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FINITE GROUPS WITH A CERTAIN NUMBER OF ELEMENTS PAIRWISE GENERATING A NON-NILPOTENT SUBGROUP†

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Abstract. Let $n > 0$ be an integer and $\mathcal{X}$ be a class of groups. We say that a group G satisfies the condition $(\mathcal{X}, n)$ whenever in every subset with $n+1$ elements of G there exist distinct elements x, y such that $\langle x, y\rangle$ is in $\mathcal{X}$. Let $\mathcal{N}$ and $\mathcal{A}$ be the classes of nilpotent groups and abelian groups, respectively. Here we prove that: (1) If G is a finite semi-simple group satisfying the condition $(\mathcal{N}, n)$, then $|G| < c^{2[\log_{21} n]n^2}[\log_{21} n]!$, for some constant c. (2) A finite insoluble group G satisfies the condition $(\mathcal{N}, 21)$ if and only if $G/Z^*(G) \cong A_5$, the alternating group of degree 5, where $Z^*(G)$ is the hypercentre of G. (3) A finite non-nilpotent group G satisfies the condition $(\mathcal{N}, 4)$ if and only if $G/Z^*(G) \cong S_3$, the symmetric group of degree 3. (4) An insoluble group G satisfies the condition $(\mathcal{A}, 21)$ if and only if $G \cong Z(G) \times A_5$, where $Z(G)$ is the centre of G. (5) If d is the derived length of a soluble group satisfying the condition $(\mathcal{A}, n)$, then $d = 1$ if $n \in \{1, 2\}$ and $d \leq 2n-3$ if $n \geq 2$.

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1. **Introduction and results**

Let $n > 0$ be an integer and $\mathcal{X}$ be a class of groups. We say that a group G satisfies the condition $(\mathcal{X}, n)$ whenever in every subset with $n+1$ elements of G there exist distinct elements x, y such that $\langle x, y\rangle$ is in $\mathcal{X}$. If $\mathcal{X}$ is subgroup-closed, then every group which is the union of n $\mathcal{X}$-subgroups satisfies the condition $(\mathcal{X}, n)$. Let $\mathcal{N}$ be the class of nilpotent groups. Tomkinson in [23] proved that if G is a finitely generated soluble group satisfying the condition $(\mathcal{N}, n)$, then $|G/Z^*(G)| < n^{n^4}$, where $Z^*(G)$ is the hypercentre of G. This result gives a bound for the size of every finite soluble centerless group satisfying the condition $(\mathcal{N}, n)$; on the other hand, Endimioni in [10] proved that if $n \leq 20$, then every finite group satisfying the condition $(\mathcal{N}, n)$ is soluble, and A_5, the alternating group of degree 5, satisfies the condition $(\mathcal{N}, 21)$. Hence for $n \leq 20$ and all soluble groups, we have a positive answer to the following question:
Does there exist a bound (depending only on n) for the size of every centerless finite group satisfying the condition $(\mathcal{N}, n)$?

Here we find a bound for the size of finite semi-simple groups satisfying the condition $(\mathcal{N}, n)$ and also for all finite centerless groups satisfying the condition $(\mathcal{N}, 21)$. We also obtain a characterization for A_5 (see Corollary 2.11, below). The main results are

Theorem 1.1. *Let G be a finite semi-simple group satisfying the condition $(\mathcal{N}, n)$. Then $|G| < c^{2[\log_{21} n]n^2}[\log_{21} n]!$, for some constant c.*

Theorem 1.2. *Let G be a finite insoluble group. Then G satisfies the condition $(\mathcal{N}, 21)$ if and only if $G/Z^*(G) \cong A_5$.*

In [10] Endimioni proved that if $n \leq 3$, then every finite group satisfying the condition $(\mathcal{N}, n)$ is nilpotent, and S_3, the symmetric group of degree 3, satisfies the condition $(\mathcal{N}, 4)$. In fact, the only non-trivial finite centerless group satisfying the condition $(\mathcal{N}, 4)$ is

S_3. In section 2, we investigate finite groups satisfying the condition $(\mathcal{N}, 4)$.

Theorem 1.3. *Let G be a non-nilpotent finite group. Then G satisfies the condition $(\mathcal{N}, 4)$ if and only if $G/Z^*(G) \cong S_3$.*

It follows from Corollaries 2.12 and 3.4 below that a finite group satisfies the condition $(\mathcal{N}, 4)$ (respectively, $(\mathcal{N}, 21)$) if and only if it is the union of 4 (respectively, 21) nilpotent subgroups. Another natural question is: "For which positive integers n is every finite group satisfying the condition $(\mathcal{N}, n)$ the union of n nilpotent subgroups?"

In section 3, we investigate (not necessarily finite) groups satisfying the condition $(\mathcal{A}, n)$, where $\mathcal{A}$ is the class of abelian groups. Indeed, in a group satisfying the condition $(\mathcal{A}, n)$, the largest set of non-commuting elements (or the largest set of elements in which no two generate an abelian subgroup) has size at most n. By a result of B.H. Neumann [19] a group satisfies the condition $(\mathcal{A}, n)$ for some $n \in \mathbb{N}$ if and only if it is centre-by-finite. In fact, Neumann answered affirmatively the following question of P. Erdös [19]: Let G be an infinite group. If there is no infinite subset of G whose elements do not mutually commute, is there then a finite bound on the cardinality of each such set of elements? Neumann [19] proved that a group has the condition of Erdös's question if and only if it is centre-by-finite. This result has initiated a great deal of research towards the determination of the structure of groups having some similar properties (for example see [1],[2],[3],[4],[5],[8],[9], [11],[13],[16],[17],[18],[22]).

Pyber in [20] gave a bound for the index of the centre of a group satisfying the condition $(\mathcal{A}, n)$. Here we characterize insoluble groups satisfying the condition $(\mathcal{A}, 21)$. Note that every group satisfying the condition $(\mathcal{A}, n)$ also satisfies the condition $(\mathcal{N}, n)$.

