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STRONGLY NIL-CLEAN CORNER RINGS

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Abstract. We show that if R is a ring with an arbitrary idempotent e such that eRe and $(1-e)R(1-e)$ are both strongly nil-clean rings, then $R/J(R)$ is nil-clean. In particular, under certain additional circumstances, R is also nil-clean. These results somewhat improves on achievements due to Diesl in J. Algebra (2013) and to Koşan-Wang-Zhou in J. Pure Appl. Algebra (2016). In addition, we also give a new transparent proof of the main result of Breaz-Calugareanu-Danchev-Micu in Linear Algebra Appl. (2013) which says that if R is a commutative nil-clean ring, then the full $n \times n$ matrix ring $\mathbb{M}_n(R)$ is nil-clean.
Keywords: Nil-clean rings, strongly nil-clean rings, idempotents, nilpotents, Jacobson radical.
MSC(2010): Primary: 16S34; Secondary: 16U60, 16D50.

1. Introduction and background

Throughout the current note, all rings R considered shall be assumed to be associative with identity element 1 which is different from the zero element 0. As usual, $Id(R)$ denotes the set of all idempotents of R and $Nil(R)$ the set of all nilpotents of R. Traditionally, $U(R)$ will denote the group of all units in R and $J(R)$ will denote the Jacobson radical of R. Notice that $1 + J(R) \subseteq U(R)$ always holds. We also use E_{ij} to denote the $n \times n$ matrix with (i, j)-entry 1 and the other entries 0. Recall that the prime (Baer-McCoy) radical $P(R)$ of a ring R is defined to be the intersection of all prime ideals in R (note that it coincides with the lower nil-radical $Nil_*(R)$). A ring R is said to be 2-*primal* if $P(R) = Nil(R)$, that is, R/P is a domain for every minimal prime ideal P of R. Note that each commutative ring and each reduced ring (i.e., a ring without nonzero nilpotent elements) must be 2-primal. Recollect also that a ring R has a *bounded index of nilpotence* provided that there exists $n \in \mathbb{N}$ such that $a^n = 0$ for every $a \in Nil(R)$. Besides, the upper (Köthe's) nil-radical $Nil^*(R)$ of R is defined as the sum of all two-sided nil ideals of R and

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so it is the largest nil-ideal of R. Furthermore, it follows that the inclusions $Nil_*(R) = P(R) \subseteq Nil^*(R) \subseteq Nil(R) \cap J(R)$ hold.

All other notions and notations, not explicitly explained herein, are standard and may be found in [7]. However, the most useful of them will be listed below.

The following fundamental concept was defined in [9].

Definition 1.1. A ring R is called *clean* if, for each $x \in R$, there exist $u \in U(R)$ and $e \in Id(R)$ such that $x = u+e$. If, in addition, the commutativity condition $ue = eu$ is satisfied, the clean ring R is said to be *strongly clean*.

It is clear that abelian (in particular, commutative) clean rings are always strongly clean.

On the other side, in [4] was introduced the following concept.

Definition 1.2. A ring R is called *nil-clean* if, for every $r \in R$, there are $q \in Nil(R)$ and $e \in Id(R)$ with $r = q + e$. If, in addition, the commutativity condition $qe = eq$ is satisfied, the nil-clean ring R is said to be *strongly nil-clean*.

It is obvious that abelian (in particular, commutative) nil-clean rings are always strongly nil-clean. Likewise, it was independently established in [6] and [3] by exploiting different ideas that a ring is strongly nil-clean if and only if it is boolean modulo its Jacobson radical which has to be nil.

It is well known that the following containment holds:

strongly nil-clean $\Rightarrow$ *nil-clean + strongly clean* $\Rightarrow$ *clean*.

There are two important and closely related directions in noncommutative ring theory investigating to what extent the ring-theoretic properties of R are preserved by its corner ring eRe, where $e \in Id(R)$, or by its full $n \times n$ matrix ring $\mathbb{M}_n(R)$, where $n \in \mathbb{N}$, and vice versa. The most important principal known results in these two subjects are the following: It was proved in [5] that if eRe and $(1-e)R(1-e)$ are clean rings, then R is a clean ring. However, it was exhibited in [10] a clean ring R for which eRe is not clean. Nevertheless, it was obtained in [2] that if R is strongly clean, then eRe is again strongly clean. Moreover, it was shown in [4, Corollary 3.26] that if R is a strongly nil-clean ring, then eRe is a strongly nil-clean ring. Likewise, this was extended in [3] to the so-called UU rings which are rings whose units are only unipotents; note that a unipotent is the sum of 1 and a nilpotent. So, a question which immediately arises is what we can say about the ring structure of R, provided that both eRe and $(1-e)R(1-e)$ are strongly nil-clean. We will somewhat settle this in the sequel.

