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DECAY ESTIMATES OF SOLUTIONS TO THE IBQ EQUATION

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ABSTRACT. In this paper we focus on the Cauchy problem for the generalized IBq equation with damped term in n-dimensional space. We establish the global existence and decay estimates of solution with $L^q(1 \le q \le 2)$ initial value, provided that the initial value is suitably small. Moreover, we also show that the solution is asymptotic to the solution u_L to the corresponding linear equation as time tends to infinity. Finally, asymptotic profile of the solution u_L to the linearized problem is also discussed.
Keywords: IBq equation, global existence, decay estimates, asymptotic profile.
MSC(2010): Primary: 35L30; Secondary: 35L75.

1. **Introduction**

We investigate the Cauchy problem for the following generalized improved Boussinesq (IBq) equation with damped term

$$u_{tt} - \Delta u_{tt} - \Delta u - \nu \Delta u_t = \Delta \Psi(u), \tag{1.1}$$

with the initial value

$$t = 0 : u = u_0(x), \;\; u_t = u_1(x). \tag{1.2}$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $t > 0$, ν is a positive constant. $\Psi(u)$ is a smooth nonlinear function with $\Psi(u) = O(|u|^{1+\sigma})(\sigma \ge 1)$ for $u \to 0$.

It is well known that the IBq equation has the form

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta(u^2). \tag{1.3}$$

Equation (1.3) is an important physical model, which approximately describes the propagation of long waves on shallow water and the dynamical and thermodynamical properties of anharmonic monatomic and diatomic chains [1].

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For the study of well-posedness, some interesting results have been established (see [14, 15]).

Owing to irreversible processes taking place within the system, the dissipation function depends on the time derivatives of the relative displacements, the authors took into account internal friction (it is called this type of friction hydrodynamical) and derived IBq equation with damped term (1.1) in [1].

Polat [11] established global existence and blow-up of solutions to (1.1)-(1.2). By the contraction mapping principle and the sharp decay estimates for the linearized problem, global existence and asymptotic behavior of solutions to (1.1)-(1.2) was established by Wang and Hu [17] provided that the initial data is suitably small. In this paper, our main purpose is to investigate sufficient conditions on global existence of solutions and obtain the optimal decay estimate of solutions to the problem (1.1)-(1.2). Moreover, we prove that our solution u is asymptotic to the solution u_L to the linearized problem and a simpler asymptotic profile of u_L is derived. The results obtained in this paper refine those in [17]. For the details see Theorem 3.2, Theorem 4.1 and Corollary 3.3-Corollary 3.6.

The study of the global existence and asymptotic behavior of solutions to wave equation with damped term has attracted lots of mathematicians' interests. We may refer to [8–10, 21] for damped wave equations and [4, 5, 7, 12, 16, 18–20, 22, 23, 25] and the references therein.

The plan of the paper is as follows. We investigate the decay property of the solution operators to (1.1)-(1.2) in Section 2. Then, in Sections 3, we prove the global existence and asymptotic decay of solutions for initial data in the space $L^q(1 \le q \le 2)$. Linear approximation of solutions is also discussed. Finally, we derive a simpler asymptotic profile which gives the approximation to the linear solution.

Notations. The Fourier transform of f is defined by

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx.$$

Its inverse transform is denoted by $\mathcal{F}^{-1}$. For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. The nonhomogeneous Sobolev space of order s is defined by $W^{s,p} = (I - \partial_x^2)^{-\frac{s}{2}} L^p$ with the norm $\|f\|_{W^{s,p}} = \|(I - \partial_x^2)^{\frac{s}{2}} f\|_{L^p}$. The homogeneous Sobolev space of order s is defined by $\dot{W}^{s,p} = (-\partial_x^2)^{-\frac{s}{2}} L^p$ with the norm $\|f\|_{\dot{W}^{s,p}} = \|(-\partial_x^2)^{\frac{s}{2}} f\|_{L^p}$. For various function spaces and notations, we may refer to [2] and [3].

2. **Linear problem**

We shall mainly investigate the decay property of solution operators in this section. Therefore, we firstly need to derive the solution formula for the problem

(1.1)-(1.2). The linearized equation of (1.1) is

$$u_{tt} - \Delta u_{tt} - \Delta u - \nu \Delta u_t = 0. \tag{2.1}$$

We apply the Fourier transform to (2.1)-(1.2). This yields

$$(1+|\xi|^2)\hat{u}_{tt} + \nu|\xi|^2\hat{u}_t + |\xi|^2\hat{u} = 0, \tag{2.2}$$

$$\hat{u}(\xi,0) = \hat{u}_0(\xi), \qquad \hat{u}_t(\xi,0) = \hat{u}_1(\xi). \tag{2.3}$$

The corresponding characteristic equation of (2.2) is

$$(1+|\xi|^2)\lambda^2 + \nu|\xi|^2\lambda + |\xi|^2 = 0.$$

We solve the above equation to get

$$\lambda_\pm(\xi) = \frac{-\nu|\xi|^2 \pm i|\xi|\sqrt{4-(4-\nu^2)|\xi|^2}}{2(1+|\xi|^2)}, \tag{2.4}$$

which are the corresponding eigenvalues. Thus we obtain the solution to the initial value problem for the second order ordinary differential equation (2.2), (2.3)

$$\hat{u}(\xi,t) = \hat{\mathfrak{S}}_1(\xi,t)\hat{u}_1(\xi) + \hat{\mathfrak{S}}_2(\xi,t)\hat{u}_0(\xi), \tag{2.5}$$

where

$$\begin{aligned} \hat{\mathfrak{S}}_1(\xi,t) &= \frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}\big(e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}\big), \\ \hat{\mathfrak{S}}_2(\xi,t) &= \frac{1}{\lambda_+(\xi)-\lambda_-(\xi)}\big(\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}\big). \end{aligned} \tag{2.6}$$

Then, we apply $\mathcal{F}^{-1}$ to (2.5) and get the solution formula to (2.1)-(1.2):

$$u(t) = \mathfrak{S}_1(t) * u_1 + \mathfrak{S}_2(t) * u_0. \tag{2.7}$$

By the Duhamel principle, it is well known that the problem (1.1)-(1.2) is equivalent to the following integral equation

$$u(t) = \mathfrak{S}_1(t) * u_1 + \mathfrak{S}_2(t) * u_0 + \int_0^t \mathfrak{S}_1(t-\tau) * (1-\Delta)^{-1}\Delta\Psi(u)(\tau)d\tau. \tag{2.8}$$

In what follows, we establish the estimate of solutions by energy method in the Fourier space.