Theorem 1.4. *Let G be an insoluble group. Then G satisfies the condition $(\mathcal{A}, 21)$ if and only if $G \cong Z(G) \times A_5$.*

We also obtain a result which is of independent interest, namely, the derived length of soluble groups satisfying the condition $(\mathcal{A}, n)$ is bounded by a function depending only on n.

Theorem 1.5. *Let G be a soluble group satisfying the condition $(\mathcal{A}, n)$ and let d be the derived length of G. Then $d = 1$ if $n \in \{1, 2\}$ and $d \leq 2n - 3$ if $n \geq 2$.*

2. Semi-simple groups satisfying the condition $(\mathcal{N}, n)$ and insoluble groups satisfying the condition $(\mathcal{N}, 21)$

Recall that a group G is semi-simple if G has no non-trivial normal abelian subgroups. If G is a finite group then we call the product of all minimal normal non-abelian subgroups of G the centerless CR-radical of G; it is a direct product of non-abelian simple groups (see page 88 of [21]).

We first prove a result on the direct product of (not necessarily finite) groups not satisfying the condition $(\mathcal{X}, n)$, for a certain class $\mathcal{X}$ of groups. This result may also be useful in other investigations on groups satisfying the condition $(\mathcal{X}, n)$. For example, if one can find a bound depending only on n for the size of finite non-abelian simple groups satisfying the condition $(\mathcal{X}, n)$, then by the aid of Lemma 2.1 below, it is easy to see that there exists a bound depending only on n for the size of every semi-simple finite group satisfying the condition $(\mathcal{X}, n)$ (for instance see Theorem 1.1).

Lemma 2.1. *Let $\mathcal{X}$ be a class of groups which is closed with respect to homomorphic images. Suppose for $i \in \{1, \ldots, t\}$ that H_i is a group not satisfying the condition $(\mathcal{X}, n_i)$. Then $H_1 \times \cdots \times H_t$ does not satisfy the condition $(\mathcal{X}, m)$, where $m = n_1 + \cdots + n_t$.*

Proof. It suffices to show that if H and K are two groups which do not satisfy $(\mathcal{X}, n)$ and $(\mathcal{X}, m)$, respectively, then $H \times K$ does not satisfy the condition $(\mathcal{X}, n+m)$. By the hypothesis, there exist $x_1, \dots, x_{n+1}$ in H and $y_1, \dots, y_{m+1}$ in K such that $\langle x_i, x_j \rangle \notin \mathcal{X}$ for $1 \le i < j \le n+1$ and $\langle y_k, y_l \rangle \notin \mathcal{X}$ for $1 \le k < l \le m+1$. Now it is easy to see that the subgroup generated by each pair of distinct elements of the set

$$\{(x_2, 1), \dots, (x_{n+1}, 1), (x_1, y_1), (x_1, y_2), \dots, (x_1, y_{m+1})\},$$

does not have the property $\mathcal{X}$. $\square$

Our next lemma is about the direct product of finite groups not satisfying $(\mathcal{N}, n)$. For finite groups, this is a better result than Lemma 2.1.

Lemma 2.2. *Suppose that H_i is a finite group not satisfying the condition $(\mathcal{N}, n_i)$ for $i \in \{1, \dots, t\}$. Then $H_1 \times \cdots \times H_t$ does not satisfy the condition $(\mathcal{N}, m)$, where $m = (n_1 + 1) \cdots (n_t + 1) - 1$.*

Proof. By the hypothesis, for every $i \in \{1, \dots, t\}$ there exists a subset X_i in H_i of size $n_i + 1$ such that no pair of its distinct elements generate a nilpotent subgroup. Now we show that the subgroup generated by each pair of distinct elements of the set $X = X_1 \times \cdots \times X_t$ is not nilpotent. Let $a = (a_1, \dots, a_t), b = (b_1, \dots, b_t)$ be two distinct elements of X. Then for some $i \in \{1, \dots, t\}$, $a_i \neq b_i$. Since $a_i, b_i \in X_i$, we have that $K := \langle a_i, b_i \rangle$ is not nilpotent. Since K is a finite non-nilpotent group, it is not an Engel group by a result of Zorn (see Theorem 12.3.4 of [21]). Therefore there exist elements $x, y \in K$ such that $[x,_n y] \neq 1$ for all $n \in \mathbb{N}$. Suppose that

$$x = a_i^{\delta_1} b_i^{\delta_2} \cdots a_i^{\delta_{r-1}} b_i^{\delta_r} \quad \text{and} \quad y = a_i^{\epsilon_1} b_i^{\epsilon_2} \cdots a_i^{\epsilon_{s-1}} b_i^{\epsilon_s}$$

where $\delta_p, \epsilon_q \in \{0, 1, -1\}$ for all $p \in \{1, \dots, r\}$ and $q \in \{1, \dots, s\}$. Suppose, for a contradiction, that $\langle a, b \rangle$ is nilpotent. Then there exists a positive integer m such that $[\bar{x},_m \bar{y}] = 1$ where

$$\bar{x} = a^{\delta_1} b^{\delta_2} \cdots a^{\delta_{r-1}} b^{\delta_r} \quad \text{and} \quad \bar{y} = a^{\epsilon_1} b^{\epsilon_2} \cdots a^{\epsilon_{s-1}} b^{\epsilon_s}.$$

But
$$[\bar{x},_m \bar{y}] = ([x_1,_m y_1], \ldots, [x_t,_m y_t]),$$
where
$$x_j = a_j^{\delta_1} b_j^{\delta_2} \cdots a_j^{\delta_{r-1}} b_j^{\delta_r} \text{ and } y_j = a_j^{\epsilon_1} b_j^{\epsilon_2} \cdots a_j^{\epsilon_{s-1}} b_j^{\epsilon_s}$$
for all $j \in \{1, \ldots, t\}$. Hence $[x,_m y] = [x_i,_m y_i] = 1$, a contradiction. This completes the proof. □

Lemma 2.3. *Let $M_1, \ldots, M_m$ be non-abelian finite simple groups. Then $M_1 \times \cdots \times M_m$ does not satisfy the condition $(\mathcal{N}, 21^m - 1)$.*

