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A HYBRID MEAN VALUE INVOLVING A NEW GAUSS SUMS AND DEDEKIND SUMS

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Abstract. In this paper, we introduce a new sum analogous to Gauss sum, then we use the properties of the classical Gauss sums and analytic method to study the hybrid mean value problem involving this new sums and Dedekind sums, and give an interesting identity for it.
Keywords: A new Gauss sums, Dedekind sums, hybrid mean value, identity, analytic method.
MSC(2010): 11L05.

1. Introduction

Let $k > 1$ be a fixed integer. For any integer a and any Dirichlet character χ mod k, the classical Gauss sums $G(\chi, a)$ is defined by

$$G(\chi, a) = \sum_{n=1}^{k} \chi(n) e\left(\frac{an}{k}\right),$$

where $e(y) = e^{2\pi i y}$. Especially, $\tau(\chi) = G(\chi, 1)$.

The various properties of $G(\chi, a)$ were investigated by many authors(see [1, 3, 5, 6]). For example, T.M. Apostol [1] proved that if χ is any primitive character modulo k, then $G(\chi, a) = \overline{\chi}(a)\tau(\chi)$ and $|\tau(\chi)| = \sqrt{k}$; and if χ is a non-primitive character modulo k and $(a, k) = 1$, then one still has $G(\chi, a) = \overline{\chi}(a)\tau(\chi)$. L. K. Hua [3] proved that if k is a square-full number (i.e., for any prime p, $p|k$ if and only if $p^2|k$), then for any non-primitive character χ mod k, we have $\tau(\chi) = 0$. C. D. Pan and C. B. Pan [6] have obtained a general conclusion: Let χ be a character mod q induced by χ^* mod q^*, then we have

$$\tau(\chi) = \chi^*\left(\frac{q}{q^*}\right)\mu\left(\frac{q}{q^*}\right)\tau(\chi^*),$$

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where $\mu(n)$ is the Möbius function.

In this paper, we introduce a new sum analogous to Gauss sums as follows:

$$G(\chi, b, c, m; q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \chi\left(a^2 + ba + c\right) \cdot e\left(\frac{ma}{q}\right),$$

where χ is a character mod q, and b, c, m are any integers with $(m, q) = 1$.

The main purpose of this paper is to study the hybrid mean value problem involving $G(\chi, b, c, m; q)$ and Dedekind sums, and give a useful identity for the new sum. For convenience, firstly, we give the definition of Dedekind sums. Let q be a natural number and h an integer prime to q. Dedekind sums $S(h, q)$ is defined as:

$$S(h, q) = \sum_{a=1}^{q} \left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

This sums describes the behaviour of the logarithm of the eta-function (see [7, 8]) under modular transformations. Several authors have studied various arithmetical properties of $S(h, q)$, eg, see [2, 4, 9–11].

Now for any odd prime p and non-principal character χ mod p, we consider the hybrid mean value involving $G(\chi, b, c, m; p)$ and $S(h, p)$:

$$\begin{aligned} &\sum_{r=0}^{p-1}\sum_{s=0}^{p-1} |G(\chi, b, r, m; p)|^2 \cdot |G(\chi, b, s, m; p)|^2 \\ &\cdot S\left((4r - b^2) \cdot \overline{4s - b^2}, p\right), \end{aligned} \tag{1.1}$$

where $\overline{n} \cdot n \equiv 1 \bmod p$, if $(n, p) = 1$. If $(n, p) = p$, we put $\overline{n} = 0$.

The main purpose of this paper is to give a computational formula for (1.1) by using properties of the classical Gauss sums and some analytic methods. That is, we prove the following:

Theorem 1.1. *Let p be an odd prime, and let χ be any non-principal character* mod p. *Then for any integers b and m with $(m,p)=1$, we have the identity*

$$\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}|G(\chi,b,r,m;p)|^2\cdot|G(\chi,b,s,m;p)|^2\cdot S\left((4r-b^2)\overline{4s-b^2},p\right)$$
$$=\begin{cases}\frac{p(p-1)(p-2)}{12}, & \text{if } \chi(-1)=\chi_2(-1);\\ \frac{p(p-1)(p-2)}{12}-\frac{2}{\pi^2}\cdot p^2\cdot\left|L\left(1,\chi\chi_2\right)\right|^2, & \text{if } \chi(-1)\neq\chi_2(-1),\end{cases}$$

where $\chi_2=\left(\frac{*}{p}\right)$ denotes the Legendre symbol.