On the other vein, in [5] it was established that if R is a clean ring, then so is $\mathbb{M}_n(R)$. Besides, in [1, Corollary 7] it was proved that if R is a commutative nil-clean ring, then the ring $\mathbb{M}_n(R)$ is nil-clean. This was extended in [6, Theorem 6.1] to 2-primal strongly nil-clean rings and in [6, Corollary 6.8] to strongly nil-clean rings of bounded index of nilpotence.

The objective of this article is to continue the investigations of these two closely related directions by giving a partial converse to the cited above Corollary 3.26 from [4], so that we shall deal in what follows with rings whose corners have the strongly nil-cleanness. Likewise, some new matrix results will be deduced as well, thus improving the aforementioned two results from [6].

2. **Main results**

We first will give a new simpler and more conceptual verification of the aforementioned fact from [4, Corollary 3.26].

Proposition 2.1. *If R is a strongly nil-clean ring, then eRe is also a strongly nil-clean ring for any idempotent e of R.*

Proof. It was proved in [3] that a ring is strongly nil-clean if and only if it is a strongly clean UU ring. Thus we can subsequently apply the cited above two facts from [2] and [3] to get the desired claim. □

Remark 2.2. We may also apply the mentioned above characterization from [6] or [3] that a ring R is strongly nil-clean if and only if $R/J(R)$ is boolean and $J(R)$ is nil. And so, with the aid of [7], we deduce that the factor-ring $eRe/J(eRe) = eRe/eJ(R)e \cong e'[R/J(R)]e'$, where $e' = e + J(R)$ is an idempotent in $R/J(R)$, is again boolean. Also, as above, eRe is a UU ring, whence $J(eRe)$ must be nil, as required.

The following technicality is our crucial tool.

Lemma 2.3. *Suppose that R is a ring with $e \in Id(R)$ for which eRe and $(1-e)R(1-e)$ are both boolean rings. Then R is nil-clean.*

Proof. Given $r \in R$, one sees that the equality $r = ere + (1-e)r(1-e) + (1-e)re + er(1-e)$ holds. Notice that both $ere \in eRe$ and $(1-e)r(1-e) \in (1-e)R(1-e)$ are orthogonal idempotents taking into account that $e(1-e) = (1-e)e = 0$, while both $(1-e)re$ and $er(1-e)$ are nilpotents bearing in mind that $[(1-e)re]^2 = (1-e)re.(1-e)re = 0 = er(1-e).er(1-e) = [er(1-e)]^2$. On the other hand, setting $t = (1-e)re + er(1-e)$ and $f = (1-e)rer(1-e) + er(1-e)re$, one observes that $t^2 = f$. But note that $(1-e)rer(1-e) \in (1-e)R(1-e)$ and $er(1-e)re \in eRe$ are both idempotents by assumption, so that the element f being a sum of two orthogonal idempotents is again an idempotent. Hence, $t^2 = f^2$, that is, $t^2 - f^2 = 0$. Moreover, one checks that $tf = (1-e)rer(1-e)re + er(1-e)rer(1-e) = ft$ and thus $(t-f)(t+f) = 0$. Since $2f = 0$ as f is an element of the sum of two boolean rings, the last equality is tantamount to $(t-f)^2 = 0$, i.e., $t \in f + Nil(R)$. Next, seeing that $r = ere + (1-e)r(1-e) + t$, we write that $r = [ere + er(1-e)re] + [(1-e)r(1-e) + (1-e)rer(1-e)] + q$, where $q \in Nil(R)$. Since $e_1 = ere + er(1-e)re = e(r + r(1-e)r)e \in eRe$ and