Proof. Since by Proposition 2 of [10], M_i does not satisfy the condition $(\mathcal{N}, 20)$ for all $i \in \{1, \ldots, m\}$, the proof follows easily from Lemma 2.2. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let R be the centerless CR-radical of G. Then R is a direct product of a finite number m of finite non-abelian simple groups and G is embedded in $\mathrm{Aut}(R)$. Then by Lemma 2.3, we have $21^m - 1 < n$ and so $m \le [\log_{21} n]$. On the other hand, since $Z(G) = 1$, by Lemma 3.3 of [23] every prime divisor of $|G|$ is less than n. Thus by Remark 5.5 of [6], there is a constant c such that the order of every non-abelian simple section of G is less than c^{n^2}. Hence $|R| < c^{n^2[\log_{21} n]}$. Now using the following well-known facts that: (a) for a finite simple group S we have $|\mathrm{Aut}(S)| < |S|^2$ and (b) if R is the product of m simple groups S_i, then G acts on these factors, the quotient group is embeddable into $\mathrm{Sym}(m)$ and the kernel K of the action is embeddable into the product of groups $\mathrm{Aut}(S_i)$; hence $|K| < |R|^2$. Thus $|G| < c^{2n^2[\log_{21} n]}[\log_{21} n]!$. □

Since in every finite group G, the quotient $G/\mathrm{Sol}(G)$ is semisimple, where $\mathrm{Sol}(G)$ is the soluble radical (the largest soluble normal subgroup) of G, we have

Corollary 2.4. *Let G be a finite group satisfying $(\mathcal{N}, n)$. Then*

$$|G/\mathrm{Sol}(G)| < c^{2n^2[\log_{21} n]}[\log_{21} n]!$$

for some constant c.

Combining the result of Tomkinson quoted in the introduction and Corollary 2.4, we obtain as a further nice corollary that in fact:

Corollary 2.5. *Let G be a finite group satisfying $(\mathcal{N}, n)$. Then*

$$|G/F(G)| < n^{n^4} c^{2n^2[\log_{21} n]}[\log_{21} n]!$$

for some constant c, where $F(G)$ is the largest nilpotent normal subgroup of G.

We need the following proposition, which is of independent interest, in the proof of Proposition 2.7.

Proposition 2.6. *Let p be a prime number, n a positive integer and r and q be two odd prime numbers dividing respectively $p^n + 1$ and $p^n - 1$. Then the number of Sylow r-subgroups (respectively, q-subgroups) of $G = \mathrm{PSL}(2, p^n)$ is $\frac{p^n(p^n-1)}{2}$ (respectively, $\frac{p^n(p^n+1)}{2}$). Also the intersection of every two distinct Sylow r-subgroups or q-subgroups is trivial.*

Proof. Our proof uses Theorems 8.3 and 8.4 in chapter II of [14]. Let q be an odd prime dividing $p^n - 1$ and let $k = \gcd(p^n - 1, 2)$. By Theorem 8.3 in Chapter II of [14], $\mathrm{PSL}(2, p^n)$ possesses a cyclic subgroup U of order $u = \frac{p^n-1}{k}$ such that

(1) The intersection of every two distinct conjugates of U is trivial.
(2) For every non-trivial element w of U, the normalizer $N_G(\langle w \rangle)$ of $\langle w \rangle$ is a dihedral group of order $2u$.

Since q is an odd prime number, q divides u, and since $|G| = \frac{p^n(p^n+1)(p^n-1)}{k}$, we have $\gcd(p^n(p^n + 1), q) = 1$. It follows that any Sylow q-subgroup of U is also a Sylow q-subgroup of G and each of them is cyclic. Therefore it follows from (2) that the number of Sylow q-subgroups of G is $\frac{p^n(p^n+1)}{2}$. Now (1) implies that the intersection of every two distinct Sylow q-subgroups of G is trivial. By a similar argument the second statement of the proposition follows from the corresponding parts of Theorem 8.4 in Chapter II of [14], namely that the group G contains a cyclic subgroup K of order $s = \frac{p^n+1}{k}$ such that

(1) The intersection of every two distinct conjugates of K is trivial.
(2) For every non-trivial element t of K, the normalizer $N_G(\langle t \rangle)$ of $\langle t \rangle$ is a dihedral group of order $2s$. $\square$

Proposition 2.7. *The only non-abelian finite simple group satisfying the condition* $(\mathcal{N}, 21)$ *is* A_5.

Proof. Suppose, for a contradiction, that there exists a non-abelian finite simple group satisfying the condition $(\mathcal{N}, 21)$ which is not isomorphic to A_5. Let G be such a group of least order. Thus every proper non-abelian simple section of G is isomorphic to A_5. Therefore by Proposition 3 of [7], G is isomorphic to one of the following:
$\mathrm{PSL}(2, 2^p)$, $p = 4$ or a prime;
$\mathrm{PSL}(2, 3^p)$, $\mathrm{PSL}(2, 5^p)$, p a prime;
$\mathrm{PSL}(2, p)$, p a prime ≥ 7;
$\mathrm{PSL}(3, 3)$, $\mathrm{PSL}(3, 5)$;
$\mathrm{PSU}(3, 4)$ (the projective special unitary group of degree 3 over the finite field of order 4^2) or
$Sz(2^p)$, p an odd prime.

