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CONVEX COMBINATIONS OF HARMONIC SHEARS OF SLIT MAPPINGS

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Abstract. In this paper, we study the convex combinations of harmonic mappings obtained by shearing a class of slit conformal mappings. Sufficient conditions for the convex combinations of harmonic mappings of this family to be univalent and convex in the horizontal direction are derived. Several examples of univalent harmonic mappings constructed by using these methods are presented to illustrate potential applications of the main results.

Keywords: Harmonic mapping, slit mapping, convex combination, shear construction.
MSC(2010): Primary: 58E20; Secondary: 30C45.

1. Introduction

A complex-valued function $f = u + iv$ defined in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is called a harmonic mapping if u and v are real harmonic functions. Let $\mathcal{H}$ denote the class of all complex-valued harmonic mappings f in $\mathbb{D}$ normalized by the conditions $f(0) = f_z(0) - 1 = f_{\bar{z}}(0) = 0$. Such mappings can be written in the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n \tag{1.1}$$

are *analytic* in $\mathbb{D}$. Moreover, a mapping $f \in \mathcal{H}$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if

$$|g'(z)| < |h'(z)| \quad (z \in \mathbb{D}). \tag{1.2}$$

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Let $\mathcal{H}_l$ be the subclass of $\mathcal{H}$ consisting of locally univalent and sense-preserving mappings. Let $\mathcal{S}_H^0$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving mappings. Let $\mathcal{K}_H^0$ and $\mathcal{C}_H^0$ be the subclasses of $\mathcal{S}_H^0$ whose image domains are convex and close-to-convex domains, respectively.

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the horizontal direction (CHD) if its intersection with each horizontal line is connected (or empty). A univalent harmonic mapping is called a CHD mapping if its range is a CHD domain. An effective way of constructing univalent harmonic mappings with given dilatations, known as the *shear construction*, was introduced by Clunie and Sheil-Small [2].

Suppose that $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ are two harmonic univalent mappings in $\mathbb{D}$, the linear combination of f_1 and f_2 is given by

$$f_3 = tf_1 + (1-t)f_2 = [th_1 + (1-t)h_2] + [t\overline{g_1} + (1-t)\overline{g_2}] = h_3 + \overline{g_3}. \tag{1.3}$$

Indeed, we observe that even if $\mathfrak{f}$ and $\mathfrak{g}$ are convex analytic functions, the convex combination $\mathfrak{h} = t\mathfrak{f} + (1-t)\mathfrak{g}$ need not be univalent (for details, see [7]). For results on the linear combinations of analytic functions, we refer to [1, 11].

Dorff [4] derived some sufficient conditions for the linear combination $f_3 = tf_1 + (1-t)f_2$ to be univalent and convex in the direction of the imaginary axis under the assumption that f_1 and f_2 have the same dilatation. Moreover, Wang *et al.* [12] proved that the linear combination $f_3 = tf_1 + (1-t)f_2$ with $h_k + g_k = \frac{z}{1-z}$ $(k = 1, 2)$ is univalent and convex in the direction of the real axis. Recently, Kumar *et al.* [6] studied the linear combinations of functions from the family of locally univalent and sense-preserving harmonic functions $f_\alpha = h_\alpha + \overline{g_\alpha}$, obtained by shearing the conformal mapping F_α defined by

$$F_\alpha(z) = h_\alpha(z) + g_\alpha(z) = \frac{z(1-\alpha z)}{1-z^2} \quad (\alpha \in [-1, 1]). \tag{1.4}$$

They also studied linear combinations of f_α and f_θ, where $f_\theta = h_\theta + \overline{g_\theta}$ is the harmonic mapping obtained by shearing the analytic vertical strip mapping

$$h_\theta(z) + g_\theta(z) = \frac{1}{2i\sin\theta} \log\left(\frac{1+ze^{i\theta}}{1+ze^{-i\theta}}\right) \quad (\theta \in (0, \pi)). \tag{1.5}$$

For other recent investigations of linear combinations of harmonic mappings, we refer to [10, 13].

In the present paper, we will consider the linear combinations of mappings of the family of locally univalent and sense-preserving harmonic mappings $f = h + \overline{g}$, by shearing the function φ defined by

$$\varphi(z) = A \log\left(\frac{1+z}{1-z}\right) + B\frac{z}{1+cz+z^2}, \tag{1.6}$$

where $A > 0$, $B > 0$ and $c \in [-2, 2]$. In [5], Dorff *et al.* have shown that φ is univalent and it maps the unit disk $\mathbb{D}$ onto a domain convex in the direction of the real axis. Harmonic shears of φ with the dilatation $\omega(z) = z^n$ $(n \geqq 2)$ is discussed by Ponnusamy *et al.* in [9]. We note that for special choices of c, the images of the unit disk $\mathbb{D}$ under φ is $\mathbb{C}$ minus four half-lines. For example, in the case $c = 0$, the mapping φ maps the unit disk $\mathbb{D}$ onto $\mathbb{C}$ minus the following four half-lines:

(1.7)
$$\mathbb{C} \setminus \left\{ x \pm \frac{A\pi}{2} i : x \in \left(-\infty, -\frac{A}{2} \log \left(\frac{\sqrt{2A+B} + \sqrt{B}}{\sqrt{2A+B} - \sqrt{B}} \right) - \frac{\sqrt{B(2A+B)}}{2} \right] \right\},$$

and

(1.8)
$$\mathbb{C} \setminus \left\{ x \pm \frac{A\pi}{2} i : x \in \left[\frac{A}{2} \log \left(\frac{\sqrt{2A+B} + \sqrt{B}}{\sqrt{2A+B} - \sqrt{B}} \right) + \frac{\sqrt{B(2A+B)}}{2}, \infty \right) \right\}.$$

The main objective of this paper is to derive sufficient conditions for the convex combination of two univalent harmonic mappings to be univalent and convex in the horizontal direction. Several examples of harmonic univalent mappings constructed by using these methods are also given to demonstrate applications of the main results.

2. Preliminary results

In order to derive the main results, we need the following lemmas.

Lemma 2.1 ([2])**.** *Let $f = h + \overline{g}$ be a locally univalent harmonic mapping in the unit disk $\mathbb{D}$. Then f is univalent in $\mathbb{D}$ and its range is a* CHD *domain if and only if $h - g$ is a conformal mapping of $\mathbb{D}$ onto a* CHD *domain.*

Lemma 2.2 (Cohn's Rule, see [3])**.** *Given a polynomial*

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \tag{2.1}$$

of degree n, let

$$f^*(z) = z^n \overline{f(1/\overline{z})} = \overline{a_n} + \overline{a_{n-1}} z + \cdots + \overline{a_0} z^n. \tag{2.2}$$

Denote by p and s the number of zeros of f inside the unit circle and on it, respectively. If $|a_0| < |a_n|$, then

$$f_1(z) = \frac{\overline{a_n} f(z) - a_0 f^*(z)}{z} \tag{2.3}$$

is of degree $n-1$ with $p_1 = p - 1$ and $s_1 = s$ the number of zeros of f_1 inside the unit circle and on it, respectively.

Lemma 2.3. *Let $f_k = h_k + \overline{g_k} \in \mathcal{H}_l (k = 1, 2)$ with*

$$h_k - g_k = \frac{1}{2} \alpha_k \log \left(\frac{1+z}{1-z} \right) + (1 - \alpha_k) \frac{z}{1+z^2} \quad (\alpha_k \in [0, 1]).$$

Also, $\omega_k = g_k'/h_k'\,(k=1,2)$ *are the dilatations of* f_1 *and* f_2*, respectively. Then the dilatation* ω *of* $f = tf_1 + (1-t)f_2\,(t\in[0,1])$ *is given by*

$$\omega(z) = \frac{t\omega_1\left[1+2(2\alpha_1-1)z^2+z^4\right](1-\omega_2)+(1-t)\omega_2\left[1+2(2\alpha_2-1)z^2+z^4\right](1-\omega_1)}{t\left[1+2(2\alpha_1-1)z^2+z^4\right](1-\omega_2)+(1-t)\left[1+2(2\alpha_2-1)z^2+z^4\right](1-\omega_1)}. \tag{2.4}$$

Proof. For $f = tf_1+(1-t)f_2 = th_1+(1-t)h_2+t\overline{g_1}+(1-t)\overline{g_2}$, we have

$$\omega(z) = \frac{tg_1'+(1-t)g_2'}{th_1'+(1-t)h_2'} = \frac{t\omega_1h_1'+(1-t)\omega_2h_2'}{th_1'+(1-t)h_2'}. \tag{2.5}$$

Since $h_k - g_k = \frac{1}{2}\alpha_k\log\left(\frac{1+z}{1-z}\right)+(1-\alpha_k)\frac{z}{1+z^2}$ and $\omega_k(z) = g_k'/h_k'\,(k=1,2)$, we see that

$$h_1'(z) = \frac{1+2(2\alpha_1-1)z^2+z^4}{(1-\omega_1)(1-z^2)(1+z^2)^2}, \tag{2.6}$$

and

$$h_2'(z) = \frac{1+2(2\alpha_2-1)z^2+z^4}{(1-\omega_2)(1-z^2)(1+z^2)^2}. \tag{2.7}$$

By substituting (2.6) and (2.7) into (2.5), we readily get (2.4). □

3. Main results

We first prove the following result.