Lemma 2.1. *Assume that u is a solution to (2.1)-(1.2). Then, the Fourier transform $\hat{u}$ of the solution u satisfies the following estimate*

$$(1+|\xi|^2)|\hat{u}_t(\xi,t)|^2 + |\xi|^2|\hat{u}(\xi,t)|^2 \le Ce^{-c\omega(\xi)t}\big((1+|\xi|^2)|\hat{u}_1(\xi)|^2 + |\xi|^2|\hat{u}_0(\xi)|^2\big), \tag{2.9}$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$, where $\omega(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.

Proof. We may prove (2.9) by the energy method in the Fourier space. The energy method in Fourier space was first developed in [13] and then used in many papers (see, for example, [5, 12, 19, 20, 25]).

Multiplying (2.2) by $\bar{\hat{u}}_t$ and taking the real part, we have

$$\frac{1}{2}\frac{d}{dt}\{(1+|\xi|^2)|\hat{u}_t|^2+|\xi|^2|\hat{u}|^2\}+\nu|\xi|^2|\hat{u}_t|^2=0. \tag{2.10}$$

We multiply (2.2) by $\bar{\hat{u}}$ and take the real part. This gives

$$\frac{1}{2}\frac{d}{dt}\{\nu|\xi|^2|\hat{u}|^2+2(1+|\xi|^2)Re(\hat{u}_t\bar{\hat{u}})\}+|\xi|^2|\hat{u}|^2-(1+|\xi|^2)|\hat{u}_t|^2=0. \tag{2.11}$$

Multiplying (2.10) and (2.11) by $2(1+|\xi|^2)$ and $\mu|\xi|^2$, respectively, and then, summing up these two equalities, we arrive at

$$\frac{d}{dt}\mathfrak{E}+\mathfrak{F}=0, \tag{2.12}$$

where

$$\mathfrak{E}=(1+|\xi|^2)^2|\hat{u}_t|^2+\{|\xi|^2(1+|\xi|^2)+\frac{1}{2}\nu^2|\xi|^4\}|\hat{u}|^2+\nu|\xi|^2(1+|\xi|^2)Re(\hat{u}_t\bar{\hat{u}}),$$
$$\mathfrak{F}=\nu|\xi|^4|\hat{u}|^2+\nu|\xi|^2(1+|\xi|^2)|\hat{u}_t|^2.$$

A simple computation shows that

$$c(1+|\xi|^2)\mathfrak{E}_0\le\mathfrak{E}\le C(1+|\xi|^2)\mathfrak{E}_0,\qquad \mathfrak{F}\ge c\omega(\xi)\mathfrak{E}_0, \tag{2.13}$$

where $\mathfrak{E}_0=(1+|\xi|^2)|\hat{u}_t|^2+|\xi|^2|\hat{u}|^2$ and $\omega(\xi)=\frac{|\xi|^2}{1+|\xi|^2}$. Consequently, we obtain $\mathfrak{F}\ge c\omega(\xi)\mathfrak{E}$. Substituting this estimate into (2.12), we have $\frac{d}{dt}\mathfrak{E}+c\omega(\xi)\mathfrak{E}\le 0$. This differential inequality is solved as $\mathfrak{E}(\xi,t)\le e^{-c\omega(\xi)t}\mathfrak{E}(\xi,0)$, which together with (2.13) proves the desired estimate (2.9). This completes the proof of Lemma 2.1. □

The estimate (2.9) and the solution formula (2.5) immediately give the following estimates for $\hat{\mathfrak{G}}_1$ and $\hat{\mathfrak{G}}_2$.

Lemma 2.2. *The following estimates*

$$\begin{aligned}&|\hat{\mathfrak{G}}_1(\xi,t)|\le C|\xi|^{-1}(1+|\xi|^2)^{\frac{1}{2}}e^{-c\omega(\xi)t},\\ &|\hat{\mathfrak{G}}_2(\xi,t)|+|\partial_t\hat{\mathfrak{G}}_1(\xi,t)|\le Ce^{-c\omega(\xi)t},\\ &|\partial_t\hat{\mathfrak{G}}_2(\xi,t)|\le C|\xi|(1+|\xi|^2)^{-\frac{1}{2}}e^{-c\omega(\xi)t},\end{aligned} \tag{2.14}$$

hold for $\xi\in\mathbb{R}^n$ and $t\ge 0$, where $\omega(\xi)=\frac{|\xi|^2}{1+|\xi|^2}$.

Thanks to (2.14), we may establish the decay estimates for $\mathfrak{G}_1(t)$ and $\mathfrak{G}_2(t)$.

