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THE GRAPH OF EQUIVALENCE CLASSES AND ISOCLINISM OF GROUPS

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Abstract. Let G be a non-abelian group and let $\Gamma(G)$ be the non-commuting graph of G. In this paper we define an equivalence relation $\sim$ on the set of $V(\Gamma(G)) = G \setminus Z(G)$ by taking $x \sim y$ if and only if $N(x) = N(y)$, where $N(x) = \{u \in G \mid x \text{ and } u \text{ are adjacent in } \Gamma(G)\}$ is the open neighborhood of x in $\Gamma(G)$. We introduce a new graph determined by equivalence classes of non-central elements of G, denoted $\Gamma_E(G)$, as the graph whose vertices are $\{[x] \mid x \in G \setminus Z(G)\}$ and join two distinct vertices $[x]$ and $[y]$, whenever $[x, y] \neq 1$. We prove that group G is AC-group if and only if $\Gamma_E(G)$ is complete graph. Among other results, we show that the graphs of equivalence classes of non-commuting graph associated with two isoclinic groups are isomorphic.
Keywords: Non-commuting graph, graph of equivalence classes, Isoclinism.
MSC(2010): Primary: 05C25; Secondary: 20F99.

1. Introduction

Let G be a group and $Z(G)$ be the center of G. The non-commuting graph $\Gamma(G)$ associated with G is the graph whose vertex set is $G \setminus Z(G)$ and two distinct elements x and y are adjacent, denoted $x - y$, if and only if $[x, y] \neq 1$. According to [2] the non-commuting graph of a finite group G was first considered by Paul Erdös in connection with the following problem. Let G be a group whose non-commuting graph $\Gamma(G)$ has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of $\Gamma(G)$? B.H. Neumann [12] answered positively to this question. In [2] and [11], some graph theoretical properties of $\Gamma(G)$ and the relations between some properties of $\Gamma(G)$ and the structure of group G were studied. Of course, there are some other ways to construct a graph associated with a given group.

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We may refer to the works of Bertram et al. [6] and Moghadamfar et al. [11] or recent papers on non-commuting graph, Engel graph and non-cyclic graph in [1, 2] and [4], respectively.

Two vertices a and b of a simple graph Γ are said to be equivalent, if their open neighborhoods are the same, i.e., $a \sim b$ if and only if $N(a) = N(b)$, where $N(a) = \{c \in V(\Gamma) \mid a$ and c are adjacent in $\Gamma\}$. One can see that $\sim$ is an equivalence relation and we denote the class of a by $[a]$. The graph of equivalence classes of Γ, denoted Γ_E, is the graph associated with Γ whose vertex set is $\{[a] : a \in V(\Gamma)\}$ and two equivalence classes $[a]$ and $[b]$ are adjacent in Γ_E if a and b are adjacent in Γ. In Section 2, we will introduce the graph of equivalence classes of the non-commuting graph $\Gamma(G)$. We will state some of basic graph theoretical properties of $\Gamma_E(G)$, for instance determining diameter, girth, dominating set, planarity of the graph and we give some relation between the graph properties of $\Gamma(G)$ and $\Gamma_E(G)$. In Section 3 of the paper, we state a connection between the graph of equivalence classes of the non-commuting graph and isoclinism of groups. We prove that the graphs of equivalence classes of two isoclinic groups are isomorphic. Moreover, we show that for any group G with $\Gamma_E(G) < \infty$, there is a finite group K such that $\Gamma_E(G) \cong \Gamma_E(K)$.

2. **Definitions and basic results**

Let $\Gamma(G)$ be the non-commuting graph of a non-abelian group G. For $x, y \in G \setminus Z(G)$, we say that $x \sim y$ if and only if $G \setminus C_G(x) = N(x) = N(y) = G \setminus C_G(y)$ if and only if $C_G(x) = C_G(y)$, where $N(x) = \{u \in G \mid x$ and u are adjacent in $\Gamma(G)\}$. It is easy to see that $\sim$ is an equivalence relation and we denote the class of x by $[x]$.

Definition 2.1. The *graph of equivalence classes of* $\Gamma(G)$, denoted $\Gamma_E(G)$, is the graph associated with G with vertex set $\{[x] : x \in G \setminus Z(G)\}$ such that two distinct vertices $[x]$ and $[y]$ are joined by an edge, denoted $[x] - [y]$, if and only if $[x, y] \neq 1$.