The theorem immediately implies the following:

Corollary 1.2. *Let p be an odd prime, and let χ be any non-principal character* mod p. *Then for any integer m with $(m,p)=1$, we have the identity*

$$\sum_{r=1}^{p-1}\sum_{s=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi\left(a^2+r\right)e\left(\frac{ma}{p}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}\chi\left(a^2+s\right)e\left(\frac{ma}{p}\right)\right|^2\cdot S\left(r\cdot\overline{s},p\right)$$
$$=\begin{cases}\frac{p(p-1)(p-2)}{12}, & \text{if } \chi(-1)=\chi_2(-1);\\ \frac{p(p-1)(p-2)}{12}-\frac{2}{\pi^2}\cdot p^2\cdot\left|L\left(1,\chi\chi_2\right)\right|^2, & \text{if } \chi(-1)\neq\chi_2(-1).\end{cases}$$

2. Some lemmas

In this section, we give some lemmas, which are necessary in the proof of the main theorem. Hereinafter, we use properties of character sums and Gauss sums which can be found in [1,5,6].

First we have the following:

Lemma 2.1. *Let p be an odd prime and let χ be any non-principal Dirichlet character* mod p. *Then for any integers b and c we have the identity*

$$\left|\sum_{a=1}^{p-1}\chi\left(a^2+ba+c\right)\cdot e\left(\frac{a}{p}\right)\right|=\left|\sum_{r=1}^{p-1}\overline{\chi}\chi_2(r)\cdot e\left(\frac{(4c-b^2)r-\overline{16}\cdot\overline{r}}{p}\right)\right|,$$

where $\overline{r}$ denotes the solution of the congruence equation $r\cdot x\equiv 1$ mod p.

Proof. By properties of Gauss sums and quadratic residue mod p we know that if $(n,p)=1$, then

$$\begin{aligned}
&\sum_{a=0}^{p-1} e\left(\frac{na^2}{p}\right) = 1+\sum_{a=1}^{p-1} e\left(\frac{na^2}{p}\right)\\
&= 1+\sum_{a=1}^{p-1}\left(1+\left(\frac{a}{p}\right)\right)\cdot e\left(\frac{na}{p}\right)\\
&= \sum_{a=0}^{p-1} e\left(\frac{na}{p}\right)+\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\cdot e\left(\frac{na}{p}\right)\\
&= \left(\frac{n}{p}\right)\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)\cdot e\left(\frac{a}{p}\right) = \left(\frac{n}{p}\right)\cdot \tau\left(\chi_2\right). \qquad (2.1)
\end{aligned}$$

Since χ is a non-principal Dirichlet character mod p, by properties of Gauss sums, (2.1) implies that

$$\begin{aligned}
&\sum_{a=1}^{p-1} \chi\left(a^2+ba+c\right)\cdot e\left(\frac{a}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\cdot\sum_{a=1}^{p-1}\sum_{r=1}^{p-1}\overline{\chi}(r)e\left(\frac{r(a^2+ba+c)}{p}\right)\cdot e\left(\frac{a}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\cdot\sum_{r=1}^{p-1}\overline{\chi}(r)\sum_{a=1}^{p-1} e\left(\frac{ra^2+(br+1)a+cr}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\cdot\sum_{r=1}^{p-1}\overline{\chi}(r)\sum_{a=0}^{p-1} e\left(\frac{\overline{4}r(2a+\overline{r}(br+1))^2+cr-\overline{4}\overline{r}(br+1)^2}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\cdot\sum_{r=1}^{p-1}\overline{\chi}(r)\sum_{u=0}^{p-1} e\left(\frac{\overline{4}ru^2+\overline{4}\cdot(4c-b^2)r-\overline{2}\cdot b-\overline{4}\cdot\overline{r}}{p}\right)\\
&= \frac{1}{\tau\left(\overline{\chi}\right)}\cdot\sum_{r=1}^{p-1}\overline{\chi}(r)\left(\frac{\overline{4}\cdot r}{p}\right)\cdot\tau\left(\chi_2\right)\cdot e\left(\frac{\overline{4}\cdot(4c-b^2)r-\overline{2}\cdot b-\overline{4}\cdot\overline{r}}{p}\right)\\
&= \frac{\tau\left(\chi_2\right)}{\tau\left(\overline{\chi}\right)}\cdot\overline{\chi}(4)\cdot e\left(\frac{-\overline{2}\cdot b}{p}\right)\cdot\sum_{r=1}^{p-1}\overline{\chi}(r)\left(\frac{r}{p}\right)\cdot e\left(\frac{(4c-b^2)r-\overline{16}\cdot\overline{r}}{p}\right). \qquad (2.2)
\end{aligned}$$