Lemma 2.3. *Assume that $1 \le q \le 2$ and k, j and l are nonnegative integers. Then*

$$\|\partial_x^k \mathfrak{S}_1(t)*f\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k-j}{2}}\|\partial_x^j f\|_{\dot{W}^{-1,q}}+Ce^{-ct}\|\partial_x^{k+l}f\|_{L^2}, \tag{2.15}$$

$$\|\partial_x^k \mathfrak{S}_2(t)*g\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k-j}{2}}\|\partial_x^j g\|_{L^q}+Ce^{-ct}\|\partial_x^{k+l}g\|_{L^2}, \tag{2.16}$$

for $0 \le j \le k$. Similarly, we have

$$\|\partial_x^k \partial_t \mathfrak{S}_1(t)*f\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1-j}{2}}\|\partial_x^j f\|_{\dot{W}^{-1,q}}+Ce^{-ct}\|\partial_x^{k+l}f\|_{L^2}, \tag{2.17}$$

$$\|\partial_x^k \partial_t \mathfrak{S}_2(t)*g\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1-j}{2}}\|\partial_x^j g\|_{L^q}+Ce^{-ct}\|\partial_x^{k+l}g\|_{L^2}, \tag{2.18}$$

for $0 \le j \le k+1$.

Proof. We only prove (2.15) and the proof of (2.16)-(2.18) is similar. It follows from the Plancherel theorem and the pointwise estimate for $\hat{\mathfrak{S}}_1$ in (2.14) that

$$\begin{aligned}\|\partial_x^k \mathfrak{S}_1(t)*f\|_{L^2}^2 &= \int_{\mathbb{R}^n}|\xi|^{2k}|\hat{\mathfrak{S}}_1(\xi,t)|^2|\hat{f}(\xi)|^2d\xi\\ &\le C\int_{\mathbb{R}^n}|\xi|^{2k-2}(1+|\xi|^2)e^{-c\omega(\xi)t}|\hat{f}(\xi)|^2d\xi\\ &\le C\int_{|\xi|\le r_0}|\xi|^{2k-2}(1+|\xi|^2)e^{-c\omega(\xi)t}|\hat{f}(\xi)|^2d\xi\\ &\quad +C\int_{|\xi|\ge r_0}|\xi|^{2k-2}(1+|\xi|^2)e^{-c\omega(\xi)t}|\hat{f}(\xi)|^2d\xi\\ &=: \mathscr{A}_1+\mathscr{A}_2,\end{aligned} \tag{2.19}$$

where $r_0>0$ is a constant. Firstly, we estimate $\mathscr{A}_1$. It follows from the Hölder inequality and the Hausdorff-Young inequality that

$$\begin{aligned}\mathscr{A}_1 &\le C\int_{|\xi|\le r_0}|\xi|^{2k-2}e^{-c|\xi|^2t}|\hat{f}(\xi)|^2d\xi\\ &\le C\|\,|\xi|^{j-1}\hat{f}\|_{L^{q'}}^2\Big(\int_{|\xi|\le r_0}|\xi|^{2(k-j)p}e^{-cp|\xi|^2t}d\xi\Big)^{\frac{1}{p}}\\ &\le C(1+t)^{-n(\frac{1}{q}-\frac{1}{2})-(k-j)}\|\partial_x^j f\|_{\dot{W}^{-1,q}}^2,\end{aligned} \tag{2.20}$$

where p,q,q' satisfy $\frac{1}{q'}+\frac{1}{q}=1$ and $\frac{2}{q'}+\frac{1}{p}=1$.

In what follows, we may estimate $\mathscr{A}_2$ as

$$\begin{aligned}\mathscr{A}_2 &\le Ce^{-ct}\int_{|\xi|\ge r_0}|\xi|^{2k}|\hat{f}(\xi)|^2d\xi \\ &\le Ce^{-ct}\int_{|\xi|\ge r_0}|\xi|^{2(k+l)}|\hat{f}(\xi)|^2d\xi \le Ce^{-ct}\|\partial_x^{k+l}f\|_{L^2}^2.\end{aligned} \tag{2.21}$$

We insert (2.20) and (2.21) into (2.19) and immediately obtain the desired estimate (2.15). Thus, the proof of Lemma 2.3 is completed. □

We also have the following decay estimate for $\mathfrak{S}_1(t)*(1-\Delta)^{-1}\Delta\Psi$.

Lemma 2.4. *Let $1\le q\le 2$, and let k, j and l be nonnegative integers. Then we have*

$$\|\partial_x^k\mathfrak{S}_1(t)*(1-\Delta)^{-1}\Delta\Psi\|_{L^2}\le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1-j}{2}}\|\partial_x^j\Psi\|_{L^q}+Ce^{-ct}\|\partial_x^{k+l}\Psi\|_{L^2} \tag{2.22}$$

and

$$\|\partial_x^k\partial_t\mathfrak{S}_1(t)*(I-\Delta)^{-1}\Delta\Psi\|_{L^2}\le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+2-j}{2}}\|\partial_x^j\Psi\|_{L^q}+Ce^{-ct}\|\partial_x^{k+l}\Psi\|_{L^2}, \tag{2.23}$$

where $0\le j\le k+1$ in (2.22) and $0\le j\le k+2$ in (2.23).

Proof. The proof is essentially the same as that of Lemma 2.3. Here we omit the details of the proof. □

3. **Decay estimates**

In this section, we shall prove the global existence and asymptotic decay of solutions to the nonlinear problem (1.1)-(1.2). We need the following lemma for composite functions, which can be found in [6, 24].

Lemma 3.1. *Assume that $f=f(v)$ is a smooth function satisfying $f(v)=O(|v|^{1+\sigma})$ for $v\to 0$, where $\sigma\ge 1$ is an integer. Let $v\in L^\infty$ and $\|v\|_{L^\infty}\le M_0$ for a positive constant M_0. Let $1\le p,\ q,\ r\le +\infty$ and $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$, and let $k\ge 0$ be an integer. Then, we have*

$$\|\partial_x^kf(v)\|_{L^p}\le C\|v\|_{L^\infty}^{\sigma-1}\|v\|_{L^q}\|\partial_x^kv\|_{L^r},$$

where $C=C(M_0)$ is a constant depending on M_0.

In this subsection, we shall prove the decay estimate of solutions to the problem (1.1)-(1.2) with $L^q(1\le q\le 2)$ initial data. The result is stated as follows.