For each prime divisor p of $|G|$, let $\nu_p(G)$ be the number of all Sylow p-subgroups of G. If p is a prime number dividing $|G|$ such that the intersection of any two distinct Sylow p-Subgroups is trivial, then by Lemma 3 of [10], $\nu_p(G) \leq 21$ (*). Now, for every prime number p and every integer $n > 0$, we have $\nu_p(\mathrm{PSL}(2, p^n)) = p^n + 1$ and the intersection of any two distinct Sylow p-subgroups is trivial (see chapter II Theorem 8.2 (b),(c) of [14]). Thus among the projective special linear groups, we only need to investigate the following groups:

$$\mathrm{PSL}(2, 3^2), \mathrm{PSL}(2, 8), \mathrm{PSL}(2, 2^4), \mathrm{PSL}(3, 3), \mathrm{PSL}(3, 5), \mathrm{PSL}(2, p)$$

for $p \in \{7, 11, 13, 17, 19\}$. Now if in Proposition 2.6, we take $q = 7$ for PSL$(2, 8)$; $q = 5$ for PSL$(2, 16)$; $r = 5$ for PSL$(2, 9)$; $q = 3$ for PSL$(2, 7)$, PSL$(2, 13)$ and PSL(2,19); and $r = 3$ for PSL$(2, 11)$ and PSL$(2, 17)$; we see, by (*), that G cannot be isomorphic with any of these groups. Therefore we must consider the groups $\mathrm{PSL}(3, 3), \mathrm{PSL}(3, 5), \mathrm{PSU}(3, 4)$ or $Sz(2^p)$, p an odd prime.

$H := \mathrm{PSL}(3, 3)$ has order $2^4 \times 3^3 \times 13$, so $\nu_{13}(H) = 1 + 13k$, for some $k > 0$ and since 14 does not divide $|H|$, $\nu_{13}(H) > 26$.

$K := \mathrm{PSL}(3, 5)$ has order $5^3 \times 2^5 \times 3 \times 31$, so $\nu_{31}(K) = 1+31k > 21$ for some $k > 0$.

$L := \mathrm{PSU}(3, 4)$ has order $2^6 \times 5^2 \times 13$ (see Theorem 10.12(d) of chapter II in [14] and note that L is the projective special unitary group of degree 3 over the finite field of order 4^2). Therefore $\nu_{13}(L) = 1 + 13k > 21$ for some $k > 0$ and since 14 does not divide $|L|$, $\nu_{13}(L) > 26$.

$M := \mathrm{Sz}(2^p)$ (p an odd prime) has order $2^{2p}(2^p - 1)(2^{2p} + 1)$ and $\nu_2(M) = 2^{2p} + 1 \geq 65$ (see Theorem 3.10 (and its proof) of chapter XI in [15]). This completes the proof by (*). $\square$

Lemma 2.8. *S_5, the symmetric group of degree 5, does not satisfy the condition $(\mathcal{N}, 21)$.*

Proof. Every subgroup generated by a pair of distinct elements of 22-element subset
{(3,4,5), (2,3,4), (2,3,4,5),(1,4,5), (2,3,5,4), (2,3,5), (2,4,5), (1,2,3), (1,2,3,4),
(1,2,4,5,3), (1,2,4,3,5), (1,2,5),(1,3,4), (1,3,4,5), (1,3,5), (1,3,2,4,5), (1,4,2),
(1,5,4,3,2), (1,5,3,2), (1,5,4,2), (1,5,2,4,3), (1,5,3,2,4)} is not nilpotent. $\square$

Remark 2.9. Here we state two properties of A_5 which we use in the sequel. Suppose that $P_1, \ldots, P_{21}$ are all the Sylow subgroups of A_5. Then
(i) For all $x_i \in P_i \backslash \{1\}$ ($i = 1, \ldots, 21$), the set $\{x_1, \ldots, x_{21}\}$ is a subset of A_5 such that no pair of its distinct elements generate a

nilpotent subgroup. (See the proof of Proposition 2 of [10]).
(ii) $A_5 = \cup_{i=1}^{21} P_i$.

We use the following fact in the sequel without any specific reference. If G is any group such that $G/Z_m(G)$ is nilpotent for some integer $m \geq 0$, then G is nilpotent. Because $Z_n(G/Z_m(G)) = G/Z_m(G)$ for some integer $n \geq 0$ and so by Theorem 5.1.11 (iv) of [21], we have $Z_{m+n}(G) = G$, which implies that G is nilpotent.

Lemma 2.10. *Let G be a finite insoluble group satisfying the condition $(\mathcal{N}, 21)$ and let $S = \mathrm{Sol}(G)$ be the soluble radical of G. Then $G/S \cong A_5$, and for all $a \in S$ and for all $x \in G \backslash S$, the subgroup $\langle a, x\rangle$ is nilpotent. In particular, $Z^*(G) = Z^*(S)$.*

Proof. Let S be the soluble radical of G and consider the semisimple group $\overline{G} = G/S$. Let $\overline{R}$ be the centerless CR-radical of $\overline{G}$. Then $\overline{R}$ is a direct product of non-abelian simple groups. Since G is insoluble, $\overline{R}$ is non-trivial. Now, by Lemma 2.3 and Proposition 2.7, $\overline{R} \cong A_5$. Since $C_{\overline{G}}(\overline{R}) = 1$, we have $\overline{G} \cong A_5$ or S_5. By Lemma 2.8, $\overline{G} \cong A_5$. Now, let $Q_1, \dots, Q_{21}$ be the Sylow subgroups of G/S. For each $i \in \{1, \dots, 21\}$, let x_iS be a non-trivial element of Q_i. Then, by Remark 2.9(i), $\langle x_i, x_j\rangle S \notin \mathcal{N}$ and so $\langle x_i, x_j\rangle \notin \mathcal{N}$ for all distinct $i, j \in \{1, \dots, 21\}$. Now, fix $k \in \{1, \dots, 21\}$ and for an arbitrary element $a \in S$ consider the elements

$$x_k, x_1, \dots, x_{k-1}, ax_k, x_{k+1}, \dots, x_{21}.$$

For $k, j \in \{1, \dots, 21\}$ and $j \neq k$, $\langle ax_k, x_j\rangle$ is not nilpotent, since $\langle ax_k, x_j\rangle S = \langle x_k, x_j\rangle S$. Since G satisfies the condition $(\mathcal{N}, 21)$, the subgroup $\langle x_k, ax_k\rangle$ is nilpotent and hence so is $\langle a, x_k\rangle$ for all $k \in \{1, \dots, 21\}$. On the other hand, the union of the subgroups $Q_1, \dots, Q_{21}$ is G/S, by Remark 2.9(ii), and so $\langle a, x\rangle$ is nilpotent for all $x \in G\backslash S$ and for all $a \in S$.