It is easy to check $[x] \mapsto C_G(x)$ establishes a one-to-one correspondence between $V(\Gamma_E(G))$ and the set of all proper centralizers of group G. Hence $|V(\Gamma_E(G))| = \#\mathrm{Cent}(G) - 1$, where $\mathrm{Cent}(G)$ denote the set of centralizers of single elements of G and $\#\mathrm{Cent}(G)$ is the size of $\mathrm{Cent}(G)$.

Recall that a clique of a graph is a set of mutually adjacent vertices, and that the maximum size of a clique of a graph Γ, the clique number of Γ, is denoted $\omega(\Gamma)$. Moreover, a clique of a graph Γ is called a maximum clique if its size is $\omega(\Gamma)$

Lemma 2.2. *Assume that* $\mathcal{A} = \{[x] : C_G(x)$ *is an abelian group*$\}$. *Then* $\mathcal{A}$ *is a clique in* $\Gamma_E(G)$.

Proof. By the structure of $\mathcal{A}$, it will be enough to prove the induced subgraph on $\mathcal{A}$ is a complete graph. Hence, suppose that $[x]$ and $[y]$ are two distinct elements of $\mathcal{A}$. We claim that $[x]$ and $[y]$ are adjacent in $\Gamma_E(G)$, or equivalently, $[x,y] \neq 1$. If not, then for every $a \in C_G(y)$, we get $[a,x] = 1$, since $x \in C_G(y)$ which is an abelian group. Thus, $a \in C_G(x)$, and so $C_G(y) \subseteq C_G(x)$. Similarly $C_G(x) \subseteq C_G(y)$, and hence $C_G(x) = C_G(y)$. But then, by definition we have $x \sim y$, which forces $[x] = [y]$, a contradiction. □

Let $\mathcal{B} = \{[x] : C_G(x)$ is minimal among all centralizers of $G\}$ i.e., if $[x] \in \mathcal{B}$ and $C_G(y) \subseteq C_G(x)$, then $C_G(y) = C_G(x)$. Assume that $\mathcal{A}$ is as in Lemma 2.2, $[x] \in \mathcal{A}$ and $C_G(y) \subseteq C_G(x)$. For every $a \in C_G(x)$, $[a,y] = 1$, since $a, y \in C_G(x)$ which is abelian, and so $a \in C_G(y)$. Therefore $[x] \in \mathcal{B}$. It follows that $\mathcal{A} \subseteq \mathcal{B}$. In the following we will give some other facts on the sets $\mathcal{A}$ and $\mathcal{B}$.

For a graph Γ and a subset S of vertices, denote by $N_\Gamma[S]$ the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$, then S is called a dominating set for Γ. The dominating number $\gamma(\Gamma)$ of Γ is the minimum size of a dominating set of the vertices of Γ.

Lemma 2.3. *Assume that G is a non-abelian group, and $\mathcal{A}$ and $\mathcal{B}$ defined as above.*

(i) *If $[y] \in \mathcal{B}$, then $[y]$ is adjacent to all elements of $\mathcal{A} \setminus \{[y]\}$ in $\Gamma_E(G)$.*
(ii) *If $[z] \in V(\Gamma_E(G)) \setminus \mathcal{B}$, then $[z]$ is not adjacent to all elements of $\mathcal{B}$ in $\Gamma_E(G)$.*
(iii) *If $\mathcal{A} = \mathcal{B}$, then every vertex of $\Gamma_E(G)$ is adjacent to at least a vertex of $\mathcal{A}$. Moreover, $\mathcal{A}$ is a maximal clique of $\Gamma_E(G)$ and if $\Gamma_E(G)$ is finite, then $\omega(\Gamma_E(G)) = |\mathcal{A}|$.*

Proof. (i) For $[y] \in \mathcal{B}$, if $[y] \in \mathcal{A}$ and $[x,y] = 1$ for some $[x] \in \mathcal{A}$, then $[x] = [y]$ which is a contradiction. Then $[y]$ is adjacent to all elements of $\mathcal{A}$. Now assume that $[y] \in \mathcal{B} \setminus \mathcal{A}$ and $[x,y] = 1$ for some $[x] \in \mathcal{A}$, then $C_G(x) \subseteq C_G(y)$ and so $C_G(x) = C_G(y)$, since $[y] \in \mathcal{B}$. Thus $[x] = [y]$, a contradiction and so $[y]$ is adjacent to all elements of $\mathcal{A}$.

(ii) Assume that $[z] \in V(\Gamma_E(G)) \setminus \mathcal{B}$. Then there is $[w] \in \mathcal{B}$ such that $C_G(w) \subseteq C_G(z)$ and so $[z,w] = 1$. This means that $[z]$ is not adjacent to $[w]$, as required.