Theorem 3.2. *Assume that $\sigma\ge 1$, $q\in[1,2]$, $n\sigma\ge q$, $s\ge[n/2]+1$, $u_0\in H^s\cap L^q$, $u_1\in H^s\cap\dot{W}^{-1,q}$ and put $N_1=\|u_0\|_{H^s\cap L^q}+\|u_1\|_{H^s\cap\dot{W}^{-1,q}}$. Then there exists a constant $\epsilon_1>0$ such that if $N_1\le\epsilon_1$, then the problem (1.1)-(1.2)* *has a unique global solution u with $u \in C^0([0,+\infty); H^s) \cap C^1([0,+\infty); H^s)$. Moreover, the solution satisfies the decay estimates*

$$\|\partial_x^k u(t)\|_{L^2} \le CN_1(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}, \tag{3.1}$$

$$\|\partial_x^k u_t(t)\|_{L^2} \le CN_1(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}\eta(t), \tag{3.2}$$

where $0 \le k \le s$ in (3.1) and (3.2) and

$$\eta(t) = \begin{cases} \log(2+t), & n\sigma = q, \\ (1+t)^{-\frac{\sigma n}{2q}+\frac{1}{2}}\log(2+t), & q < n\sigma \le 2q, \\ (1+t)^{-\frac{1}{2}}, & n\sigma > 2q. \end{cases} \tag{3.3}$$

Proof. We shall prove Theorem 3.2 by the contraction mapping principle. For this purpose, we define the function space

$$X = \{u \in C^0([0,+\infty); H^s) : \|u\|_X < \infty\},$$

where

$$\|u\|_X = \sum_{k=0}^{s} \sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{2}(\frac{1}{q}-\frac{1}{2})+\frac{k}{2}}\|\partial_x^k u(\tau)\|_{L^2}. \tag{3.4}$$

For any $R > 0$, we also define

$$X_R = \{u \in X : \|u\|_X \le R\}.$$

Here R depends on the norm of the initial value, which is chosen in the proof of Theorem 3.2. For any $u \in X_R$, we obtain by using the Gagliardo-Nirenberg inequality

$$\|u\|_{L^\infty} \le C\|\partial_x^{s_0}u\|_{L^2}^{\theta}\|u\|_{L^2}^{1-\theta},$$

where $s_0 = [n/2]+1$ and $\theta = \frac{n}{2s_0}$. By the definition of X_R, we have

$$\|u(t)\|_{L^\infty} \le CR(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{n}{4}}, \tag{3.5}$$

provided that $s \ge [n/2]+1$.

For any $u \in X_R$, we define the mapping

$$\Phi(u) = \mathfrak{S}_1(t) * u_1 + \mathfrak{S}_2(t) * u_0 + \int_0^t \mathfrak{S}_1(t-\tau) * (1-\Delta)^{-1}\Delta\Psi(u)(\tau)d\tau. \tag{3.6}$$

Now, by applying ∂_x^k to (3.6) and taking the L^2 norm, we arrive at

$$\begin{aligned} \|\partial_x^k\Phi(u)(t)\|_{L^2} \le\ & \|\partial_x^k\mathfrak{S}_1(t) * u_1\|_{L^2} + \|\partial_x^k\mathfrak{S}_2(t) * u_0\|_{L^2} \\ & + \int_0^t \|\partial_x^k\mathfrak{S}_1(t-\tau) * (1-\Delta)^{-1}\Delta\Psi(u)(\tau)\|_{L^2}d\tau \\ =:\ & I + J + K, \end{aligned} \tag{3.7}$$

where $0 \le k \le s$.

In what follows, we estimate I, J, K, respectively. Firstly, by (2.15) with $j = 0$ and $l = 0$, we obtain

$$I \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}\|u_1\|_{\dot{W}^{-1,q}} + e^{-ct}\|\partial_x^k u_1\|_{L^2} \le CN_1(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}.$$

For the term J, we apply (2.16) with $j = 0$ and $l = 0$ to get

$$J \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}\|u_0\|_{L^q} + e^{-ct}\|\partial_x^k u_0\|_{L^2} \le CN_1(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}.$$

To estimate the term K. We firstly make estimates for the nonlinear term. By Lemma 3.1, the Gagliardo-Nirenberg inequality and (3.5), we deduce that

$$\begin{aligned} \|\Psi(u)(\tau)\|_{L^q} &\le C\|u\|_{L^\infty}\|u\|_{L^{2q}}^2 \le C\|u\|_{L^\infty}^{\sigma-1+\frac{2(q-1)}{q}}\|u\|_{L^2}^{\frac{2}{q}} \\ &\le CR^{1+\sigma}(1+\tau)^{-(\sigma-1)\left(\frac{n}{2}(\frac{1}{q}-\frac{1}{2})+\frac{n}{4}\right)-\frac{n}{2q}} \\ &\le CR^{1+\sigma}(1+\tau)^{-\frac{\sigma n}{2q}}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \|\partial_x^k\Psi(u)(\tau)\|_{L^2} &\le C\|u\|_{L^\infty}^{\sigma}\|\partial_x^k u\|_{L^2} \\ &\le CR^{1+\sigma}(1+\tau)^{-(\sigma+1)\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{n\sigma}{4}-\frac{k}{2}} \\ &\le CR^{1+\sigma}(1+\tau)^{-\frac{\sigma n}{2q}-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}. \end{aligned} \tag{3.9}$$

