A^{1-\nu}XB^\nu\|_2^2 &\leqslant \|(1-\nu)AX - \nu XB\|_2^2 - r_0^2\|AX - XB\|_2^2 \\ &\quad - \sum_{k=1}^{t-1} r_k \|A^{1-\frac{m_k}{2^k}} XB^{\frac{m_k}{2^k}} - A^{1-\frac{m_k+1}{2^k}} XB^{\frac{m_k+1}{2^k}}\|_2^2 \\ &\leqslant \overline{K_t}\|A^{1-\nu}XB^\nu\|_2^2. \end{aligned} \tag{4.1}$$

Proof. Let $Y = U^*XV = (y_{ij})$. Then

$$(1-\nu)AX - \nu XB = U((\lambda_i \nabla_\nu \mu_j) \circ Y)V^*, \tag{4.2}$$

$$AX - XB = U((\lambda_i - \mu_j) \circ Y)V^*, \tag{4.3}$$

$$A^{1-\nu}XB^\nu = U((\lambda_i \sharp_\nu \mu_j) \circ Y)V^*, \tag{4.4}$$

and

(4.5)

$$A^{1-\frac{m_k}{2^k}}XB^{\frac{m_k}{2^k}} - A^{1-\frac{m_k+1}{2^k}}XB^{\frac{m_k+1}{2^k}} = U((\lambda_i\sharp_{\frac{m_k}{2^k}}\mu_j - \lambda_i\sharp_{\frac{m_k+1}{2^k}}\mu_j)\circ Y)V^*.$$

Utilizing the unitarily invariant property of $\|.\|_2$ and Theorem 2.6, we have

$$\begin{aligned}
\|(1-\nu)AX - \nu XB\|_2^2 &= \sum_{i,j=1}^{n}(\lambda_i\nabla_\nu\mu_j)^2|y_{ij}|^2 \qquad \text{(by inequality (4.2))}\\
&\geqslant \sum_{i,j=1}^{n}\left\{K\left((\frac{\mu_j}{\lambda_i})^{\frac{1}{2^{t-1}}},2\right)^{r_t}(\lambda_i\sharp_\nu\mu_j)^2 + r_0^2(\lambda_i-\mu_j)^2\right\}|y_{ij}|^2\\
&\quad + \sum_{i,j=1}^{n}\left\{\sum_{k=1}^{t-1}r_k(\lambda_i\sharp_{\frac{m_k}{2^k}}\mu_j - \lambda_i\sharp_{\frac{m_k+1}{2^k}}\mu_j)^2\right\}|y_{ij}|^2 \qquad \text{(by Theorem 2.6)}\\
&= \sum_{i,j=1}^{n}K\left((\frac{\mu_j}{\lambda_i})^{\frac{1}{2^{t-1}}},2\right)^{r_t}(\lambda_i\sharp_\nu\mu_j)^2|y_{ij}|^2 + \sum_{i,j=1}^{n}r_0^2(\lambda_i-\mu_j)^2|y_{ij}|^2\\
&\quad + \sum_{i,j=1}^{n}\left\{\sum_{k=1}^{t-1}r_k(\lambda_i\sharp_{\frac{m_k}{2^k}}\mu_j - \lambda_i\sharp_{\frac{m_k+1}{2^k}}\mu_j)^2|y_{ij}|^2\right\}\\
&\geqslant \underline{K_t}\sum_{i,j=1}^{n}(\lambda_i\sharp_\nu\mu_j)^2|y_{ij}|^2 + \sum_{i,j=1}^{n}r_0^2(\lambda_i-\mu_j)^2|y_{ij}|^2\\
&\quad + \sum_{k=1}^{t-1}\left\{\sum_{i,j=1}^{n}r_k(\lambda_i\sharp_{\frac{m_k}{2^k}}\mu_j - \lambda_i\sharp_{\frac{m_k+1}{2^k}}\mu_j)^2|y_{ij}|^2\right\}\\
&= \underline{K_t}\|A^{1-\nu}XB^\nu\|_2^2 + r_0^2\|AX-XB\|_2^2 \qquad \text{(by inequalities (4.3)-(4.5))}\\
&\quad + \sum_{k=1}^{t-1}r_k\|A^{1-\frac{m_k}{2^k}}XB^{\frac{m_k}{2^k}} - A^{1-\frac{m_k+1}{2^k}}XB^{\frac{m_k+1}{2^k}}\|_2^2.
\end{aligned}$$

This completes the proof of the left hand side of (4.1). By the same ideas, we can prove the right hand side. □

Theorem 4.2. *Suppose $A, B, X \in \mathbb{M}_n$ such that A and B are positive matrices and $\nu \in (0,1)$ and*

$$\underline{K_t} = \min\left\{K\left((\frac{\mu_j}{\lambda_i})^{\frac{1}{2^{t-1}}},2\right)^{r_t} : i,j = 1,2,\ldots,n\right\}.$$

Then

$$\begin{aligned}
\|(1-\nu)AX - \nu XB\|_2^2 \leqslant \underline{K_t}^{-1}\|A^{1-\nu}XB^\nu\|_2^2 + R_0^2\|AX-XB\|_2^2\\
- \sum_{k=1}^{\infty}r_k\|A^{1-\frac{m_k}{2^k}}XB^{\frac{m_k}{2^k}} - A^{1-\frac{m_k+1}{2^k}}XB^{\frac{m_k+1}{2^k}}\|_2^2.
\end{aligned}$$

Proof. In view of Theorem 2.7, using the same ideas as in the proof of Theorem 4.1, we can obtain the desired result. □

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References

[1] M. Bakherad, M. Krnic and M. S. Moslehian, Reverse Young-type inequalities for matrices and operators, *Rocky Mountain J. Math.* **46** (2016), no. 4, 1089–1105.
[2] C. Conde, Young type inequalities for positive operators, *Ann. Funct. Anal.* **4** (2013), no. 2, 144–152.
[3] T. Furuta, Invitation to Linear Operators: Form Matrix to Bounded Linear Operators on a Hilbert Space, Taylor and Francis, 2002.
[4] T. Furuta and M. Yanagide, Generalized means and convexity of inversion for positive operators, *Amer. Math. Monthly* **105** (1998) 258–259.
[5] C.J. He and L.M. Zou, Some inequalities involving unitarily invariant norms, *Math. Inequal. Appl.* **12** (2012), no. 4, 767–776.
[6] O. Hirzallah and F. Kittaneh, Matrix Young inequalities for the Hilbert-Schmidt norm, *Linear Algebra Appl.* **308** (2000), no. 1-3, 77–84.
[7] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrices, *J. Math. Anal. Appl.* **361** (2010), no. 1, 262–269.
[8] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra* **59** (2011), no. 9, 1031–1037.
[9] W. Liao and J. Wu, Improved Young and Heinz inequalities with the Kantorovich constant, *J. Math. Inequal.* **10** (2016), no. 2, 559–570.
[10] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), no. 2, 467–479.
[11] A. Salemi and A. Sheikh Hosseini, On reversing of the modified Young inequality , *Ann. Funct. Anal.* **5** (2014), no. 1, 70–76.
[12] J. Wu and J. Zhao, Operator inequalities and reverse inequalities related to the Kittaneh-Manasrah inequalities, *Linear Multilinear Algebra* **62** (2014), no. 7, 884–894.
[13] H.L. Zuo, G.H. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.* **5** (2011), no. 4, 551–556.

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