Theorem 3.1. *Let* $f_k = h_k+\overline{g_k}\in\mathcal{H}_l$ $(k=1,2)$ *with*

$$h_1-g_1=\psi \quad \text{and} \quad h_2-g_2=\lambda\psi \quad (\lambda>0). \tag{3.1}$$

If ψ *is univalent and convex in the horizontal direction, then* $f = tf_1+(1-t)f_2\,(t\in[0,1])$ *is univalent and convex in the horizontal direction.*

Proof. Since $h_1-g_1=\psi$ and $h_2-g_2=\lambda\psi$, we have

$$\begin{aligned} h-g &= [th_1+(1-t)h_2]-[tg_1+(1-t)g_2] \\ &= t\,(h_1-g_1)+(1-t)\,(h_2-g_2) \\ &= [t+\lambda(1-t)]\psi, \end{aligned} \tag{3.2}$$

which is convex in the horizontal direction. Thus, by Lemma 2.1, it suffices to show that f is locally univalent and sense-preserving. If ω_1, ω_2 and ω are the dilatations of f_1, f_2 and f, respectively, then we get

$$\omega = \frac{tg_1'+(1-t)g_2'}{th_1'+(1-t)h_2'} = \frac{t\omega_1h_1'+(1-t)\omega_2h_2'}{th_1'+(1-t)h_2'}. \tag{3.3}$$

From (3.1) and the definitions of ω_1 and ω_2, we see that

$$h_1' = \frac{\psi'}{1-\omega_1}, \tag{3.4}$$

and

$$h_2' = \frac{\lambda\psi'}{1-\omega_2}. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we obtain

$$\omega = \frac{t\omega_1(1-\omega_2)+\lambda(1-t)\omega_2(1-\omega_1)}{t(1-\omega_2)+\lambda(1-t)(1-\omega_1)}. \tag{3.6}$$

Note that

$$\begin{aligned} \Re\left(\frac{1+\omega}{1-\omega}\right) =& \Re\left(\frac{1+\frac{t\omega_1(1-\omega_2)+\lambda(1-t)\omega_2(1-\omega_1)}{t(1-\omega_2)+\lambda(1-t)(1-\omega_1)}}{1-\frac{t\omega_1(1-\omega_2)+\lambda(1-t)\omega_2(1-\omega_1)}{t(1-\omega_2)+\lambda(1-t)(1-\omega_1)}}\right) \\ =&\Re\left(\frac{t(1+\omega_1)(1-\omega_2)+\lambda(1-t)(1-\omega_1)(1+\omega_2)}{[t+\lambda(1-t)](1-\omega_1)(1-\omega_2)}\right) \\ =&\frac{t}{t+\lambda(1-t)}\Re\left(\frac{1+\omega_1}{1-\omega_1}\right)+\frac{\lambda(1-t)}{t+\lambda(1-t)}\Re\left(\frac{1+\omega_2}{1-\omega_2}\right) \\ >&0, \end{aligned} \tag{3.7}$$

we see that $|\omega| < 1$, which implies that f is locally univalent and sense-preserving. This completes the proof of Theorem 3.1. □

Suppose that $A \geqq 0$, $B \geqq 0$, $A+B>0$ and $c\in[-2,2]$. Let $f = h+\overline{g}\in\mathcal{H}_l$ with

$$\varphi = h-g = A\log\left(\frac{1+z}{1-z}\right)+B\frac{z}{1+cz+z^2}. \tag{3.8}$$

Since φ is univalent and convex in the horizontal direction, by Lemma 2.1, we know that f is univalent and convex in the horizontal direction. In view of Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let* $f_k = h_k+\overline{g_k}\in\mathcal{H}_l$ $(k=1,2)$ *with*

$$h_1-g_1 = A\log\left(\frac{1+z}{1-z}\right)+B\frac{z}{1+cz+z^2}, \tag{3.9}$$

and

$$h_2-g_2 = \lambda A\log\left(\frac{1+z}{1-z}\right)+\lambda B\frac{z}{1+cz+z^2} \quad (\lambda>0). \tag{3.10}$$

Then $f = tf_1+(1-t)f_2\,(t\in[0,1])$ *is univalent and convex in the horizontal direction.*

Another way to construct desired univalent harmonic mappings is taking the linear combination of two harmonic mappings with the same dilatation ω, it is easy to get the following result, and we choose to omit the details of proof.

Theorem 3.3. *Let* $f_k = h_k + \overline{g_k} \in \mathcal{H}_l$ $(k = 1, 2)$ *with*

$$h_k - g_k = A_k \log\left(\frac{1+z}{1-z}\right) + B_k \frac{z}{1 + c_k z + z^2}. \tag{3.11}$$

If $c_1 = c_2$ *and* $\omega_1 = \omega_2$*, then* $f = tf_1 + (1-t)f_2\,(t \in [0,1])$ *is univalent and convex in the horizontal direction.*

If two harmonic mappings have different dilatations and they are sheared by different conformal mappings, it seems to be difficult to guarantee the univalency of their linear combinations. Next, we will discuss certain special cases.

Taking $A = \frac{1}{2}\alpha$, $B = 1-\alpha$ and $c = 0$, we obtain the conformal mapping φ_α defined by

$$\varphi_\alpha(z) = \frac{1}{2}\alpha \log\left(\frac{1+z}{1-z}\right) + (1-\alpha)\frac{z}{1+z^2} \quad (\alpha \in [0,1]). \tag{3.12}$$

The above conformal mapping φ_α is constructed in [8].