We divide K into two parts and write $K = K_1 + K_2$, where K_1 and K_2 are corresponding to the time intervals $[0, t/2]$ and $[t/2, t]$, respectively. Applying (2.22) with $j = 0$ and $l = 0$ to the term K_1 and using (3.8)-(3.9), we arrive at

$$\begin{aligned} K_1 &\le C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}\|\Psi(u)(\tau)\|_{L^q}d\tau \\ &\quad + C\int_0^{t/2}e^{-c(t-\tau)}\|\partial_x^k\Psi(u)(\tau)\|_{L^2}d\tau \\ &\le CR^{1+\sigma}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}\int_0^{t/2}(1+\tau)^{-\frac{n\sigma}{2q}}d\tau + CR^{1+\sigma}e^{-ct} \\ &\le \begin{cases} CR^{1+\sigma}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}(1+t)^{-\frac{\sigma n}{2q}+\frac{1}{2}}, & n\sigma \ne 2q, \\ CR^{1+\sigma}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}\log(2+t)(1+t)^{-\frac{1}{2}}, & n\sigma = 2q. \end{cases} \end{aligned} \tag{3.10}$$

Finally, we estimate the term K_2 on the time interval $[t/2, t]$. We apply (2.22) with $q = 2$, $j = k$ and $l = 0$ and use (3.9). This yields

$$\begin{aligned} K_2 &\le C\int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^k\Psi(u)(\tau)\|_{L^2}d\tau \\ &\le CR^{1+\sigma}\int_{t/2}^{t}(1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{\sigma n}{2q}-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}d\tau \\ &\le CR^{1+\sigma}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}(1+t)^{-\frac{n\sigma}{2q}+\frac{1}{2}}. \end{aligned} \tag{3.11}$$

By inserting all these estimates into (3.7), we obtain

$$\|\Phi(u)\|_X \leq CN_1 + CR^{1+\sigma}.$$

Consequently, we take $R = 2CN_1$ and obtain

$$\|\Phi(u)\|_X \leq 2CN_1 = R,$$

provided that N_1 is suitably small. This implies that $\Phi(u) \in X_R$, provided that $u \in X_R$.

For any $\bar{u}, \bar{\bar{u}} \in X_R$, we obtain from (3.6)

$$\Phi(\bar{u}) - \Phi(\bar{\bar{u}}) = \int_0^t \mathfrak{S}_1(t-\tau) * (1-\Delta)^{-1}\Delta\Big(\Psi(\bar{u}) - \Psi(\bar{\bar{u}})\Big)(\tau)d\tau. \tag{3.12}$$

To estimate $\|\Phi(\bar{u}) - \Phi(\bar{\bar{u}})\|_{X_R}$, we make estimate for the following nonlinear term. By Lemma 3.1 and 3.5, we arrive at

(3.13)

$$\begin{aligned}
&\|\Phi(\bar{u}) - \Phi(\bar{\bar{u}})\|_{L^q}\\
&\quad\leq C(\|\bar{u}\|_{L^\infty} + \|\bar{\bar{u}}\|_{L^\infty})^{\sigma-1}(\|\bar{u}\|_{L^{\frac{2q}{2-q}}} + \|\bar{\bar{u}}\|_{L^{\frac{2q}{2-q}}})\|\bar{u} - \bar{\bar{u}}\|_{L^2}\\
&\quad\leq C(\|\bar{u}\|_{L^\infty} + \|\bar{\bar{u}}\|_{L^\infty})^{\sigma-1}(\|\bar{u}\|_{L^\infty}^{2-\frac{2}{q}}\|\bar{u}\|_{L^2}^{\frac{2}{q}-1} + \|\bar{\bar{u}}\|_{L^\infty}^{2-\frac{2}{q}}\|\bar{\bar{u}}\|_{L^2}^{\frac{2}{q}-1})\|\bar{u} - \bar{\bar{u}}\|_{L^2}\\
&\quad\leq CR^\sigma(1+\tau)^{-(1+\sigma)\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{n}{4}(\sigma+1-\frac{2}{q})}\|\bar{u} - \bar{\bar{u}}\|_X\\
&\quad\leq CR^\sigma(1+\tau)^{-\frac{\sigma n}{2q}}\|\bar{u} - \bar{\bar{u}}\|_X
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_x^k\Big(\Phi(\bar{u}) - \Phi(\bar{\bar{u}})\Big)\|_{L^2}\\
&\qquad\leq C(\|\bar{u}\|_{L^\infty} + \|\bar{\bar{u}}\|_{L^\infty})^{\sigma-1}\Big\{(\|\bar{u}\|_{L^\infty} + \|\bar{\bar{u}}\|_{L^\infty})\|\partial_x^k(\bar{u} - \bar{\bar{u}})\|_{L^2}\\
&\qquad\quad+(\|\partial_x^k\bar{u}\|_{L^\infty} + \|\partial_x^k\bar{\bar{u}}\|_{L^\infty})\|\bar{u} - \bar{\bar{u}}\|_{L^2}\Big\}\\
&\qquad\leq CR^\sigma(1+\tau)^{-(1+\sigma)\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{n}{4}\sigma-\frac{k}{2}}\|\bar{u} - \bar{\bar{u}}\|_X\\
&\qquad\leq CR^\sigma(1+\tau)^{-\frac{\sigma n}{2q}-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}\|\bar{u} - \bar{\bar{u}}\|_X.
\end{aligned} \tag{3.14}$$

It follows from (2.22), (3.13) and (3.14) that

(3.15)

$$
\begin{aligned}
&\|\partial_x^k\Big(\Phi(\bar{u})-\Phi(\bar{\bar{u}})\Big)\|_{L^2}\\
\leq\ & C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}\|\Phi(\bar{u})-\Phi(\bar{\bar{u}})\|_{L^q}d\tau\\
&+C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}\|\partial_x^k(\Phi(\bar{u})-\Phi(\bar{\bar{u}}))\|_{L^2}d\tau\\
&+C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}\|\partial_x^k(\Phi(\bar{u})-\Phi(\bar{\bar{u}}))\|_{L^2}d\tau\\
\leq\ & CR^\sigma\|\bar{u}-\bar{\bar{u}}\|_X(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{\sigma n}{2q}}d\tau\\
&+CR^\sigma e^{-ct}\|\bar{u}-\bar{\bar{u}}\|_X\\
&+CR^\sigma\|\bar{u}-\bar{\bar{u}}\|_X(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}-\frac{\sigma n}{2q}}\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1}{2}}d\tau\\
\leq\ & \begin{cases} CR^\sigma(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}(1+t)^{-\frac{\sigma n}{2q}+\frac{1}{2}}\|\bar{u}-\bar{\bar{u}}\|_X, & n\sigma\neq 2q,\\ CR^\sigma(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}}\log(2+t)(1+t)^{-\frac{1}{2}}\|\bar{u}-\bar{\bar{u}}\|_X, & n\sigma=2q.\end{cases}
\end{aligned}
$$