Since S is finite, $Z^*(S) = Z_m(S)$ for some $m \in \mathbb{N}$. Now for all $a \in Z_m(S)$ and for all $b \in S$, the subgroup $T := \langle a, b\rangle$ is nilpotent, since $TZ_m(S)/Z_m(S) \cong T/(T\cap Z_m(S))$ is cyclic and $T\cap Z_m(S) \leq Z_m(T)$.

Thus $\langle a, x\rangle$ is nilpotent for all $a \in Z^*(S)$ and for all $x \in G$. Since G is finite, a is a right Engel element for all $a \in Z^*(S)$ (see Theorem 12.3.7 of [21]) and so $Z^*(S) \leq Z^*(G)$. Hence $Z^*(S) = Z^*(G)$. This completes the proof. $\Box$

Proof of Theorem 1.2. Suppose that G satisfies the condition $(\mathcal{N}, 21)$ and suppose, for a contradiction, that G is a counterexample of least order subject to $G/Z^*(G) \ncong A_5$. Let $S = \mathrm{Sol}(G)$ be the soluble radical of G. We claim that $Z(S) = 1$. For if $Z(S) \neq 1$ then $G/Z(S)$ is a finite insoluble group satisfying the condition $(\mathcal{N}, 21)$ and since $|G/Z(S)| < |G|$ and the soluble radical of $G/Z(S)$ is $S/Z(S)$, we have that the assertion of Theorem 1.2 is true for the group $G/Z(S)$, i.e.

$$\frac{G/Z(S)}{Z^*(G/Z(S))} \cong A_5. \qquad (*)$$

Now Lemma 2.10 implies that $Z^*(S/Z(S)) = Z^*(G/Z(S))$. On the other hand

$$Z^*(S/Z(S)) = Z^*(S)/Z(S) = Z^*(G)/Z(S),$$

by Lemma 2.10 (note that for a finite group K we have $Z^*(K) = Z_m(K)$ for some integer $m > 0$). Thus it follows from $(*)$ that $G/Z^*(G) \cong A_5$ which is a contradiction. Hence $Z(S) = 1$, which implies that $Z^*(S) = 1$.

Now, let $x \in G \backslash S$ be such that $x^2 \in S$. Thus for all $b \in S$, we have $bx \in G\backslash S$ and $(bx)^2 \in S$. By Lemma 2.10, $\langle bx, a\rangle$ is nilpotent for all $a \in S$, and so also is $\langle (bx)^2, a\rangle$. Therefore $(bx)^2$ is a right Engel element of S and so $(bx)^2 \in Z^*(S) = 1$. Thus for all $b \in S$, $(bx)^2 = 1$. Now, again by Lemma 2.10, $\langle bx, x\rangle = \langle b, x\rangle$ is nilpotent and so is $\langle b, x^2\rangle$. Thus as before $x^2 = 1$. Therefore $D := \langle b, x\rangle$ is a finite dihedral group which is nilpotent and so $|D|$ is a power of 2 and b is a 2-element. Hence S is a 2-group, and since $Z(S) = 1$, we conclude that S must be trivial. Therefore, by Lemma 2.10, $Z^*(G) = 1$ and $G/Z^*(G) = G/S \cong A_5$, a contradiction.

Conversely, suppose that $G/Z^*(G) \cong A_5$. By Remark 2.9(ii),

$$\frac{G}{Z^*(G)} = \bigcup_{i=1}^{21} \frac{P_i}{Z^*(G)},$$

where $P_1/Z^*(G), \ldots, P_{21}/Z^*(G)$ are the Sylow subgroups of $G/Z^*(G)$. But G is finite, so $Z^*(G) = Z_m(G)$ for some $m \in \mathbb{N}$. Since $Z_m(G) \leq Z_m(P_i)$ for all $i \in \{1, \ldots, 21\}$ and $P_i/Z_m(G)$ is nilpotent, we conclude that each P_i is nilpotent. Now the proof is complete, since $G = \cup_{i=1}^{21} P_i$. $\Box$

From Theorem 1.2 we have a nice characterization for A_5.

Corollary 2.11. *The only finite centerless insoluble group satisfying the condition $(\mathcal{N}, 21)$ is A_5.*

Theorem 1.2 also gives us the following consequences.

Corollary 2.12. *A finite insoluble group satisfies the condition $(\mathcal{N}, 21)$ if and only if it is covered by 21 nilpotent subgroups.*

Corollary 2.13. *Let G be a finite group satisfying the condition $(\mathcal{N}, 21)$. If the centerless CR-radical of G is non-trivial, then $G \cong A_5 \times Z^*(G)$.*

Proof. Let R be the centerless CR-radical of G. Then R is a non-trivial direct product of some non-abelian simple groups and so by Lemma 2.3 and Proposition 2.7, $R \cong A_5$. Since R is simple, $R \cap Z^*(G) = 1$. But, by Theorem 1.2, $|G| = |Z^*(G)||A_5|$, and so $G \cong A_5 \times Z^*(G)$. $\Box$

Remark 2.14. We note that not every finite insoluble group satisfying the condition $(\mathcal{N}$,21) is necessarily isomorphic to a direct product as in Corollary 2.13. For example if $K := \mathrm{SL}(2,5)$ then $K/Z(K) \cong A_5$ and so K satisfies the condition ($\mathcal{N}$,21), by Theorem 1.2. However we conjecture that every finite insoluble group satisfying the condition $(\mathcal{N},21)$ is a direct product of a nilpotent group and a group isomorphic to either A_5 or $\mathrm{SL}(2,5)$.

3. Finite groups satisfying the condition $(\mathcal{N},4)$

In this section, we investigate finite groups satisfying the condition $(\mathcal{N},4)$, and give the proof of Theorem 1.3.