(iii) Suppose that $\mathcal{A} = \mathcal{B}$. If there is $[z] \in V(\Gamma_E(G))$ such that $[z]$ is not adjacent to elements in $\mathcal{A}$, then $[x,z] = 1$ for all $[x] \in \mathcal{A} = \mathcal{B}$ and so $\bigcup_{[x]\in\mathcal{A}} C_G(x) \subseteq C_G(z)$. On the other hand, there is a non-central element $w \in G$ such that $[z]$ and $[w]$ are adjacent. Thus $w \notin C_G(z)$ and so $[w]$ is adjacent to all elements of $\mathcal{A} = \mathcal{B}$, which will contradict part (ii). Therefore, $\mathcal{A}$ is a dominating set for $\Gamma_E(G)$. Moreover, assume that $\mathcal{A} \cup \{[y]\}$ is a clique of $\Gamma_E(G)$, where $[y] \notin \mathcal{A}$. Then there is $[x] \in \mathcal{B} = \mathcal{A}$ such that $C_G(x) \subseteq C_G(y)$ and so $[x,y] = 1$, a contradiction. Therefore, $\mathcal{A}$ is a maximal clique of $\Gamma_E(G)$. Now suppose that $\Gamma_E(G)$ is finite and Y is a maximum clique of $\Gamma_E(G)$ such that $|\mathcal{A}| < |Y|$. Then there are $[x] \in \mathcal{B} = \mathcal{A}$ and $[y_1], [y_2] \in Y$ such that $C_G(x) \subseteq C_G(y_i)$, for $i = 1, 2$. Therefore, $y_1, y_2 \in C_G(x)$ and so $[y_1, y_2] = 1$, which is a contradiction. Hence $\omega(\Gamma_E(G)) = |\mathcal{A}|$ and the proof is complete. □

A non-abelian group is called an AC-group if the centralizer of every non-central element is abelian.

Theorem 2.4. *A non-abelian group G is an AC-group if and only if $\Gamma_E(G)$ is a complete graph. In particular, if G is an AC-group with $n = \#\mathrm{Cent(G)} < \infty$, then $\Gamma_E(G) \cong K_{n-1}$, where K_{n-1} is a complete graph with $n-1$ vertices.*

Proof. Suppose that $\Gamma_E(G)$ is a complete graph and $a, b \in C_G(x) \setminus Z(G)$. Then $[a] = [x] = [b]$ and so $[a, b] = 1$. Therefore, $C_G(x)$ is an abelian group. Now, assume that G is an AC-group. Then $V(\Gamma_E(G)) = \mathcal{A}$ and the result follows from Lemma 2.2. Furthermore, assume that $\#\mathrm{Cent}(G) = n < \infty$. If $C_G(x_1), C_G(x_2), \ldots, C_G(x_{n-1})$ are all proper centralizers of G, then $[x_i, x_j] \neq 1$ for all $1 \leq i \neq j \leq n$ and so $V(\Gamma_E(G)) = \{[x_1], [x_2], \ldots, [x_{n-1}]\}$ is the maximum clique of $\Gamma_E(G)$. Thus $\Gamma_E(G) \cong K_{n-1}$. □

Corollary 2.5. *Let G and H be two non-abelian groups. If $\Gamma_E(G) \cong \Gamma_E(H)$, then G is an AC-group if and only if H is an AC-group.*

Corollary 2.6.

(i) *Let $G = D_{2n} = \langle a, b | a^n = b^2 = (ab)^2 = 1\rangle$ be the dihedral group of order $2n$. Then*
$$\Gamma_E(G) \cong \begin{cases} K_{n+1} & n \text{ is odd} \\ K_{\frac{n}{2}+1} & n \text{ is even} \end{cases}$$

(ii) *If $G = Q_{4n}$ is the generalized quaternion group of order $4n$, then $\Gamma_E(G) \cong K_{n+1}$.*

Proof. It is easy to check that $\mathrm{Cent}(G) = \{G, C_G(a), C_G(a^i b),\ 0 \leq i \leq n-1\}$, when n is odd and $\mathrm{Cent}(G) = \{G, C_G(a), C_G(a^i b),\ 0 \leq i \leq n/2 - 1\}$ when n is even. Furthermore $\mathrm{Cent}(G) = \{G, C_G(a), C_G(a^i b),\ 0 \leq i \leq n-1\}$ when $G = Q_{4n}$. Now the results follows from Theorem 2.4. □

From Theorem 2.4 we note that in some cases the graph of the equivalence classes of groups are not complete. The smallest counterexample is the symmetric group S_4. It is easy to check that $C_G((1\ 2)(3\ 4))$ and $C_G((1\ 3)(2\ 4))$ are non-abelian and distinct, and the vertices $[(1\ 2)(3\ 4)]$ and $[(1\ 3)(2\ 4)]$ are not adjacent in $\Gamma_E(S_4)$.