For any non-principal character χ mod p, we have $|\tau(\chi)| = |\tau(\chi_2)| = \sqrt{p}$. Note that $|\overline{\chi}(4)| = \left|e\left(\frac{-\overline{2}\cdot b}{p}\right)\right| = 1$. Thus, by (2.2) we have

$$\left|\sum_{a=1}^{p-1} \chi\left(a^2+ba+c\right)\cdot e\left(\frac{a}{p}\right)\right| = \left|\sum_{r=1}^{p-1} \overline{\chi}\chi_2(r)\cdot e\left(\frac{(4c-b^2)r-\overline{16}\cdot\overline{r}}{p}\right)\right|.$$

This proves Lemma 2.1. □

Lemma 2.2. *Let p be an odd prime and let χ be any non-real character* mod p. *Then for any integer n with $(n,p)=1$ and χ_1 mod p, we have the identity*

$$\sum_{m=1}^{p-1} \chi(m)\left|\sum_{a=1}^{p-1} \chi_1(a)\cdot e\left(\frac{ma+n\overline{a}}{p}\right)\right|^2 = \frac{\overline{\chi}(n)\chi_1(-1)\tau^2(\chi)\tau(\chi\chi_1)\tau(\chi\overline{\chi_1})}{\tau(\chi^2)}.$$

Proof. Let p be an odd prime. Then by properties of the Gauss sums, for any non-real character χ mod p we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \chi(m)\left|\sum_{a=1}^{p-1} \chi_1(a)\cdot e\left(\frac{ma+n\overline{a}}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1} \chi(m) \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi_1(a\overline{b}) e\left(\frac{ma-mb+n(\overline{a}-\overline{b})}{p}\right) \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi_1(a) \sum_{m=1}^{p-1} \chi(m) e\left(\frac{mb(a-1)+n\overline{b}(\overline{a}-1)}{p}\right) \\
&= \tau(\chi)\sum_{a=1}^{p-1} \chi_1(a)\overline{\chi}(a-1)\sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{\overline{b}n(\overline{a}-1)}{p}\right) \\
&= \tau(\chi)\sum_{a=1}^{p-1} \chi_1(a)\overline{\chi}(a-1)\sum_{b=1}^{p-1} \chi(b) e\left(\frac{bn(\overline{a}-1)}{p}\right) \\
&= \overline{\chi}(n)\tau^2(\chi)\sum_{a=1}^{p-1} \chi_1(a)\overline{\chi}((a-1)(\overline{a}-1)). \qquad (2.3)
\end{aligned}$$

On the other hand, since χ is a non-real character mod p, so $\chi^2 \neq \chi_p^0$, where χ_p^0 is the principal character mod p. Therefore, again by properties of the Gauss sums we have

$$\tau(\chi\chi_1)\tau(\chi\overline{\chi_1}) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\chi_1(a)\chi(b)\overline{\chi_1}(b)\ e\left(\frac{a+b}{p}\right)$$
$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\chi_1(a)\chi(-b)\overline{\chi_1}(-b)\ e\left(\frac{a-b}{p}\right)$$
$$= \chi(-1)\chi_1(-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\chi_1(a)\chi^2(b)\ e\left(\frac{b(a-1)}{p}\right)$$
$$= \chi(-1)\chi_1(-1)\sum_{a=1}^{p-1}\chi(a)\chi_1(a)\sum_{b=1}^{p-1}\chi^2(b)\ e\left(\frac{b(a-1)}{p}\right)$$
$$= \chi(-1)\chi_1(-1)\tau(\chi^2)\sum_{a=1}^{p-1}\chi(a)\chi_1(a)\overline{\chi}^2(a-1)$$
$$= \chi_1(-1)\tau(\chi^2)\sum_{a=1}^{p-1}\chi_1(a)\overline{\chi}\left((a-1)(\overline{a}-1)\right). \tag{2.4}$$