Equation (3.15) implies that

$$\|\Phi(\bar{u})-\Phi(\bar{\bar{u}})\|_{X_R}\leq CR^\sigma\leq\frac{1}{2}\|\bar{u}-\bar{\bar{u}}\|_X,$$

provided that $n\sigma\geq q$ and N_1 is suitably small. Therefore, according to the contraction mapping principle, the problem (1.1)-(1.2) has a unique global solution u with $u\in X_R$. Moreover, the solution u satisfies (3.1).

In what follows, we prove $u_t\in C([0,+\infty);H^s)$ and (3.2). For this purpose, we differentiate (2.8) with respect to t to obtain

(3.16)

$$u_t(t)=\partial_t\mathfrak{S}_1(t)*u_1+\partial_t\mathfrak{S}_2(t)*u_0+\int_0^t\partial_t\mathfrak{S}_1(t-\tau)*(1-\Delta)^{-1}\Delta\Psi(u)(\tau)d\tau.$$

We apply ∂_x^k to (3.16) and take the L^2 norm to get

$$
\begin{aligned}
&\|\partial_x^k u_t(t)\|_{L^2}\leq\|\partial_x^k\partial_t\mathfrak{S}_1(t)*u_1\|_{L^2}+\|\partial_x^k\partial_t\mathfrak{S}_2(t)*u_0\|_{L^2}\\
&\quad+\int_0^t\|\partial_x^k\partial_t\mathfrak{S}_1(t-\tau)*(1-\Delta)^{-1}\Delta\Psi(u)(\tau)\|_{L^2}d\tau=:I'+J'+K',
\end{aligned}\tag{3.17}
$$

where $0\leq k\leq s$. By using (2.17) and (2.18) with $j=0$ and $l=0$, respectively, we obtain

$$I_1'\leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}\|u_1\|_{\dot{W}^{-1,q}}+e^{-ct}\|\partial_x^k u_1\|_{L^2}\leq CN_1(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}.$$

and

$J' \leq C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}\|u_0\|_{L^1} + e^{-ct}\|\partial_x^k u_0\|_{L^2} \leq CN_1(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}.$

In what follows, we estimate K' and divide K' into two sections, the first section and second section is correspond to the time intervals $[0, t/2]$ and $[t/2, t]$, respectively. Equation (2.23) with $j = 0$ and $l = 0$ and equations (3.8)-(3.9) give

$$
\begin{aligned}
K' \leq\ & C\int_0^{t/2}(1+t-\tau)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+2}{2}}\|\Psi(u)(\tau)\|_{L^q}d\tau + C\int_0^{t/2}e^{-c(t-\tau)}\|\partial_x^k\Psi(u)(\tau)\|_{L^2}d\tau \\
& + C\int_{t/2}^{t}(1+t-\tau)^{-1}\|\partial_x^k\Psi(u)(\tau)\|_{L^2}d\tau \\
\leq\ & CR^{1+\sigma}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+2}{2}}\int_0^{t/2}(1+t-\tau)^{-\frac{\sigma n}{2q}}d\tau + CR^{1+\sigma}e^{-ct} \\
& + CR^{1+\sigma}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k}{2}-\frac{\sigma n}{2q}}\int_{t/2}^{t}(1+t-\tau)^{-1}d\tau \\
\leq\ & CR^{1+\sigma}(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1}{2}}\eta(t),
\end{aligned}
$$

where the function $\eta(t)$ is defined by (3.3). Combining the above estimates and noting $R = 2CN_1$, we obtain $u_t \in C([0,+\infty); H^s)$ the desired estimate (3.2) for $0 \leq k \leq s$. This completes the proof of Theorem 3.2. □

The proof of Theorem 3.2 immediately implies that the following corollaries hold.

Corollary 3.3. *Assume that $n\sigma \geq 1$, $s \geq [n/2]+1$, $u_0 \in H^s \cap L^1$, $u_1 \in H^s \cap \dot{W}^{-1,1}$ and put $N_1 = \|u_0\|_{H^s\cap L^1} + \|u_1\|_{H^s\cap\dot{W}^{-1,1}}$. Then there exists a constant $\epsilon_2 > 0$ such that if $N_1 \leq \epsilon_2$, then the problem (1.1)-(1.2) has a unique global solution u with $u \in C^0([0,+\infty); H^s) \cap C^1([0,+\infty); H^s)$. Moreover, for $0 \leq k \leq s$, the solution satisfies the decay estimates*

$$\|\partial_x^k u(t)\|_{L^2} \leq CN_1(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \tag{3.18}$$

$$\|\partial_x^k u_t(t)\|_{L^2} \leq CN_1(1+t)^{-\frac{n}{4}-\frac{k+1}{2}}\eta_1(t), \tag{3.19}$$

where $\eta_1(t)$ is given by (3.3) with $q = 1$.