Proposition 2.7. *Assume that G is a non-abelian group. Then $diam(\Gamma_E(G)) \leqslant 2$ and $girth(\Gamma_E(G)) = 3$. In particular $\Gamma_E(G)$ is connected.*

Proof. Let $[x]$ and $[y]$ be two distinct vertices of $\Gamma_E(G)$. If $[x] - [y]$ then $d([x], [y]) = 1$. Thus we may assume that $[x] \nsim [y]$. Since x, y are non-central, there exist $[x'], [y'] \in V(\Gamma_E(G))$ such that $\{[x], [x']\}$ and $\{[y], [y']\}$

are edges. If $[y] - [x']$ or $[x] - [y']$ then $d([x],[y]) = 2$. Otherwise the vertex $[x'y']$ is adjacent to both $[x]$ and $[y]$ and again $d([x],[y]) = 2$. Therefore, $diam(\Gamma_E(G)) \leqslant 2$. Moreover, for every edge $\{[x],[y]\}$ of $\Gamma_E(G)$, $\{[x],[y],[xy]\}$ is a triangle. Hence the girth of $\Gamma_E(G)$ is 3. □

A subset X of the vertices of a graph Γ is called an independent set if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the independence number of Γ and denoted by $\alpha(\Gamma)$.

Proposition 2.8. *Let G be a non-abelian group.*

(i) *If $\mathcal{C}$ is a dominating set for $\Gamma(G)$, then $\overline{\mathcal{C}} = \{[x] : x \in \mathcal{C}\}$ is a dominating set for $\Gamma_E(G)$.*

(ii) *If $\mathcal{D}$ is an independent set of $\Gamma(G)$, then $\overline{\mathcal{D}} = \{[x] : x \in \mathcal{D}\}$ is an independent set of $\Gamma_E(G)$.*

Proof. (i) Let $[x]$ be a vertex of $\Gamma_E(G)$ that is not in $\overline{\mathcal{C}}$ so $x \notin \mathcal{C}$. Therefore there is $y \in \mathcal{C}$ such that $[x,y] \neq 1$ and this means that there is $[y] \in \overline{\mathcal{C}}$ such that $[x] - [y]$.
(ii) If $[x]$ and $[y]$ are two elements of $\overline{\mathcal{D}}$, then $x, y \in \mathcal{D}$ and this means that $[x,y] = 1$. Therefore $[x]$ and $[y]$ are not adjacent. □

By Proposition 2.8 one can see that $\gamma(\Gamma_E(G)) \leq \gamma(\Gamma(G))$ and $\alpha(\Gamma_E(G)) \leq \alpha(\Gamma(G))$. In the following, we will prove that the graphs $\Gamma(G)$ and $\Gamma_E(G)$ have the same clique number and vertex chromatic number.

Let $k > 0$ be an integer. A k-vertex coloring of a graph Γ is an assignment of k colors to the vertices of Γ such that no two adjacent vertices have the same color. The vertex chromatic number $\chi(\Gamma)$ of a graph Γ, is the minimum k for which Γ has a k-vertex coloring.

Proposition 2.9. *Let G be a finite non-abelian group. Then $\omega(\Gamma_E(G)) = \omega(\Gamma(G))$, $\chi(\Gamma_E(G)) = \chi(\Gamma(G))$.*

Proof. Since $\Gamma_E(G)$ is isomorphic to a subgraph of $\Gamma(G)$, then $\omega(\Gamma_E(G)) \leq \omega(\Gamma(G))$. Assume that $\omega(\Gamma(G)) = n$ and $\{x_1, x_2, \ldots, x_n\}$ is a maximum clique of $\Gamma(G)$. Then for every $1 \leqslant i \neq j \leqslant n$, $[x_i, x_j] \neq 1$ and so, $[x_i]$ and $[x_j]$ are adjacent for all $1 \leqslant i \neq j \leqslant n$. This means that $\{[x_1],[x_2],\ldots,[x_n]\}$ is a clique in $\Gamma_E(G)$ and so $\omega(\Gamma(G)) \leq \omega(\Gamma_E(G))$.