Combining (2.3) and (2.4) we may immediately deduce the identity

$$\sum_{m=1}^{p-1}\chi(m)\left|\sum_{a=1}^{p-1}\chi_1(a)\cdot e\left(\frac{ma+n\overline{a}}{p}\right)\right|^2 = \frac{\overline{\chi}(n)\chi_1(-1)\tau^2(\chi)\tau(\chi\chi_1)\tau(\chi\overline{\chi_1})}{\tau(\chi^2)}.$$

This proves Lemma 2.2. □

Lemma 2.3. *Let p be an odd prime. Then for any integer n with $(n,p)=1$ and χ_1 mod p, we have the identities*

$$\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=1}^{p-1}\chi_1(a)\cdot e\left(\frac{ma+n\overline{a}}{p}\right)\right|^2$$
$$= \begin{cases} \chi_2(-1)\chi_2(n)\tau^2(\chi_2)(p-2), & \text{if } \chi_1 \text{ is the Legendre symbol;} \\ -\chi_2(-1)\chi_2(n)\tau^2(\chi_2), & \text{otherwise.}\end{cases}$$

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)\cdot e\left(\frac{ma+n\overline{a}}{p}\right)\right|^2$$
$$= \begin{cases} p^2-p-1, & \text{if } \chi_1 \text{ is the principal character mod } p; \\ p(p-2), & \text{if } \chi_1 \text{ is not the principal character mod } p,\end{cases}$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre symbol.

Proof. If $\chi = \chi_2$ is the Legendre symbol, then by the method used in the proof of Lemma 2.2 we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=1}^{p-1} \chi_1(a) \cdot e\left(\frac{ma + n\bar{a}}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} \chi_2(m) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_1(a\bar{b}) e\left(\frac{ma - mb + n(\bar{a} - \bar{b})}{p}\right) \\
&= \tau(\chi_2) \sum_{a=1}^{p-1} \chi_1(a) \chi_2(a-1) \sum_{b=1}^{p-1} \chi_2(b)\ e\left(\frac{\bar{b}n(\bar{a}-1)}{p}\right) \\
&= \tau(\chi_2) \sum_{a=1}^{p-1} \chi_1(a) \chi_2(a-1) \sum_{b=1}^{p-1} \chi_2(b)\ e\left(\frac{bn(\bar{a}-1)}{p}\right) \\
&= \chi_2(-1)\chi_2(n)\tau^2(\chi_2) \sum_{a=1}^{p-1} \chi_1(a) \chi_2((a-1)^2\bar{a}) \\
&= \chi_2(-1)\chi_2(n)\tau^2(\chi_2) \left(\sum_{a=1}^{p-1} \chi_1(a)\chi_2(a) - 1 \right). \qquad (2.5)
\end{aligned}$$

If χ_1 is the Legendre symbol, then

$$\sum_{a=1}^{p-1} \chi_1(a)\chi_2(a) = p - 1. \qquad (2.6)$$

If χ_1 is not the Legendre symbol, then

$$\sum_{a=1}^{p-1} \chi_1(a)\chi_2(a) = 0. \qquad (2.7)$$

Combining (2.5), (2.6) and (2.7) we immediately deduce the first formula of Lemma 2.3.

To prove the second formula of Lemma 2.3, by the trigonometric identity

$$\sum_{a=1}^{q} e\left(\frac{ma}{q}\right) = \begin{cases} q, & \text{if } q \mid m; \\ 0, & \text{if } q \nmid m, \end{cases}$$

we have

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi_1(a)\cdot e\left(\frac{ma+n\bar{a}}{p}\right)\right|^2\\
&=\sum_{m=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a\bar{b})\ e\left(\frac{m(a-b)+n(\bar{a}-\bar{b})}{p}\right)\\
&=\sum_{m=1}^{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)\ e\left(\frac{mb(a-1)+\bar{b}n(\bar{a}-1)}{p}\right)\\
&=\sum_{a=1}^{p-1}\chi_1(a)\sum_{m=1}^{p-1}\sum_{b=1}^{p-1}\ e\left(\frac{m(a-1)+\bar{b}(\bar{a}-1)}{p}\right)\\
&=(p-1)^2+\sum_{a=2}^{p-1}\chi_1(a)\left(\sum_{m=1}^{p-1}e\left(\frac{m(a-1)}{p}\right)\right)\left(\sum_{b=1}^{p-1}e\left(\frac{b(\bar{a}-1)}{p}\right)\right)\\
&=(p-1)^2+\sum_{a=2}^{p-1}\chi_1(a)\\
&=(p-1)^2+\sum_{a=1}^{p-1}\chi_1(a)-1. \qquad (2.8)
\end{aligned}
$$