$e_2 = (1-e)r(1-e)+(1-e)rer(1-e) = (1-e)(r+rer)(1-e) \in (1-e)R(1-e)$ are both idempotents whose product $e_1.e_2 = e_2.e_1$ is zero, one can conclude that $e_1 + e_2 = e'$ is again an idempotent. Consequently, since $r = e' + q$ with $e' \in Id(R)$ and $q \in Nil(R)$, we finally obtain by definition that R is nil-clean, as claimed. □

Remark 2.4. It is worthwhile noticing that it cannot be expected such a ring R to be strongly nil-clean. In fact, it was demonstrated in [4] that every unit in a strongly nil-clean ring must be a unipotent. However, in the matrix ring $\mathbb{M}_2(\mathbb{F}_2)$ over the boolean ring $\mathbb{F}_2$, which is actually a field, the matrix unit $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ cannot be a unipotent because the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is never a nilpotent. In fact, in other words $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is a unit with inverse $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

Moreover, it is fairly clear in the proof of Lemma 2.3 that $e'(t-f) \neq (t-f)e'$, so that this once again substantiates our claim above that R need not be strongly nil-clean.

We are now ready to deduce one of our main statements. Specifically, the following statement is true:

Theorem 2.5. *Suppose that R is a ring with $e \in Id(R)$ for which eRe and $(1-e)R(1-e)$ are both strongly nil-clean rings. Then $R/J(R)$ is a nil-clean ring.*

Proof. According to either [3] or [6], accomplished with [7], for any $h \in Id(R)$, we derive that the quotient ring $hRh/J(hRh) = hRh/hJ(R)h \cong h'(R/J(R))h'$ with $h' = h + J(R) \in Id(R/J(R))$, is boolean. So, Lemma 2.3 applies to get that $R/J(R)$ is nil-clean, as expected. □

A direct consequence is the following one.

Corollary 2.6. *Suppose that R is a ring with nil Jacobson radical. If both eRe and $(1-e)R(1-e)$ are strongly nil-clean rings, then R is nil-clean.*

Proof. Combining Theorem 2.5 and [4], we are set. □

As other valuable consequences we derive the following assertions. Before doing that, we need the following key formula.

Lemma 2.7. *For every ring R and every idempotent e the following equality is valid:*

$$P(eRe) = eP(R)e.$$

Proof. First, observe that if P is any prime ideal of R, then either $ePe = eRe$, or ePe is a prime ideal of eRe. Hence, $eP(R)e$ is an intersection of some of the prime ideals of eRe, so it is a semiprime ideal of eRe. This shows that $P(eRe) \subseteq eP(R)e$.

To get the reverse inclusion, it is enough to show that $eP(R)e \subseteq Q$ for any prime ideal Q of eRe. We shall obtain this by showing that $Q = ePe$ for some prime ideal P of R. To prove that, notice that the set $X = eRe \setminus Q$ is what McCoy called in [8] an "m-system" of eRe: it is nonempty, and for any $x, y \in X$, there is some $a \in eRe$ such that $xay \in X$. Note that X is also an m-system in R, and that X is disjoint from the ideal RQR. Let $P \supseteq RQR$ be an ideal maximal with respect to being disjoint from X. In [8] was proved that any such ideal must be prime. Since P is disjoint from X, we must have $P \cap eRe = Q$, and therefore $ePe = Q$, as wanted. □

Remark 2.8. The same formula can also be easily deduced from [7, Exercises 10.17, 10.18(A)]. For a quick outline, Exercise 10.17 says that $P(R)$ for each ring R is just the set of "strongly nilpotent elements" in R, which is usually attributed to Levitzki; recall that an element a is called *strongly nilpotent* if there exists a non-negative integer k such that $ar_1ar_2a \cdots ar_{k-1}a = 0$ for all choices of $r_i \in R$. Using this, it follows as in Exercise 10.18(A) that $eRe \cap P(R) \subseteq P(eRe)$. Here, $eRe \cap P(R)$ is trivially seen to be just $eP(R)e$, so we have already $eP(R)e \subseteq P(eRe)$. As for the reverse inclusion, we just apply Exercise 10.17 again with a small twist, and thus we are done.

We now have all the information needed to prove the following.