Now, let $\chi(\Gamma_E(G)) = t$ and $\chi(\Gamma(G)) = k$. By [2, Lemma 4.1] k is the minimum number of abelian subgroups of G whose union is G, then G is covered by abelian subgroups $A_1, \ldots, A_k$. Assume that $V(\Gamma_E(G)) = \{[x_1],\ldots,[x_n]\}$, $T = \{x_1,\ldots,x_n\}$ and $T_i = \{[x] : x \in T \cap A_i\}$ for $1 \leq i \leq k$. Then the vertices of $\Gamma_E(G)$ in T_i are independent and so $t \leqslant k$. Now suppose that $B_1, \ldots, B_t$ are independent subsets of $V(\Gamma_E(G))$ such that $\bigcup_{j=1}^{t} B_j = V(\Gamma_E(G))$. Then $A_j = \langle \bigcup_{x \in B_j} [x], Z(G) \rangle$ is an abelian subgroup of G, for $1 \leq j \leq t$ and G is covered by these t abelian subgroups. It follows that $k \leq t$. □

Proposition 2.10. *If* $\Gamma(G) \cong \Gamma(H)$*, then* $\Gamma_E(G) \cong \Gamma_E(H)$.

Proof. Let $\varphi : V(\Gamma(G)) \longrightarrow V(\Gamma(H))$ be a bijective map such that for every two distinct elements $x, y \in V(\Gamma(G))$, we have $[x,y] = 1$ if and only if $[\varphi(x), \varphi(y)] = 1$. Define $\psi : V(\Gamma_E(G)) \longrightarrow V(\Gamma_E(H))$ such that $\psi([x]) = [\varphi(x)]$. One can check that ψ is a bijective map and $[x] - [y]$ in $\Gamma_E(G)$ if and only if $\psi([x]) = [\varphi(x)] - [\varphi(y)] = \psi([y])$ in $\Gamma_E(H)$. □

Proposition 2.11. *Let* G *or* H *be non-abelian AC-groups. Then* $\omega(\Gamma_E(G \times H)) = \omega(\Gamma_E(G))\omega(\Gamma_E(H))$.

Proof. Assume that G is an AC-group and $V(\Gamma_E(G)) = \{[x_1], [x_2], \dots, [x_n]\}$ and $\{[y_1], [y_2], \dots, [y_m]\}$ are maximum clique of $\Gamma_E(G)$ and $\Gamma_E(H)$, respectively. We show that $\Omega = \{[(x_i, y_j)] : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ is a maximum clique of $\Gamma_E(G \times H)$. Suppose, for a contradiction, that $\Omega \neq \Omega \cup \{[(u,v)]\}$ is a clique of $\Gamma_E(G \times H)$. Then $[(x_i, y_j)] - [(u,v)]$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

Case 1. $[u] = [x_t]$ or $[v] = [y_s]$ for some $1 \leqslant t \leqslant n$ and $1 \leqslant s \leqslant m$. It is clear that $\{[y_1], [y_2], \dots, [y_m], [v]\}$ or $\{[x_1], [x_2], \dots, [x_n], [u]\}$ is a clique of $\Gamma_E(H)$ or $\Gamma_E(G)$, respectively, which is a contradiction.

Case 2. $[u] \neq [x_i]$ for all $1 \leqslant i \leqslant n$. Since G is an AC-group, $[u]$ is adjacent to $[x_i]$ for every $1 \leqslant i \leqslant n$ and so $\{[x_1], [x_2], \dots, [x_n], [u]\}$ is a clique of $\Gamma_E(G)$, a contradiction. □

Proposition 2.12. *Let* H *be a finite subgroup of* G*. Then* $\Gamma_E(G) \cong \Gamma_E(H)$ *if and only if* $G = HZ(G)$.

Proof. Define $\varphi : V(\Gamma_E(G)) \longrightarrow V(\Gamma_E(H))$ by $\varphi([hz]) = [h]$, which is a bijective map. Since $[h_1 z_1, h_2 z_2] \neq 1$ if and only if $[h_1, h_2] \neq 1$, then $[h_1 z_1]$ and $[h_2 z_2]$ are adjacent in $\Gamma_E(G)$ if and only if $[h_1]$ and $[h_2]$ are adjacent in $\Gamma_E(H)$ and so φ is a graph isomorphism.