Now the second formula of Lemma 2.3 follows from (2.8). □

Lemma 2.4. *Let $q>2$ be an integer. Then for any integer a with $(a,q)=1$, we have the identity*

$$S(a,q)=\frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2,$$

where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to character $\chi \bmod d$.

Proof. See [11, Lemma 2]. □

3. Proof of theorem 1.1

In this section, we complete the proof of theorem 1.1. First, if $(m,p)=1$, then by properties of a reduced residue system mod p we have

$$\begin{aligned}
&\left|\sum_{a=1}^{p-1}\chi\left(a^2+ba+c\right)\cdot e\left(\frac{ma}{p}\right)\right|\\
&=\left|\sum_{a=1}^{p-1}\chi\left(\overline{m}^2a^2+\overline{m}ba+c\right)\cdot e\left(\frac{a}{p}\right)\right|\\
&=\left|\chi^2\left(\overline{m}\right)\sum_{a=1}^{p-1}\chi\left(a^2+mba+cm^2\right)\cdot e\left(\frac{a}{p}\right)\right|\\
&=\left|\sum_{a=1}^{p-1}\chi\left(a^2+mba+cm^2\right)\cdot e\left(\frac{a}{p}\right)\right|. \qquad (3.1)
\end{aligned}$$

From Lemma 2.4 and the definition of $S(a,p)$ we have

$$S(a,p)=\frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2 \qquad (3.2)$$

and

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2=\frac{\pi^2(p-1)}{p}\cdot S(1,p)\\
&=\frac{\pi^2(p-1)}{p}\cdot\sum_{a=1}^{p-1}\left(\frac{a}{p}-\frac{1}{2}\right)^2=\frac{\pi^2}{12}\cdot\frac{(p-1)^2(p-2)}{p^2}. \qquad (3.3)
\end{aligned}$$

If r passes through a complete residue system mod$p$, then $4m^2r-m^2b^2$ also passes through a complete residue system mod p. So if $p\equiv 1 \bmod 4$, then for any non-principal even character χ_1 mod p with $\chi_1(-1)=\chi_2(-1)$, $\chi_1\chi_2$ is also an even character mod p. For any odd character χ mod p, $\chi\chi_1\chi_2$ and $\chi\overline{\chi}_1\chi_2$ are both non-principal characters mod p and $\chi_2(-1)=1$. From (3.1), (3.2), Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}
&\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}|G(\chi_1,b,r,m;p)|^2\cdot|G(\chi_1,b,s,m;p)|^2\\
&\cdot S\left((4r-b^2)\cdot\overline{4s-b^2},p\right)\\
&=\frac{1}{\pi^2}\cdot\frac{p}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{r=0}^{p-1}\chi\left(4r-b^2\right)|G(\chi_1,b,r,m;p)|^2\right|^2\cdot|L(1,\chi)|^2 \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\sum_{r=0}^{p-1}\chi\left(4r-b^2\right)|G(\chi_1,b,r,m;p)|^2\right|^2\\
&=\left|\sum_{r=0}^{p-1}\chi\left(4r-b^2\right)\left|\sum_{a=1}^{p-1}\overline{\chi}_1\chi_2(a)\cdot e\left(\frac{(4m^2r-m^2b^2)a-\overline{16}\cdot\overline{a}}{p}\right)\right|^2\right|^2\\
&=\left|\sum_{r=0}^{p-1}\chi\left(r\right)\left|\sum_{a=1}^{p-1}\overline{\chi}_1\chi_2(a)\cdot e\left(\frac{ra-\overline{16}\cdot\overline{a}}{p}\right)\right|^2\right|^2\\
&=\left|\frac{\chi(-16)\overline{\chi}_1(-1)\overline{\chi}_1\chi_2(-1)\tau^2(\chi)\tau(\chi\overline{\chi}_1\chi_2)\tau(\chi\chi_1\chi_2)}{\tau(\chi^2)}\right|^2=p^2. \qquad (3.5)
\end{aligned}$$