Corollary 3.4. *Assume that $n\sigma \geq 2$, $s \geq [n/2]+1$, $u_0 \in H^s$, $u_1 \in H^s \cap \dot{H}^{-1}$ and put $N_1 = \|u_0\|_{H^s} + \|u_1\|_{H^s\cap\dot{H}^{-1}}$. Then there exists a constant $\epsilon_3 > 0$ such that if $N_1 \leq \epsilon_3$, then the problem (1.1)-(1.2) admits a unique global solution u with $u \in C^0([0,+\infty); H^s) \cap C^1([0,+\infty); H^s)$. Moreover, for $0 \leq k \leq s$, the solution satisfies the decay estimates*

$$\|\partial_x^k u(t)\|_{L^2} \leq CN_1(1+t)^{-\frac{k}{2}}, \tag{3.20}$$

$$\|\partial_x^k u_t(t)\|_{L^2} \leq CN_1(1+t)^{-\frac{k+1}{2}}\eta_2(t), \tag{3.21}$$

where $\eta_2(t)$ is given by (3.3) with $q = 2$.

The estimates of K_1 and K_2 in (3.10) and (3.11) immediately imply the following corollaries hold.

Corollary 3.5. *Let $n\sigma \geq 2$ and assume that the same conditions of Corollary 3.3 hold. Then the solution u of the problem (1.1)-(1.2), which is constructed in Corollary 3.3, can be approximated by the solution u_L to the linearized problem (2.1)-(1.2) as $t \to \infty$. More precisely, we have the following asymptotic relations for $0 \leq k \leq s$:*

$$\|\partial_x^k (u - u_L)(t)\|_{L^2} \leq C N_1^{1+\sigma} (1+t)^{-\frac{n}{4}-\frac{k}{2}} \eta_3(t),$$
$$\|\partial_x^k (u - u_L)_t(t)\|_{L^2} \leq C N_1^{1+\sigma} (1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \eta(t)$$

where $u_L(t) := \mathfrak{S}_1(t) * u_1 + \mathfrak{S}_2(t) * u_0$ is the linear solution, $\eta(t)$ are defined in (3.3) with $q = 1$ and $\eta_3(t)$ is defined by

$$\eta_3(t) = \begin{cases} (1+t)^{-\frac{\sigma n}{2}+\frac{1}{2}} \log(2+t), & n\sigma \neq 2, \\ (1+t)^{-\frac{1}{2}} \log(2+t), & n\sigma = 2. \end{cases} \tag{3.22}$$

Corollary 3.6. *Let $n\sigma \geq 3$ and assume that the same conditions of Corollary 3.4 hold. Then the solution u of the problem (1.1)-(1.2), which is constructed in Corollary 3.4, is asymptotic to the solution u_L to the problem (2.1)-(1.2) as $t \to \infty$. In fact, for $0 \leq k \leq s$, we have*

$$\|\partial_x^k (u - u_L)(t)\|_{L^2} \leq C N_1^{1+\sigma} (1+t)^{-\frac{k}{2}} \eta_4(t),$$
$$\|\partial_x^k (u - u_L)_t(t)\|_{L^2} \leq C N_1^{1+\sigma} (1+t)^{-\frac{k+1}{2}} \eta(t),$$

where $u_L(t) := \mathfrak{S}_1(t) * u_1 + \mathfrak{S}_2(t) * u_0$ is the linear solution and $\eta(t)$ are defined by (3.3) with $q = 2$ and $\eta_4(t)$ is defined by

$$\eta_4(t) = \begin{cases} (1+t)^{-\frac{\sigma n}{4}+\frac{1}{2}} \log(2+t), & n\sigma \neq 4, \\ (1+t)^{-\frac{1}{2}} \log(2+t), & n\sigma = 4. \end{cases} \tag{3.23}$$

4. Asymptotic profile of solutions to the linear problem

In this section, our aim is to build a simpler asymptotic profile of the solution u_L.

Noting that $u_L(t) = \mathfrak{S}_1(t) * u_1 + \mathfrak{S}_2(t) * u_0$. In the Fourier space, we get $\hat{u}_L(\xi, t) = \hat{\mathfrak{S}}_1(\xi, t)\hat{u}_1(\xi) + \hat{\mathfrak{S}}_2(\xi, t)\hat{u}_0(\xi)$. It follows from the mean value theorem that

$$
(4.1)\qquad \begin{cases} e^{-\frac{\nu|\xi|^2 t}{2(1+|\xi|^2)}} = e^{-\frac{\nu}{2}|\xi|^2 t} + \bar{K}_1, \\ \sin \dfrac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2} = \sin(|\xi|t) + \bar{K}_2, \\ \cos \dfrac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2} = \cos(|\xi|t) + \bar{K}_3, \\ \frac{1}{\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}} = 1 + \bar{K}_4, \end{cases}
$$

where

$$
\bar{K}_1 = \frac{\nu|\xi|^4 t}{2(1+|\xi|^2)} e^{-\frac{\nu}{2}|\xi|^2[\frac{\theta_1}{1+|\xi|^2}+(1-\theta_1)]t},
$$

$$
\bar{K}_2 = -\frac{|\xi|^3(|\xi|^2+\frac{12-\nu^2}{4})t}{(1+|\xi|^2)(\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}+1+|\xi|^2)} \cos\Big[\frac{|\xi|\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}t}{1+|\xi|^2}\theta_2 + (1-\theta_2)|\xi|t\Big],
$$

$$
\bar{K}_3 = \frac{|\xi|^3(|\xi|^2+\frac{12-\nu^2}{4})t}{(1+|\xi|^2)(\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}+1+|\xi|^2)} \sin\Big[\frac{|\xi|\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}t}{1+|\xi|^2}\theta_3 + (1-\theta_3)|\xi|t\Big],
$$

$$
\bar{K}_4 = \frac{(4-\nu^2)|\xi|^2}{8(1-\frac{4-\nu^2}{4}|\xi|^2\theta_4)^{\frac{3}{2}}},
$$

with $\theta_i (i = 1, 2, 3, 4) \in (0, 1)$.

We only prove the first equality in (4.1). In fact, let

$$
F(x) = e^{-\frac{\nu|\xi|^2}{2}[\frac{x}{1+|\xi|^2}+(1-x)]t}.
$$

We apply the mean value theorem to $F(x)$ and obtain

$$
F(1) - F(0) = F'(\theta_1), \quad \theta_1 \in (0, 1),
$$

which implies the first equality in (4.1) holds.