Conversely, assume that $V(\Gamma_E(H)) = \{[h_1], [h_2], \dots, [h_n]\}$, so that we may have $V(\Gamma_E(G)) = \{[h_1], [h_2], \dots, [h_n]\}$ and $Z(H) \subseteq Z(G)$. Since the map $\varphi : \frac{H}{Z(H)} \longrightarrow \frac{G}{Z(G)}$ by $\varphi(hZ(H)) = hZ(G)$ is an isomorphism, for every $g \in G$ there exists $h \in H$ such that $gZ(G) = \varphi(hZ(H)) = hZ(G)$. Thus $h^{-1}g \in Z(G)$ and so $g = hz \in HZ(G)$ for some $z \in Z(G)$ and the proof is complete. □

Corollary 2.13. $\Gamma_E(G \times A) \cong \Gamma_E(G)$ *if and only if* A *is an abelian group.*

Proposition 2.14. *Let* N *be a normal subgroup of* G*. Then* $\Gamma_E(G) \cong \Gamma_E(G/N)$*, if* $N \cap G' = 1$.

Proof. Assume that $N \cap G' = 1$. Then the map $\varphi : V(\Gamma_E(G)) \longrightarrow V(\Gamma_E(G/N))$ by $\varphi([x]) = [xN]$ is a bijection. Now assume that $[x]$ is not adjacent to $[y]$ in $\Gamma_E(G)$. Since $[x,y] = 1$, then $[xN, yN] = N$ and so $[xN] \nsim [yN]$. On the other hand, if $[xN]$ and $[yN]$ are not adjacent in $\Gamma_E(G/N)$, then $[xN, yN] = 1_{G/N}$. Thus $[x,y] \in N \cap G' = 1$ and so $[x]$ and $[y]$ are not adjacent in $\Gamma_E(G)$. □

Proposition 2.15. *Assume that* $\Gamma_E(G_1) \cong \Gamma_E(H_1)$ *and* $\Gamma_E(G_2) \cong \Gamma_E(H_2)$. *Then* $\Gamma_E(G_1 \times G_2) \cong \Gamma_E(H_1 \times H_2)$.

Proof. Let $\varphi_i : V(\Gamma_E(G_i)) \longrightarrow V(\Gamma_E(H_i))$ be a graph isomorphism for $i = 1, 2$. Then it is easy to see that $\psi : V(\Gamma_E(G_1 \times G_2)) \longrightarrow V(\Gamma_E(H_1 \times H_2))$ such that $\psi([(x,y)]) = \big[(\varphi_1(x), \varphi_2(y))\big]$, where $C_{H_1}(\varphi_1(x)) := \varphi_1(C_{G_1}(x))$ and $C_{H_2}(\varphi_2(y)) := \varphi_2(C_{G_2}(y))$ is a graph isomorphism between $\Gamma_E(G_1 \times G_2)$ and $\Gamma_E(H_1 \times H_2)$. □

3. Isoclinism classes and the graph of equivalences classes

The notion of isoclinism of groups was introduced by Philip Hall [9] as the following

Definition 3.1. Let G and H be two groups; a pair (φ, ψ) is called an *isoclinism* from G to H if :
(1) φ is an isomorphism from $G/Z(G)$ to $H/Z(H)$;
(2) ψ is an isomorphism from G' to H' ;
(3) the following diagram is commutative:

$$\begin{array}{ccc} \dfrac{G}{Z(G)} \times \dfrac{G}{Z(G)} & \xrightarrow{\varphi\times\varphi} & \dfrac{H}{Z(H)} \times \dfrac{H}{Z(H)} \\ {\scriptstyle a_G}\Big\downarrow & & {\scriptstyle a_H}\Big\downarrow \\ G' & \xrightarrow{\psi} & H' \end{array}$$

where, $a_G(g_1Z(G), g_2Z(G)) = [g_1, g_2]$ and $a_H(h_1Z(H), h_2Z(H)) = [h_1, h_2]$.

If there is an isoclinism from G to H, we say that G and H are isoclinic and denote it by $G \sim H$. One may easily check that the groups Q_8 and D_8 are isoclinic but are not isomorphic.

In the following, we prove that the graphs of equivalences classes of non-commuting graphs associated with two isoclinic groups are isomorphic.

Theorem 3.2. *Assume that* G *and* H *are two isoclinic groups. Then* $\Gamma_E(G) \cong \Gamma_E(H)$.