Theorem 3.4. *Let* $f_k = h_k + \overline{g_k} \in \mathcal{H}_l$ $(k = 1, 2)$ *with*

$$h_k - g_k = \frac{1}{2}\alpha_k \log\left(\frac{1+z}{1-z}\right) + (1-\alpha_k)\frac{z}{1+z^2} \quad (\alpha_k \in [0,1]). \tag{3.13}$$

Then $f = tf_1 + (1-t)f_2\,(t \in [0,1])$ *is univalent and convex in the horizontal direction provided that* f *is locally univalent and sense-preserving.*

Proof. Since

$$h_k - g_k = \frac{1}{2}\alpha_k \log\left(\frac{1+z}{1-z}\right) + (1-\alpha_k)\frac{z}{1+z^2},$$

we have

$$\begin{aligned} h - g &= [th_1 + (1-t)h_2] - [tg_1 + (1-t)g_2] \\ &= t(h_1 - g_1) + (1-t)(h_2 - g_2) \\ &= \frac{t\alpha_1 + (1-t)\alpha_2}{2} \log\left(\frac{1+z}{1-z}\right) + [1 - (t\alpha_1 + (1-t)\alpha_2)]\frac{z}{1+z^2}. \end{aligned} \tag{3.14}$$

Because $t\alpha_1 + (1-t)\alpha_2 \in [0,1]$ for all $t \in [0,1]$, we see that $h - g$ is univalent and convex in the horizontal direction. Thus, if $f = h + \overline{g}$ is locally univalent and sense-preserving, by Lemma 2.1, we deduce that f is univalent and maps $\mathbb{D}$ onto a domain convex in the horizontal direction. □

Theorem 3.5. *Let* $f_k = h_k + \overline{g_k} \in \mathcal{H}_l$ $(k = 1, 2)$ *with*

$$h_k - g_k = \frac{1}{2}\alpha_k \log\left(\frac{1+z}{1-z}\right) + (1-\alpha_k)\frac{z}{1+z^2} \quad (\alpha_k \in [0,1]). \tag{3.15}$$

Also, let $\omega_1 = -z^2$ *and* $\omega_2 = z^2$ *be the dilatations of* f_1 *and* f_2*, respectively. Then* $f = tf_1 + (1-t)f_2\,(t \in [0,1])$ *is univalent and convex in the horizontal direction provided that* $\alpha_1 \geqq \alpha_2$.

Proof. If $\alpha_1 = \alpha_2$, the result follows from Theorem 3.1. Now, we only need to prove the case $\alpha_1 > \alpha_2$. By virtue of Theorem 3.4, it is sufficient to prove that the dilatation ω of f satisfies $|\omega| < 1$ in $\mathbb{D}$. Substituting $\omega_1 = -z^2$ and $\omega_2 = z^2$ into (2.4), we get

(3.16)

$$\omega(z) = z^2 \frac{z^6 + [-2t - 1 + 4t\alpha_1 + 4(1-t)\alpha_2]\, z^4 + [4t - 1 - 4t\alpha_1 + 4(1-t)\alpha_2]\, z^2 + 1 - 2t}{1 + [-2t - 1 + 4t\alpha_1 + 4(1-t)\alpha_2]\, z^2 + [4t - 1 - 4t\alpha_1 + 4(1-t)\alpha_2]\, z^4 + (1-2t)z^6}.$$

Let

$$\begin{aligned} \eta(z) =& z^6 + [-2t - 1 + 4t\alpha_1 + 4(1-t)\alpha_2]\, z^4 \\ &+ [4t - 1 - 4t\alpha_1 + 4(1-t)\alpha_2]\, z^2 + 1 - 2t, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \eta^*(z) &= 1 + [-2t - 1 + 4t\alpha_1 + 4(1-t)\alpha_2]\, z^2 \\ &\quad + [4t - 1 - 4t\alpha_1 + 4(1-t)\alpha_2]\, z^4 + (1-2t)z^6 \\ &= z^6\overline{\eta(1/\overline{z})}. \end{aligned} \tag{3.18}$$

Then, we may write $\omega(z) = z^2 \frac{\eta(z)}{\eta^*(z)}$. Thus, if $z_0\,(z_0 \neq 0)$ is a zero of η, then $1/\overline{z_0}$ is a zero of η^*. Subsequently, we know that

$$\omega(z) = z^2 \frac{(z+\mu_1)(z+\mu_2)\cdots(z+\mu_6)}{(1+\overline{\mu}_1 z)(1+\overline{\mu}_2 z)\cdots(1+\overline{\mu}_6 z)}. \tag{3.19}$$