When $|\xi| \le \epsilon$, where ϵ is a small positive constant, we obtain from (4.1)

$$
\begin{aligned}
\hat{\mathfrak{S}}_1(\xi, t) &= \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \\
&= \frac{1+|\xi|^2}{|\xi|\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}} e^{-\frac{\nu|\xi|^2 t}{2(1+|\xi|^2)}} \sin \frac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2}. \\
&= \frac{1}{|\xi|} e^{-\frac{\nu}{2}\xi^2 t} \sin|\xi|t + \bar{J}_1.
\end{aligned}
$$

and

$$\begin{aligned}\hat{\mathfrak{S}}_2(\xi,t) &= \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} \\ &= \frac{v|\xi|}{2\sqrt{1-\frac{4-\nu^2}{4}|\xi|^2}} e^{-\frac{\nu|\xi|^2 t}{2(1+|\xi|^2)}} \sin\frac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2} + \\ & e^{-\frac{\nu|\xi|^2 t}{2(1+|\xi|^2)}} \cos\frac{|\xi|\sqrt{1-\frac{4-\nu^2}{|\xi|^2}|\xi|^2}t}{1+|\xi|^2} \\ &= e^{-\frac{\nu}{2}\xi^2 t}\cos|\xi|t + \bar{J}_2.\end{aligned}$$

When $|\xi| \le \epsilon$, $\bar{J}_1$ and $\bar{J}_2$ satisfy

$$|\bar{J}_1| \le C(1+|\xi|^2 t)e^{-c|\xi|^2 t},$$

and

$$|\bar{J}_2| \le C(|\xi|+|\xi|^3 t)e^{-c|\xi|^2 t}.$$

Taking

$$\hat{\mathfrak{S}}_1^0(\xi,t) = \frac{1}{|\xi|}e^{-\frac{\nu}{2}|\xi|^2 t}\sin|\xi|t, \qquad \hat{\mathfrak{S}}_2^0(\xi,t) = e^{-\frac{\nu}{2}|\xi|^2 t}\cos|\xi|t. \tag{4.2}$$

Then

$$|(\hat{\mathfrak{S}}_1 - \hat{\mathfrak{S}}_1^0)(\xi,t)| \le Ce^{-c|\xi|^2 t}, \qquad |(\hat{\mathfrak{S}}_2 - \hat{\mathfrak{S}}_2^0)(\xi,t)| \le C|\xi|e^{-c|\xi|^2 t}, \tag{4.3}$$

for $|\xi| \le \epsilon$. We now define $\bar{u}_L$ by

$$U_L(t) = \mathfrak{S}_1^0(t) * u_1 + \mathfrak{S}_2^0(t) * u_0. \tag{4.4}$$

This $\bar{u}_L$ gives an asymptotic profile of the linear solution u_L. In fact we have:

Theorem 4.1. *Let $n \ge 1$ and $s \ge 0$. Assume that $u_0 \in H^s \cap L^1$ and $u_1 \in H^s \cap \dot{W}^{-1,1}$, and put $N_1 = \|u_0\|_{H^s\cap L^1} + \|u_1\|_{H^s\cap\dot{W}^{-1,1}}$. Let u_L be the linear solution and let $\bar{u}_L$ be defined by (4.4). Then we have*

$$\|\partial_x^k(u_L - U_L)(t)\|_{L^2} \le CN_1(1+t)^{-\frac{n}{4}-\frac{k+1}{2}} \tag{4.5}$$

for $0 \le k \le s$.

Proof. Since $(u_L - U_L)(t) = (\mathfrak{S}_1 - \mathfrak{S}_1^0)(t) * u_1 + (\mathfrak{S}_2 - \mathfrak{S}_2^0)(t) * u_0$, for the proof of (4.5), it suffices to show the following estimates:

$$\|\partial_x^k(\mathfrak{S}_1 - \mathfrak{S}_1^0)(t) * u_1\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1-j}{2}}\|\partial_x^j u_1\|_{\dot{W}^{-1,q}} + Ce^{-ct}\|\partial_x^{k+l}u_1\|_{L^2},$$

$$\|\partial_x^k(\mathfrak{S}_2 - \mathfrak{S}_2^0)(t) * u_0\|_{L^2} \le C(1+t)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-\frac{k+1-j}{2}}\|\partial_x^j u_0\|_{L^q} + Ce^{-ct}\|\partial_x^{k+l}u_0\|_{L^2},$$

where $1 \le q \le 2$, and k, j and l are nonnegative integers such that $0 \le j \le k+1$. These estimates can be proved similarly as in the proof of Lemma 2.3 by using (4.3) for $|\xi| \le \epsilon$ and (2.14) and (4.2) for $|\xi| \ge \epsilon$. We omit the details. □

Remark 4.2. By the Euler formula and (4.4), it is easy to see that

$$\hat{U}_L = e^{-\frac{|\nu|}{2}|\xi|^2 t}\frac{\sin(|\xi|t)}{|\xi|}\hat{u}_1 + e^{-\frac{|\nu|}{2}|\xi|^2 t}\cos(|\xi|t)\hat{u}_0. \tag{4.6}$$

Let

$$\hat{v}(\xi, t) = \frac{\sin(|\xi|t)}{|\xi|},$$

then

$$\hat{v}_t(\xi, t) = \cos(|\xi|t).$$

It is well-known that v is the fundamental solution to the following free wave equation

$$\begin{cases} v_{tt} - \Delta v = 0, \\ t = 0: \quad v = 0, \ v_t = \delta, \end{cases} \tag{4.7}$$

where $\delta(x)$ is the usual Dirac measure. According to the above analysis, we believe that $\bar{u}_L$ may be approximated by the solution to the free wave equation (4.7) and the initial value u_0, u_1. For this problem, we shall investigate it in future.

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