Proof. Let (φ, ψ) be an isoclinism from G to H, where $\varphi : G/Z(G) \longrightarrow H/Z(H)$ and $\psi : G' \longrightarrow H'$ are isomorphisms and if $\varphi(g_1Z(G)) = h_1Z(H)$, $\varphi(g_2Z(G)) = h_2Z(H)$, then $\psi([g_1, g_2]) = [h_1, h_2]$. Now assume that $C_G(x) \in \mathrm{Cent}(G)$ such that $\varphi(C_G(x)/Z(G)) = K/Z(H)$, we will show that $K = C_H(y)$, where $\varphi(xZ(G)) = yZ(H)$. For $k \in K$ there is $g \in C_G(x)$ such that $\varphi(gZ(G)) = kZ(H)$. Since $[x, g] = 1$ and by commutativity

$$[y,k] = a_H\big(\varphi \times \varphi(xZ(G), gZ(G))\big) = \psi\big(a_G(xZ(G), gZ(G))\big) = \psi([x,g]),$$

then $[y,k]=1$ and so $k\in C_H(y)$. Conversely, for $h\in C_H(y)$ there is $g\in G$ such that $\varphi(gZ(G))=hZ(H)$ and

$$\psi([x,g])=\psi\big(a_G(xZ(G),gZ(G))\big)=a_H\big(\varphi\times\varphi(xZ(G),gZ(G))\big)=[y,k]=1.$$

Therefore $g\in C_G(x)$ and so $h\in K$, as required. Thus $\#\mathrm{Cent}(G)=\#\mathrm{Cent}(H)$ and so $\theta: V(\Gamma_E(G))\to V(\Gamma_E(H))$ by $\theta([x])=[y]$ is a bijective map, where $\varphi(C_G(x)/Z(G))=C_H(y)/Z(H)$.

Furthermore if $[a]$ and $[b]$ are adjacent in $\Gamma_E(G)$, then $C_G(a)\neq C_G(b)$ and $[a,b]\neq 1$. Since (φ,ψ) is an isoclinism, $[a']$ and $[b']$ are adjacent in $\Gamma_E(H)$, where $\varphi(C_G(a)/Z(G))=C_H(a')/Z(H)$ and $\varphi(C_G(b)/Z(G))=C_H(b')/Z(H)$. In the same way we can show that, $[a']-[b']$ in $\Gamma_E(H)$ implies that $[a]-[b]$ in $\Gamma_E(G)$ and so $\Gamma_E(G)\cong\Gamma_E(H)$. □

Theorem 3.3. *Let G be a group such that $\Gamma_E(G)$ is finite. Then there is a finite group K such that $Z(K)\subseteq K'$, $K\sim G$ and $\Gamma_E(K)\cong\Gamma_E(G)$.*

Proof. Thanks to [10, Proposition 2.5] there is a group K isoclinic to G such that $Z(K)\subseteq K'$. On the other hand, since $\Gamma_E(G)$ is finite, $\omega(\Gamma(G))=\omega(\Gamma_E(G))<\infty$ and so by the main theorem of [13], we have $[K:Z(K)]=[G:Z(G)]\leq c^{\omega(\Gamma(G))}$ for some constant c. Now, Schur's Theorem follows that K' and so $Z(K)$ is finite. Therefore K is finite and the result follows from Theorem 3.2. □

Theorem 3.4. *$\Gamma_E(G)\cong K_3$ if and only if $G\sim D_8$.*

Proof. It is clear that if G is isoclinic to D_8, then $\Gamma_E(G)\cong K_3$.
Now assume that $\Gamma_E(G)\cong K_3$. By Theorem 3.3, we might as well assume that G is a finite group such that $Z(G)\subseteq G'$ and by Theorem 2.4, G is an AC-group. Since group G is the union of its proper centralizers, there are proper centralizers $C_G(x)$, $C_G(y)$, $C_G(z)$ of G such that $G=C_G(x)\cup C_G(y)\cup C_G(z)$. On the other hand, $[xy,x]\neq 1$ and $[xy,y]\neq 1$. Thus, $C_G(xy)=C_G(z)$ and so $G=C_G(x)\cup C_G(y)\cup C_G(xy)$. We claim that for every $u\in G\smallsetminus Z(G)$, $u^2\in Z(G)$. Suppose, for a contradiction, that u^2 is not central and for example $u\in C_G(x)$, then $C_G(x)=C_G(u)$ and so $[u^2y,x]\neq 1$ and $[u^2y,y]\neq 1$. Hence, $C_G(u^2y)=C_G(xy)$, since G has four distinct centralizers. Moreover, $C_G(xy)=C_G(uy)$, since $[uy,x]\neq 1$ and $[uy,y]\neq 1$. Then $C_G(u^2y)=C_G(uy)$ and so $uy=yu$, which is a contradiction. Therefore, $G/Z(G)$ is an elementary abelian 2-group. On the other hand, one can see that for every $u\in G\smallsetminus Z(G)$ if for example $u\in C_G(x)$, then $uZ(G)=xZ(G)$ and so $[G:Z(G)]=4$, which implies that $G/Z(G)\cong C_2\times C_2$ and so $Z(G)=G'$. Now by [14, Corollary 3.1], $G'\cong C_2$ and one can see that G is isoclinic to D_8, as required. □