Note that the function $\psi(z) = \frac{z+\beta}{1+\overline{\beta}z}$ maps $\overline{\mathbb{D}} = \{z : |z| \leqq 1\}$ onto $\overline{\mathbb{D}}$ for $|\beta| \leqq 1$. Thus, to prove $|\omega(z)| < 1$, we only need to show that $|\mu_k| \leqq 1\,(k = 1, 2, \ldots, 6)$. The cases $t = 0$ and $t = 1$ are obviously true. We now suppose that $t \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, then it is clear that $|1 - 2t| < 1$. By applying Cohn's rule on η, it is sufficient to show that all zeros of η_1 lie inside or on the circle $|z| = 1$, where

$$\begin{aligned} \eta_1(z) &= \frac{\eta(z) - (1-2t)\eta^*(z)}{z} \\ &= 4t(1-t)z\left[z^4 + 2(\alpha_1 + \alpha_2 - 1)z^2 + 1 - 2\alpha_1 + 2\alpha_2\right]. \end{aligned} \tag{3.20}$$

Define

$$\gamma(z) = z^4 + 2(\alpha_1 + \alpha_2 - 1)z^2 + 1 - 2\alpha_1 + 2\alpha_2, \tag{3.21}$$

and $\gamma^*(z) = z^4\overline{\gamma(1/\overline{z})}$. Note that if $\alpha_1 = 1$ and $\alpha_2 = 0$, then all zeros of γ lies on the circle $|z| = 1$. Otherwise, we see that $|1 - 2\alpha_1 + 2\alpha_2| < 1$ for $0 < \alpha_2 < \alpha_1 < 1$. Then it is sufficient to show that all zeros of γ lie inside or on the circle $|z| = 1$. By applying Cohn's rule on γ again, we get

$$\begin{aligned}\gamma_1(z) &= \frac{\gamma(z) - (1 - 2\alpha_1 + 2\alpha_2)\gamma^*(z)}{z} \\ &= 4(\alpha_1 - \alpha_2)z\left[(1 - \alpha_1 + \alpha_2)z^2 + \alpha_1 + \alpha_2 - 1\right].\end{aligned} \tag{3.22}$$

Clearly, if $z_1\,(z_1 \neq 0)$ and $z_2\,(z_2 \neq 0)$ are the zeros of γ_1, then $z_1 + z_2 = 0$ and $z_1z_2 = \frac{\alpha_1+\alpha_2-1}{1-\alpha_1+\alpha_2}$. Thus, $|z_k| \leqq 1\,(k = 1, 2)$ is equivalent to $4\alpha_2(1 - \alpha_1) \geqq 0$, which is true for $\alpha_k \in [0, 1]\,(k = 1, 2)$. Hence, all zeros of γ_1, γ, η_1, η lie in or on the unit circle $|z| = 1$.

For the case $t = \frac{1}{2}$, we see that

$$\eta(z) = z^2\left[z^4 + 2(\alpha_1 + \alpha_2 - 1)z^2 + 1 - 2\alpha_1 + \alpha_2\right] = z^2\gamma(z). \tag{3.23}$$

It is easy to see that all zeros of η also lie in or on the unit circle $|z| = 1$. We thus complete the proof of Theorem 3.5. □

By applying the similar method as in the proof of Theorem 3.5, we get the following result.

Corollary 3.6. *Let $f_k = h_k + \overline{g_k} \in \mathcal{H}_l(k = 1, 2)$ with*

$$h_k - g_k = \frac{1}{2}\alpha_k \log\left(\frac{1+z}{1-z}\right) + (1 - \alpha_k)\frac{z}{1+z^2} \quad (\alpha_k \in [0, 1]). \tag{3.24}$$

Also, let $\omega_1 = -z^2$ and $\omega_2 = z^4$ be the dilatations of f_1 and f_2, respectively. Then $f = tf_1 + (1 - t)f_2\,(t \in [0, 1])$ is univalent and convex in the horizontal direction provided that $\alpha_1 \geqq \alpha_2$.

Theorem 3.7. *Let $f_k = h_k + \overline{g_k} \in \mathcal{H}_l\ (k = 1, 2)$ with $h_1 - g_1 = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ and $h_2 - g_2 = \frac{z}{1+z^2}$. Also, let $\omega_1(z) = -z^2$ and $\omega_2(z) = \frac{z^2+a}{1+az^2}\,(a \in (-1, 1))$ are the dilatations of f_1 and f_2, respectively. Then $f = tf_1 + (1 - t)f_2\,(t \in [0, 1])$ is univalent and convex in the horizontal direction.*

Proof. Noting that $h_1 - g_1 = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ and $h_2 - g_2 = \frac{z}{1+z^2}$, we have

$$h - g = \frac{t}{2}\log\left(\frac{1+z}{1-z}\right) + (1 - t)\frac{z}{1+z^2}. \tag{3.25}$$