Lemma 3.1. *Let G be a finite $\{2,3\}$-group. If G satisfies the condition $(\mathcal{N},4)$, then G is 2-nilpotent.*

Proof. Suppose that G is a counterexample of least order. Thus by a result of Itô (see Theorem 5.4 on page 434 of [14]), G is a minimal non-nilpotent group and G has a unique Sylow 2-subgroup P and a cyclic Sylow 3-subgroup Q such that $\Phi(Q) \leq Z(G)$ and $\Phi(P) \leq Z(G)$ (see Theorem 5.2 on page 281 of [14]). If $Z(G) \neq 1$ then $G/Z(G)$ is nilpotent and so G is nilpotent, a contradiction. Thus $Z(G) = 1$ and so $|Q| = 3$ and P is an elementary abelian 2-group.

Let $Q = \langle a \rangle$. Then $C_P(a) \leq Z(G)$, and so $C_P(a) = 1$. On the other hand by Lemma 3.4 of [23], $|P : C_P(a)| \leq 4$ and so $|P| \leq 4$. If $|P| = 4$ then $G \cong A_4$, the alternating group of degree 4. But A_4 does not satisfy the condition $(\mathcal{N},4)$; thus $|P| = 2$. Therefore $G \cong S_3$, a contradiction, since S_3 is 2-nilpotent. This completes the proof. □

Lemma 3.2. *Let $G = RX$ be an extension of an elementary abelian 3-group R by an abelian 2-group X such that X acts faithfully on R and $R = [R,X]$. If G satisfies the condition $(\mathcal{N},4)$, then $|X| \leq 2$ and $|R| \leq 3$.*

Proof. The proof follows from the argument of Lemma 3.7 of [23]. □

We are now ready to give a proof for Theorem 1.3, the outline of which is in fact a refinement of that of Theorem C in [23] for $n = 4$.

Proof of Theorem 1.3. Suppose that G satisfies the condition $(\mathcal{N},4)$. By factoring out $Z^*(G)$, we may assume that G is a finite non-trivial group with trivial centre satisfying the condition $(\mathcal{N},4)$. We note that G is a $\{2,3\}$-group by Lemma 3.3 of [23].

Let $H_p/O_{p'}(G)$ be the hypercentre of $G/O_{p'}(G)$, for $p = 2,3$. Then, since G is finite, there is a positive integer m such that $[H_{p,m}\,G] \leq O_{p'}(G)$ for $p = 2,3$. Hence

$$[H_2 \cap H_{3,m}\,G] \leq O_{2'}(G) \cap O_{3'}(G) = 1,$$

and so $H_2 \cap H_3 \leq Z^*(G) = 1$. But $O_{2'}(G) = O_3(G)$ and by Lemma 3.1, is the unique Sylow 3-subgroup of G. Thus $G/O_{2'}(G)$ is a 2-group and so $G = H_2$. Therefore $H_3 = 1$ and so $O_2(G) = 1$. Hence $P = \mathrm{Fitt}(G) = O_3(G)$. Let $\overline{G} = G/\Phi(P)$ and $\overline{P} = P/\Phi(P)$, thus $\overline{G}/\overline{P}$ acts faithfully on the GF(3)-vector space $\overline{P}$ (see [12], Theorem 6.3.4). We note that $\overline{P}$ is an elementary abelian normal 3-subgroup of $\overline{G}$, that $\overline{P} = O_3(\overline{G})$, and that $C_{\overline{G}}(\overline{P}) = \overline{P}$. Let $Q/\overline{P}$ be the socle of $\overline{G}/\overline{P}$, so that $Q/\overline{P}$ is an abelian 2-subgroup. We may write $Q = \overline{P}X$, where X is an abelian 2-subgroup of Q. Let $R = [\overline{P}, Q]$, so that $\overline{P} = R \times C_{\overline{P}}(Q)$. If $C = C_{\overline{G}}(R)$ then $C \cap Q$ centralizes $R \times C_{\overline{P}}(Q) = \overline{P}$ and so $C \cap Q = \overline{P}$. It follows that $C_{\overline{G}}(R) = \overline{P}$ and so $\overline{G}/\overline{P}$ acts faithfully on R. Now R and X satisfy the conditions of Lemma 3.2 and so $|R| \leq 3$. Since $\overline{G}/\overline{P}$ acts faithfully on R, the order of G/P is no more than 2. Let T be a Sylow 2-subgroup of G; then $|T| \leq 2$ and hence T is cyclic and by Lemma 3.4 of [23], $|P : C_P(T)| \leq 3$. Now, we have $[C_P(T),_m G] = [C_P(T),_m P] = 1$ for some $m \in \mathbb{N}$. Thus $C_P(T) \leq Z^*(G) = 1$ and so $|G| = |T||P| \leq 2 \times 3 = 6$. Therefore $G \cong S_3$.

Conversely, suppose that $G/Z^*(G) \cong S_3$. Since S_3 is covered by 4 abelian subgroups, G is also covered by 4 nilpotent subgroups. This completes the proof. □

Corollary 3.3. *Every finite group satisfying the condition* ($\mathcal{N}$,4) *is supersoluble. The alternating group A_4 satisfies the condition* ($\mathcal{N}$,5).

Proof. Let G be a finite group satisfying the condition ($\mathcal{N}$,4). By Proposition 1 of [10], $G = H \times K$, where H is a nilpotent $\{2,3\}'$-group and K is a $\{2,3\}$-group. If K is nilpotent, then there is nothing to prove. Assume that K is not nilpotent. By Theorem 1.3, $K/Z^*(K) \cong S_3$ and so K is supersoluble. Thus G is also a supersoluble group.

The group A_4 is the union of its five Sylow subgroups , so A_4 satisfies the condition ($\mathcal{N}$,5). □

Corollary 3.4. *A finite group satisfies the condition* ($\mathcal{N}$,4) *if and only if it is the union of four nilpotent subgroups.*

Proof. Let G be a finite group satisfying the condition ($\mathcal{N}$,4). Then by Theorem 1.3, $G/Z^*(G)$ is the union of 4 nilpotent subgroups and hence so is G. The converse is clear. □

4. Finite groups satisfying the condition ($\mathcal{A}$,n)

Now suppose that $\mathcal{A}$ is the class of abelian groups. Then every group satisfying the condition ($\mathcal{A}$,n) also satisfies the condition ($\mathcal{N}$,n). The converse is not true, since as we have observed already, $\mathrm{SL}(2,5)$ satisfies the condition ($\mathcal{N}$,21). However $\mathrm{SL}(2,5)$ does not satisfy the condition ($\mathcal{A}$,21).