Theorem 3.5. *$\Gamma_E(G)\cong K_4$ if and only if G is isoclinic to S_3 or an extra special group of order 27.*

Proof. If $G \sim S_3$, then $\Gamma_E(G) \cong \Gamma_E(S_3) \cong K_4$ by Theorem 3.2. Moreover, assume that E is an extra special group of order 27 and G is isoclinic to E. By using GAP, we can see that E has four abelian proper centralizers and so $\Gamma_E(G) \cong \Gamma_E(E) \cong K_4$. For the converse, let $\Gamma_E(G) \cong K_4$. From Theorem 3.3, we may assume, up to isoclinism, that G is a finite group such that $Z(G) \subseteq G'$. Then we have $\#\mathrm{Cent(G)} = 5$ and so $G/Z(G)$ is isomorphic to $C_3 \times C_3$ or S_3 by [5, Fact 5]. First assume that $G/Z(G) \cong S_3$ and S is a non-central Sylow p-subgroup of G (p is a prime). Then $SZ(G)/Z(G) \in \mathrm{Syl}_p(G/Z(G))$ and Sylow subgroups of $G/Z(G) \cong S_3$ are cyclic groups of order 2 or 3. Thus, $SZ(G)/Z(G)$ is cyclic which implies that $SZ(G)$ is abelian and so is S. Hence all Sylow subgroups of G are abelian and by [7, Corollary 4.5], $G' \cap Z(G) = 1$ and this implies that $Z(G) = 1$. Then we conclude that $G \sim S_3$, as required. Now let $G/Z(G) \cong C_3 \times C_3$. By a similar argument as Theorem 3.4, we can see that $Z(G) = G'$ and $G' \cong C_3$. It remains to prove that G is isoclinic to an extra special group of order 27. Let $G/Z(G) = \langle xZ(G)\rangle \times \langle yZ(G)\rangle$ and E be an extra special group of order 27 i.e., $E' = Z(E) \cong C_3$ and $E/Z(E) \cong C_3 \times C_3 = \langle aZ(E)\rangle \times \langle bZ(E)\rangle$, where $x, y \in G$ and $a, b \in E$.

Define $\varphi : G/Z(G) \to E/Z(E)$ by the rules $\varphi(xZ(G)) = aZ(E)$ and $\varphi(yZ(G)) = bZ(E)$, which is an isomorphism. Furthermore, one can check that $G' =< [x, y] >$ and $E' =< [a, b] >$ and so the map $\psi : G' \to E'$ by the rule $\psi([x, y]) = [a, b]$ is an isomorphism. Now we prove that the following diagram is commutative,

$$\begin{array}{ccc} \dfrac{G}{Z(G)} \times \dfrac{G}{Z(G)} & \xrightarrow{\varphi\times\varphi} & \dfrac{E}{Z(E)} \times \dfrac{E}{Z(E)} \\ {\scriptstyle a_G}\downarrow & & {\scriptstyle a_E}\downarrow \\ G' & \xrightarrow{\psi} & E' \end{array}$$

For any $0 \le t, s, u, v \le 2$, we have

$$\begin{aligned} a_E\varphi \times \varphi(x^t y^s Z(G), x^u y^v Z(G)) &= a_E(a^t b^s Z(E), a^u b^v Z(E)) \\ &= [a^t b^s, a^u b^v] = [a, b]^{tv-su} \\ &= \psi([x, y]^{tv-su}) = \psi([x^t y^s, x^u y^v]) \\ &= \psi a_G(x^t y^s Z(G), x^u y^v Z(G)). \end{aligned}$$

Therefore G and E are isoclinic by Definition 3.1 as required.

□

Corollary 3.6. *Let G be an AC-group. Then $\Gamma_E(G)$ is planar if and only if G is isoclinic to S_3, D_8 or an extra special group of order 27.*

Proposition 3.7. *If $\omega(\Gamma_E(G)) \leqslant 20$, then G is solvable.*

Proof. By Theorem 3.3, we may assume that G is a finite group with $\omega(\Gamma_E(G)) = \omega(\Gamma(G)) \leqslant 20$. Therefore, the result follows from [3, Theorem 1.4]. □

It must be note that A_5 is an AC-group with $\#\mathrm{Cent}(A_5) = 22$ and so $\omega(\Gamma_E(A_5)) = 21$, by Theorem 2.4. Therefore, the above upper bound is sharp.

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