Thus, $h - g$ is univalent and convex in the horizontal direction. Hence, it suffices to show that the dilatation ω of f satisfies $|\omega| < 1$ in $\mathbb{D}$. Since $h_1 - g_1 = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$, $h_2 - g_2 = \frac{z}{1+z^2}$, $\omega_1(z) = -z^2$ and $\omega_2(z) = \frac{a+z^2}{1+az^2}$, we obtain

$$h_1'(z) = \frac{1}{(1+z^2)(1-z^2)} \tag{3.26}$$

and

$$h_2'(z) = \frac{1+az^2}{(1-a)(1+z^2)^2}. \tag{3.27}$$

By observing that

$$\omega(z)=\frac{t\omega_1 h'+(1-t)\omega_2 h_2'}{th_1'+(1-t)h_2'}, \tag{3.28}$$

we get

$$\omega(z)=-\frac{(1-at)z^4+(1-a)(2t-1)z^2+a(t-1)}{1-at+(1-a)(2t-1)z^2+a(t-1)z^4}. \tag{3.29}$$

Suppose that

$$\varrho(z)=(1-at)z^4+(1-a)(2t-1)z^2+a(t-1), \tag{3.30}$$

and

$$\varrho^*(z)=1-at+(1-a)(2t-1)z^2+a(t-1)z^4. \tag{3.31}$$

Then

$$\omega(z)=-\frac{\varrho(z)}{\varrho^*(z)}=-\frac{(z+\lambda_1)(z+\lambda_2)(z+\lambda_3)(z+\lambda_4)}{\left(1+\overline{\lambda}_1 z\right)\left(1+\overline{\lambda}_2 z\right)\left(1+\overline{\lambda}_3 z\right)\left(1+\overline{\lambda}_4 z\right)}. \tag{3.32}$$

We only need to show that $|\lambda_k|\leqq 1\,(k=1,2,3,4)$. Since $|a(t-1)|<|1-at|$ provided that $a\in(-1,1)$, by applying Cohn's rule on ϱ, it is sufficient to show that all zeros of ϱ_1 lie inside or on the circle $|z|=1$, where

$$\varrho_1(z)=\frac{(1-at)\varrho(z)-a(t-1)\varrho^*(z)}{z}=(1+a-2at)(1-a)z(z^2+2t-1). \tag{3.33}$$

It is easy to see that all zeros of ϱ_1 lie in or on the unit circle $|z|=1$. Thus, Theorem 3.7 is proved.

□

4. **Examples**

In this section, we give several examples to illustrate potential applications of the main results.

Example 4.1. Let $f_1=h_1+\overline{g_1}$, where $h_1-g_1=\frac{1}{10}\log\left(\frac{1+z}{1-z}\right)+\frac{1}{5}\frac{z}{(1+z)^2}$ and $\omega_1=z$. Then

$$h_1=\frac{1}{10}\frac{z}{1-z}+\frac{1}{20}\log\left(\frac{1+z}{1-z}\right)+\frac{1}{10}\frac{z(2+z)}{(1+z)^2}, \tag{4.1}$$

and

$$g_1=\frac{1}{10}\frac{z}{1-z}-\frac{1}{20}\log\left(\frac{1+z}{1-z}\right)+\frac{1}{10}\frac{z^2}{(1+z)^2}. \tag{4.2}$$

Suppose that $f_2 = h_2 + \overline{g_2}$, where $h_2 - g_2 = \log\left(\frac{1+z}{1-z}\right) + \frac{2z}{(1+z)^2}$ and $\omega_2 = \frac{2z+i}{2-iz}$. Then we have

$$h_2 = \frac{8+7i}{16}\log(1+z) - \frac{1+2i}{2}\log(1-z) + \frac{(7+2i)z+3z^2}{4(1+z)^2} + \frac{9i}{16}\log\left[\frac{2-i-(2+i)z}{2-i}\right], \tag{4.3}$$

and

$$g_2 = \frac{8+7i}{16}\log(1+z) - \frac{1+2i}{2}\log(1-z) - \log\left(\frac{1+z}{1-z}\right) + \frac{(-1+2i)z+3z^2}{4(1+z)^2} + \frac{9i}{16}\log\left[\frac{2-i-(2+i)z}{2-i}\right]. \tag{4.4}$$

By Theorem 3.1, we see that $f_3 = \frac{1}{2}f_1 + \frac{1}{2}f_2$ is convex in the horizontal direction. The images of $\mathbb{D}$ under $f_k\,(k = 1, 2, 3)$ are shown in Figures 1–3, respectively.

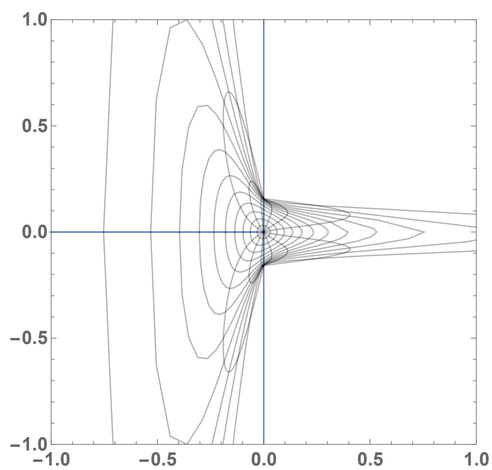

FIGURE 1. The image of f_1.