Now combining (3.3), (3.4) and (3.5) we have

$$\begin{aligned}
&\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}|G(\chi_1,b,r,m;p)|^2\cdot|G(\chi_1,b,s,m;p)|^2\\
&\cdot S\left((4r-b^2)\cdot\overline{4s-b^2},p\right)\\
&=\frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}p^2\cdot|L(1,\chi)|^2=\frac{p(p-1)(p-2)}{12}. \qquad (3.6)
\end{aligned}$$

If $p\equiv 1 \bmod 4$ and χ_1 is an odd character mod p, then $\chi_1(-1)\neq\chi_2(-1)$, $\chi_1\chi_2$ and $\overline{\chi}_1\chi_2$ are both odd characters mod p. By the method in the proof of

(3.6) we have

$$\begin{aligned}
&\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}\left|G(\chi_1,b,r,m;p)\right|^2\cdot\left|G(\chi_1,b,s,m;p)\right|^2\\
&\cdot S\left((4r-b^2)\cdot\overline{4s-b^2},p\right)\\
&=\frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1\\ \chi\neq\chi_1\chi_2,\ \overline{\chi}_1\chi_2}} p^2\cdot|L(1,\chi)|^2+\frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot p\\
&\qquad\cdot\left|L\left(1,\chi_1\chi_2\right)\right|^2+\frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot p\cdot\left|L\left(1,\overline{\chi}_1\chi_2\right)\right|^2\\
&=\frac{p(p-1)(p-2)}{12}-\frac{2}{\pi^2}\cdot p^2\cdot\left|L\left(1,\chi_1\chi_2\right)\right|^2. \tag{3.7}
\end{aligned}$$

Now combining (3.6) and (3.7) we may immediately deduce the identity

$$\begin{aligned}
&\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}\left|G(\chi,b,r,m;p)\right|^2\left|G(\chi,b,s,m;p)\right|^2 S\left((4r-b^2)\overline{4s-b^2},p\right)\\
&=\begin{cases}\frac{p(p-1)(p-2)}{12}, & \text{if } \chi(-1)=\chi_2(-1);\\ \frac{p(p-1)(p-2)}{12}-\frac{2}{\pi^2}\cdot p^2\cdot\left|L\left(1,\chi\chi_2\right)\right|^2, & \text{if } \chi(-1)\neq\chi_2(-1).\end{cases} \tag{3.8}
\end{aligned}$$

If $p\equiv 3 \bmod 4$, then χ_2 is an odd character mod p. Then for any odd character χ_1 mod p, $\chi_1(-1)=\chi_2(-1)$, $\chi_1\chi_2$ and $\overline{\chi}_1\chi_2$ are both even characters mod p. By the method employed in the proof of (3.6) and the first formula in Lemma 2.3 we have

$$\begin{aligned}
&\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}\left|G(\chi_1,b,r,m;p)\right|^2\cdot\left|G(\chi_1,b,s,m;p)\right|^2\\
&\cdot S\left((4r-b^2)\cdot\overline{4s-b^2},p\right)\\
&=\frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} p^2\cdot|L(1,\chi)|^2=\frac{p(p-1)(p-2)}{12}. \tag{3.9}
\end{aligned}$$

For any non-principal even character χ_1 mod p, $\chi_1(-1)\neq\chi_2(-1)$, $\chi_1\chi_2$ and $\overline{\chi}_1\chi_2$ are both odd characters mod p, and they are different from the Legendre symbol χ_2. Thus by the method used in the proof of (3.7) we deduce that

$$\begin{aligned}
&\sum_{r=0}^{p-1}\sum_{s=0}^{p-1}\left|G(\chi_1,b,r,m;p)\right|^2\cdot\left|G(\chi_1,b,s,m;p)\right|^2\\
&\cdot S\left((4r-b^2)\cdot\overline{4s-b^2},p\right)\\
&=\frac{p(p-1)(p-2)}{12}-\frac{2}{\pi^2}\cdot p^2\cdot\left|L\left(1,\chi_1\chi_2\right)\right|^2. \qquad (3.10)
\end{aligned}$$

Now theorem 1.1 follows from (3.8), (3.9) and (3.10). This completes the proof of the theorem.

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