Theorem 2.9. *Suppose that R is a ring with $e \in Id(R)$ for which eRe and $(1-e)R(1-e)$ are both 2-primal strongly nil-clean rings. Then R is nil-clean.*

Proof. Firstly, we shall show that if f is either e or $1-e$, then $Nil^*(fRf) = fNil^*(R)f$. In fact, since $P(R) \subseteq Nil^*(R) \subseteq J(R)$, with Lemma 2.7 at hand combined with the fact from [4] that strongly nil-clean rings have nil Jacobson radicals, we deduce that $Nil^*(fRf) = P(fRf) = fP(R)f \subseteq fNil^*(R)f \subseteq fJ(R)f = J(fRf) \subseteq Nil^*(fRf)$, as desired. In particular, $J(fRf) = Nil^*(fRf)$.

Further, by what we have obtained above, one sees by [3] or [11] that the factor-ring $fRf/Nil^*(fRf) = fRf/fNil^*(R)f \cong f'(R/Nil^*(R))f'$ with $f' = f + Nil^*(R) \in Id(R/Nil^*(R))$, is boolean. Now Lemma 2.3 allows us to infer that $R/Nil^*(R)$ is nil-clean. Hence, again by [4], we conclude that R is nil-clean, as stated. □

As a direct consequence, we also arrive at the following.

Corollary 2.10. *Suppose that R is a ring with $e \in Id(R)$ for which eRe and $(1-e)R(1-e)$ are both commutative nil-clean rings. Then R is nil-clean.*

Remark 2.11. As a different proof of Theorem 2.9 we may also use Theorem 2.5 by deriving also that $J(R)$ is nil.

Using ordinary induction arguments in the key Lemma 2.3, all statements concerning corners eRe and $(1-e)R(1-e)$ can be expanded to a system of mutually orthogonal idempotents $\{e_i\}_{i=1}^n$ with $1 = e_1 + \cdots + e_n$ such that all corners e_iRe_i are as above in the case of two idempotents (compare with [5], too).

With this at hand, as an immediate pivotal consequence, we now yield the generalization of [6, Theorem 6.1] discussed above.

Corollary 2.12. *Let R be a 2-primal strongly nil-clean ring. Then $\mathbb{M}_n(R)$ is nil-clean for each $n \geq 1$.*

Proof. Knowing that $R \cong E_{11}\mathbb{M}_n(R)E_{11} \cong \cdots \cong E_{nn}\mathbb{M}_n(R)E_{nn}$ for any $n \geq 1$, where $\{E_{ii}\}_{i=1}^n$ forms a complete system of matrix idempotents (i.e., a set of matrix orthogonal idempotents with sum 1), it suffices to apply the generalized form of Theorem 2.9 to get the wanted claim. □

As a direct consequence, we obtain an independent direct verification of [1, Corollary 7] as promised above.

Corollary 2.13. *Let R be a commutative nil-clean ring. Then, for any $n \geq 1$, $\mathbb{M}_n(R)$ is nil-clean.*

3. Left-open problems

We close this work with two problems of some interest.

Problem 1. If R is clean, respectively nil-clean, and $e \in Id(R)$, does it follow that eRe is also clean, respectively nil-clean, provided eRe is commutative?

Problem 2. If R is a ring and $e \in Id(R)$ for which eRe and $(1-e)R(1-e)$ are both strongly nil-clean, is it true that R is nil-clean?

In particular, if R is strongly nil-clean, is then $\mathbb{M}_n(R)$ nil-clean?

Notice that these queries can be settled at once in the affirmative, assuming that the formula $Nil^*(eRe) = eNil^*(R)e$ is true for any ring R and any idempotent e. However, this theme is closely related to the well-known famous Köthe's conjecture. In fact, all one can say – unless the Köthe Conjecture is proved – is that $eNil^*(R)e \subseteq Nil^*(eRe)$. One condition equivalent to the conjecture is that if I is a nil-ideal of a ring S, then $\mathbb{M}_n(I)$ is a nil-ideal of $\mathbb{M}_n(S)$. If this fails, there should be a ring S with $I = Nil^*(S)$ nil but $\mathbb{M}_n(I)$ not nil for some $n \in \mathbb{N}$. Furthermore, we are able to arrange this so that $Nil^*(\mathbb{M}_n(S)) = 0$, and then we get a negative answer with $e = E_{11}$.

Moreover, we know in view of [4, Example 4.5] that $\mathbb{M}_n(R)$ need not be strongly nil-clean.

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