Example 4.2. Let $f_4 = h_4 + \overline{g_4}$, where $h_4 - g_4 = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1+z^2}$ and $\omega_4 = -z^2$. Then we have

$$h_4 = \frac{1}{8}\frac{z}{1+z^2} + \frac{1}{4}\frac{z}{(1+z^2)^2} + \frac{1}{8}\log\left(\frac{1+z}{1-z}\right) - \frac{3i}{16}\log\left(\frac{1+iz}{1-iz}\right), \tag{4.5}$$

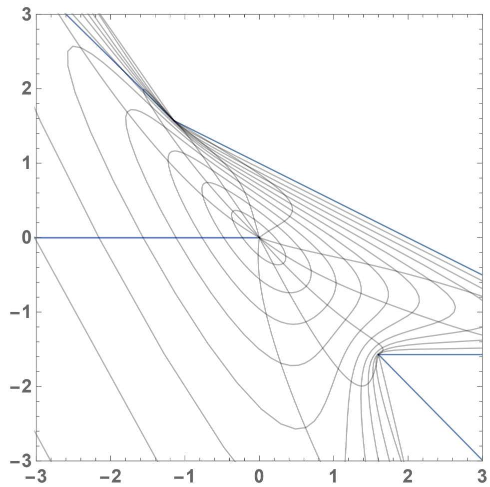

FIGURE 2. The image of f_2.

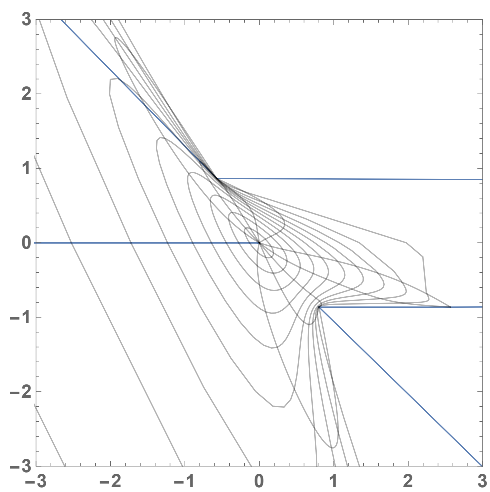

FIGURE 3. The image of $f_3 = \frac{1}{2}f_1 + \frac{1}{2}f_2$.

and

$$g_4 = -\frac{3}{8}\frac{z}{1+z^2} + \frac{1}{4}\frac{z}{(1+z^2)^2} - \frac{1}{8}\log\left(\frac{1+z}{1-z}\right) - \frac{3i}{16}\log\left(\frac{1+iz}{1-iz}\right). \tag{4.6}$$

Suppose that $f_5 = h_5 + \overline{g_5}$, where $h_5 - g_5 = \frac{1}{6}\log\left(\frac{1+z}{1-z}\right) + \frac{2}{3}\frac{z}{1+z^2}$ and $\omega_5 = z^4$. Then we get

(4.7)
$$h_5 = \frac{1}{6}\frac{z}{(1+z^2)^2} + \frac{1}{4}\frac{z}{1+z^2} + \frac{1}{12}\frac{z}{1-z^2} + \frac{1}{12}\log\left(\frac{1+z}{1-z}\right) - \frac{i}{6}\log\left(\frac{1+iz}{1-iz}\right),$$

and

(4.8)
$$g_5 = \frac{1}{6}\frac{z}{(1+z^2)^2} - \frac{5}{12}\frac{z}{1+z^2} + \frac{1}{12}\frac{z}{1-z^2} - \frac{1}{12}\log\left(\frac{1+z}{1-z}\right) - \frac{i}{6}\log\left(\frac{1+iz}{1-iz}\right).$$

By Corollary 3.6, we see that $f_6 = \frac{9}{10}f_4 + \frac{1}{10}f_5$ is convex in the horizontal direction. The images of $\mathbb{D}$ under $f_k\,(k = 4,5,6)$ with $t = \frac{9}{10}$ are shown in Figures 4–6, respectively.

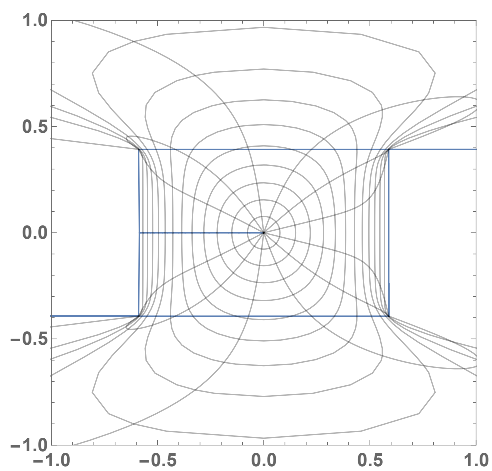

FIGURE 4. The image of f_4.