Lemma 4.1. $\mathrm{SL}(2,5)$ *does not satisfy the condition* ($\mathcal{A}$,21).

Proof. Let $P_1,\dots,P_5$ be the Sylow 2-subgroups of $\mathrm{SL}(2,5)$, $Q_1,\dots,$ Q_{10} the Sylow 3-subgroups of $\mathrm{SL}(2,5)$, and $R_1,\dots,R_6$ the Sylow 5-subgroups of $\mathrm{SL}(2,5)$. We note for each $i=1,\dots,5$ that P_i is a quaternion group of order 8 and $Z(P_i) = Z(\mathrm{SL}(2,5))$ (see, for example, Theorem 8.10 in chapter II of [14]). Let $x_i \in P_i \backslash Z(P_i)$

$(i = 1, \ldots, 5)$, $y_j \in Q_j \backslash \{1\}$ $(j = 1, \ldots, 10)$ and $z_k \in R_k \backslash \{1\}$ $(k = 1, \ldots, 6)$. Then since $\mathrm{SL}(2,5)/Z(\mathrm{SL}(2,5)) \cong A_5$, it follows from Remark 2.9(i) that no two distinct elements of the set

$$\{x_1, \ldots, x_5, y_1, \ldots, y_{10}, z_1, \ldots, z_6\}$$

commute. Now since P_1 is a quaternion group of order 8 and $x_1 \in P_1 \backslash Z(P_1)$, there exists an element $x \in P_1 \backslash Z(P_1)$ such that $x_1 x \neq x x_1$. On the other hand, as above, no two distinct elements in

$$\{x, x_2 \ldots, x_5, y_1, \ldots, y_{10}, z_1, \ldots, z_6\}$$

commute. Therefore no two distinct elements in the set

$$\{x, x_1, \ldots, x_5, y_1, \ldots, y_{10}, z_1, \ldots, z_6\}$$

commute, which completes the proof. □

Lemma 4.2. *Let G be a finite group satisfying the condition $(\mathcal{A}$,21). If there exists a central subgroup B of G of order no more than 2 such that $G/B \cong A_5$, then $G \cong B \times A_5$.*

Proof. Since $G/B \cong A_5$ it follows that $G = G'B$ and $G'/(B \cap G') \cong A_5$. Therefore if $G' \cap B = 1$ then the proof is complete. So suppose, for a contradiction, that $G' \cap B \neq 1$. Thus $|B| = 2$. According to the Universal Coefficients Theorem (see Theorem 11.4.18 of [21]) the central extension $B \rightarrowtail G \twoheadrightarrow G/B$ determines a homomorphism $\delta : M(G/B) \to B$ so that $\mathrm{Im}\delta = G' \cap B$, where $M(G/B)$ is the Shur multiplicator of G/B (see for example Exercise 10 on page 354 of [21]). But we know that the Shur multiplicator of the alternating group A_5 is $\mathbb{Z}_2$. Hence $G' \cap B = B$ and so $B \leq G'$. It follows that G is a perfect group of order 120. But it is well-known that the only perfect group of order 120 is $\mathrm{SL}(2,5)$. Now Lemma 4.1 gives a contradiction and the proof is complete. □

We need the following lemma in the proof of Theorem 1.4.

Lemma 4.3. *Let G be a group satisfying the condition $(\mathcal{A}$,$n)$ $(n > 1)$. Then for any normal non-abelian subgroup N of G, the quotient G/N satisfies the condition $(\mathcal{A}$,$n-1)$.*

Proof. Suppose, for a contradiction, that $G/N \notin (\mathcal{A},n-1)$. Then there exist elements $x_1, \ldots, x_n$ in G such that $[x_i, x_j] \notin N$ for all distinct $i, j \in \{1, \ldots, n\}$ $(*)$. Let a, b be two distinct arbitrary elements of N and consider the subset $X = \{ax_1, \ldots, ax_n, bx_1\}$. By the hypothesis, there exist two distinct commuting elements in X. But, by $(*)$, the only commuting pair of elements of X are bx_1 and ax_1. Therefore for all $a, b \in N$, we have $ax_1b = bx_1a$ $(**)$ and in particular for $b = 1$, we have $ax_1 = x_1a$ for all $a \in N$. Thus for all $x, y \in N$ we have

$$xyx_1 = xx_1y = yx_1x = yxx_1$$

(the middle equality follows from $(**)$) and so $xy = yx$. Hence N is abelian, a contradiction. $\square$

Proof of Theorem 1.4. Suppose that $G \cong Z(G) \times A_5$. Then G is covered by 21 abelian subgroups as A_5 has this property, by Remark 2.9(ii).

Now, suppose that G satisfies the condition $(\mathcal{A},21)$. Then by a famous Theorem of B. H. Neumann [19], $G/Z(G)$ is finite. Thus, by Theorem 1.2,

$$\frac{G/Z(G)}{Z^*(G/Z(G))} \cong G/Z^*(G) \cong A_5.$$

If $H := Z^*(G)$ is not abelian, then Lemma 4.3 shows that A_5 satisfies the condition $(\mathcal{A},20)$, which contradicts Proposition 2 of [10]. Thus H is abelian; we show that in fact $H = Z(G)$. To prove this let $Q_1, \ldots, Q_{21}$ be the Sylow subgroups of $\overline{G} := G/H$. For each $i \in \{1, \ldots, 21\}$, let x_iH be a non-trivial element of Q_i. Then $[x_i, x_j] \notin H$ and so $[x_i, x_j] \neq 1$ for all distinct $i, j \in \{1, \ldots, 21\}$, by Remark 2.9(i). Now fix $k \in \{1, \ldots, 21\}$ and consider the elements

$$x_k, x_1, \ldots, x_{k-1}, ax_k, x_{k+1}, \ldots, x_{21},$$

for an arbitrary element $a \in H$. Then for $j \in \{1, \ldots, 21\}$ and $j \neq k$, we have $[ax_k, x_j] \neq 1$, since $[ax_k, x_j]H = [x_k, x_j]H$. Since G satisfies the condition $(\mathcal{A},21)$, $[x_k, ax_k] = 1$ and so $[a, x_k] = 1$ for all $k \in \{1, \ldots, 21\}$. Since the union of $Q_1, \ldots, Q_{21}$ is $\overline{G}$, by Remark