Example 4.3. Let $f_7 = h_7 + \overline{g_7}$, where $h_7 - g_7 = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ and $\omega_7 = -z^2$. Then we have

$$h_7 = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right), \tag{4.9}$$

and

$$g_7 = -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right). \tag{4.10}$$

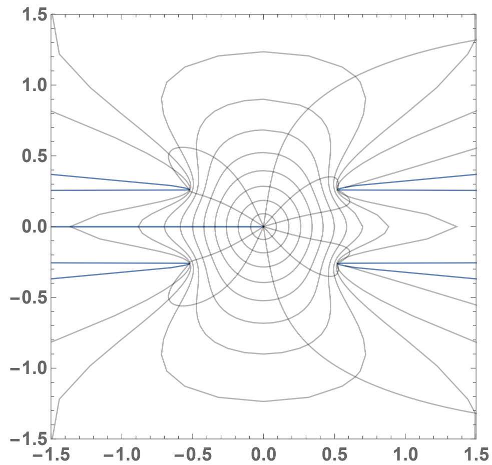

FIGURE 5. The image of f_5.

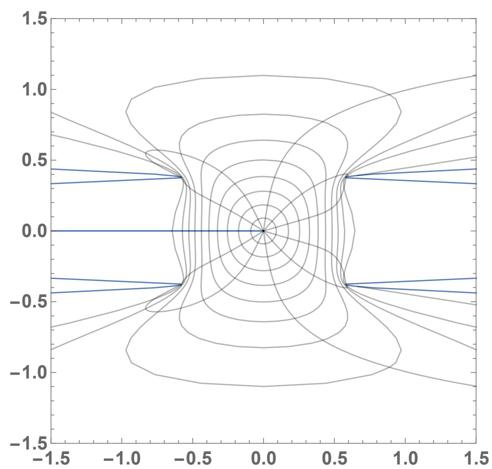

FIGURE 6. The image of $f_6 = \frac{9}{10} f_4 + \frac{1}{10} f_5$.

Suppose that $f_8 = h_8 + \overline{g_8}$, where $h_8 - g_8 = \frac{z}{1+z^2}$ and $\omega_8 = \frac{1+2z^2}{2+z^2}$. Then we get

$$h_8 = \frac{1}{2}\frac{z}{1+z^2} - \frac{3i}{4}\log\left(\frac{1+iz}{1-iz}\right), \tag{4.11}$$

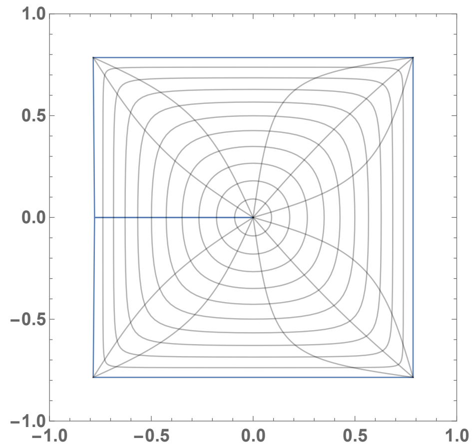

FIGURE 7. The image of f_7.

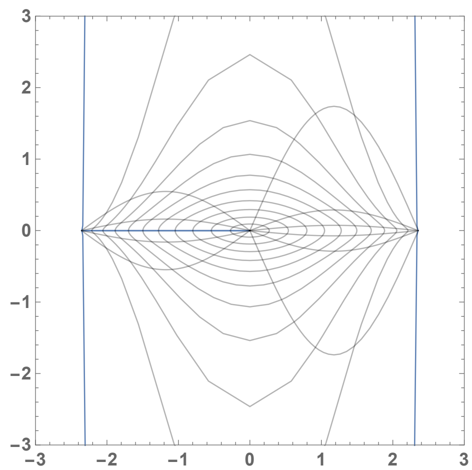

FIGURE 8. The image of f_8.

and

$$g_8 = -\frac{1}{2}\frac{z}{1+z^2} - \frac{3i}{4}\log\left(\frac{1+iz}{1-iz}\right). \tag{4.12}$$

By Theorem 3.7, we see that $f_9 = \frac{9}{10}f_7 + \frac{1}{10}f_8$ is univalent and convex in the horizontal direction. The images of $\mathbb{D}$ under $f_k\,(k = 7, 8, 9)$ with $t = \frac{9}{10}$ are shown in Figures 7–9, respectively.

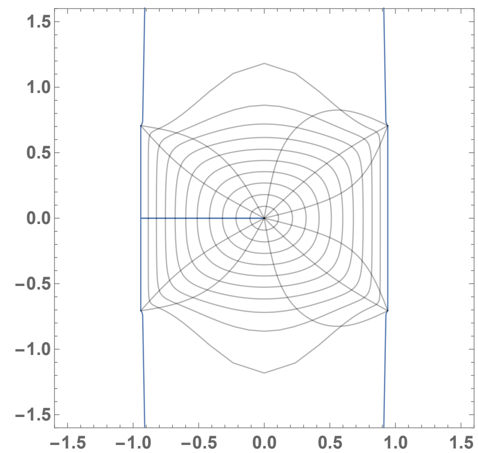

Figure 9. The image of $f_9 = \frac{9}{10} f_7 + \frac{1}{10} f_8$.

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