2.9(ii), we have $[a, x] = 1$ for all $x \in G \backslash H$ and for all $a \in H$. Therefore $H = Z(G)$.

Now by the same argument as in Lemma 4.2, by considering the central extension $Z(G) = H \rightarrowtail G \twoheadrightarrow \overline{G}$, we have that $K = G' \cap Z(G)$ is of order no more than 2, $G = G'Z(G)$ and $G'/K \cong A_5$. Thus Lemma 4.2 implies that there is a subgroup L of G' such that $G' = K \times L$ and $L \cong A_5$. Therefore $G = G'Z(G) = LKZ(G) = LZ(G)$ and it is clear that $L \cap Z(G) = 1$. Therefore $G = L \times Z(G) \cong A_5 \times Z(G)$. $\square$

We end this paper by proving Theorem 1.5.

Proof of Theorem 1.5. We first prove that if $n = 2$, then G is abelian. Consider two distinct elements $x, y \in G$. Then $X = \{x, y, xy\}$ is a subset of size 3. Thus by the hypothesis two distinct elements of X commute. But commutativity of each pair of distinct elements of X implies the commutativity of x and y. Hence G is abelian.

Now suppose that $n \geq 2$ and use induction on n. If $n = 2$ then G is abelian and $d = 1$. So let $n > 2$. Then $2 < 2n - 3$. Thus we may assume that $d > 2$. Therefore G^{d-2} is not abelian and so G/G^{d-2} satisfies the condition $(\mathcal{A}, n - 1)$ by Lemma 4.3. Thus by induction the derived length of G/G^{d-2} is at most $2(n-1)-3$ and so $d - 2 \leq 2(n - 1) - 3$. Hence $d \leq 2n - 3$. $\square$

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References

[1] A. Abdollahi, A characterization of infinite 3-abelian groups, *Arch. Math.* (*Basel*) **73** (1999), 104-108.

[2] A. Abdollahi, Some Engel conditions on infinite subsets of certain groups, *Bull. Austral. Math. Soc.* **62** (2000), 141-148.

[3] A. Abdollahi, Finitely generated soluble groups with an Engel condition on infinite subsets, *Rend. Sem. Mat. Univ. Padova* **103** (2000), 47-49.

[4] A. Abdollahi and B. Taeri, A condition on finitely generated soluble groups, *Comm. Algebra* **27** (1999), 5633-5638.

[5] A. Abdollahi and B. Taeri, Some conditions on infinite subsets of infinite groups, *Bull. Malays. Math. Soc.* (*2*) **22** no. 1 (1999), 87-93.

[6] L. Babai, A. J. Goodman and L. Pyber, Groups without faithful transitive permutation representations of small degree, *J. Algebra* **195** (1997), 1-29.

[7] R. D. Blyth and D. J. S. Robinson, Insoluble groups with the rewriting property $\mathbf{P}_8$, *J. Pure Appl. Algebra* **72** (1991), 251-263.

[8] C. Delizia, Finitely generated soluble groups with a condition on infinite subsets, *Instit. Lombardo Accad. Sci. Lett. Rend. A* **128** (1994), 201-208.

[9] C. Delizia, On a certain residually finite groups, *Comm. Algebra* **24** (1996), 3531-3535.

[10] G. Endimioni, Groupes finis satisfaisant la condition $(\mathcal{N}, n)$, *C. R. Acad. Sci. Paris Ser. I* **319** (1994), 1245-1247.

[11] G. Endimioni, Groups covered by finitely many nilpotent subgroups, *Bull. Austral. Math. Soc.* **50** (1994), 459-464.

[12] D. Gorenstein, *Finite groups,* (Harper and Row, New York, 1968).

[13] J. R. J. Groves, A conjecture of Lennox and Wiegold concerning supersoluble groups, *J. Austral. Math. Soc. Ser. A* **35** (1983), 218-220.

[14] B. Huppert, *Endliche Gruppen I,* Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[15] B. Huppert and N. Blackburn, *Finite groups III,* Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[16] J. C. Lennox and J. Wiegold, Extensions of a problem of Paul Erdös on groups, *J. Austral. Math. Soc. Ser. A* **31** (1981), 459-463.

[17] P. Longobardi and M. Maj, Finitely generated soluble groups with an Engel condition on infinite subsets, *Rend. Sem. Mat. Univ. Padova* **89** (1993), 97-102.

[18] P. Longobardi, M. Maj and A. H. Rhemtulla, Infinite groups in a given variety and Ramsey's theorem, *Comm. Algebra* **20** (1992), 127-139.

[19] B. H. Neumann, A problem of Paul Erdös on groups, *J. Austral. Math. Soc. Ser. A* **21** (1976), 467-472.

[20] L. Pyber, The number of pairwise non-commuting elements and the index of the centre in a finite group, *J. London Math. Soc.* **35** (1987), 287-295.

[21] D. J. S. Robinson, *A course in the theory of groups,* 2nd Ed., Springer-Verlag, Berlin-New York, 1995.

[22] N. Trabelsi, Characterization of nilpotent-by-finite groups, *Bull. Austral. Math. Soc.* **61** (2000), 33-38.

[23] M. J. Tomkinson, Hypercentre-by-finite groups, *Publ. Math. Debrecen* **40** (1992), 313-321.

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